Chapter 6

LINEAR EQUATIONS FOR SMALL FIELDS SUPERPOSED ON FINITE BIASING FIELDS

The theory of linear piezoelectricity assumes infinitesimal deviations from an ideal reference state of the material in which there are no pre-existing mechanical and/or electrical fields (initial or biasing fields). The presence of biasing fields makes a material apparently behave like a different material, and renders the linear theory of piezoelectricity invalid. The behavior of electroelastic bodies under biasing fields can be described by the theory for infinitesimal incremental fields superposed on finite biasing fields, which is a consequence of the nonlinear theory of electroelasticity. Knowledge of the behavior of electroelastic bodies under biasing fields is important in many applications including the buckling of thin electroelastic structures, frequency stability of piezoelectric resonators, acoustic wave sensors based on frequency shifts due to biasing fields, characterization of nonlinear electroelastic materials by propagation of small-amplitude waves in electroelastic bodies under biasing fields, and electrostrictive ceramics which operate under a biasing electric field. This chapter presents the theory for small fields superposed on biasing fields in an electroelastic body and some of its applications.

1. A NONLINEAR SPRING

The basic concept of small fields superposed on finite biasing fields can be well explained by a simple nonlinear spring. Consider the following spring-mass system (see Figure 6.1-1). When the spring is stretched by x, the force in the spring is $kx + k'x^2$, where k and k' are linear and nonlinear spring constants.

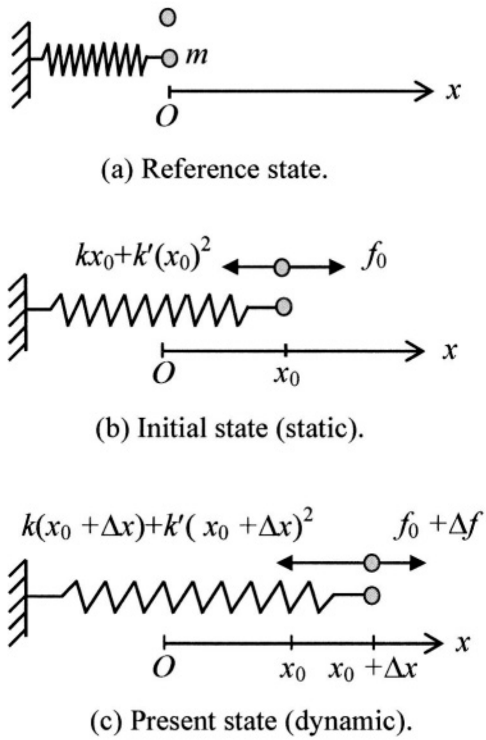

Figure 6.1-1. Reference, initial, and present states of a nonlinear spring-mass system.

The reference state in Figure 6.1-1 (a) is the natural state of the spring when there is no force and stretch in it. Under an initial, constant force f_0 the mass m is in equilibrium with an initial stretch x_0 in the spring (see Figure 6.1-1 (b)) such that

$$f_0 = kx_0 + k'x_0^2. \quad (6.1\text{-}1)$$

Then a small, dynamic, incremental force Δf is applied, and the mass is in small amplitude motion around x_0 with position $x_0 + \Delta x$ (see Figure 6.1-1 (c)). Since both Δf and Δx are small, we want to derive a linear relation between them. In the state in Figure 6.1-1 (c) the equation of motion for the mass is

$$\begin{aligned} m\frac{d^2}{dt^2}(x_0 + \Delta x) &= (f_0 + \Delta f) - [k(x_0 + \Delta x) + k'(x_0 + \Delta x)^2] \\ &= (f_0 + \Delta f) - [kx_0 + k\Delta x + k'x_0^2 + k'2x_0\Delta x + k'(\Delta x)^2] \quad (6.1\text{-}2) \\ &= (f_0 - kx_0 - k'x_0^2) + [\Delta f - k\Delta x - k'2x_0\Delta x - k'(\Delta x)^2]. \end{aligned}$$

Using (6.1-1) and the smallness of Δx,

$$m\frac{d^2}{dt^2}(\Delta x) = \Delta f - k\Delta x - k'2x_0\Delta x - k'(\Delta x)^2$$
$$\cong \Delta f - (k + k'2x_0)\Delta x \quad (6.1\text{-}3)$$
$$= \Delta f - k^e \Delta x,$$

where

$$k^e = k + 2k'x_0 \quad (6.1\text{-}4)$$

is an effective linear spring constant at the initial stretch x_0. Thus at the state with an initial stretch, the nonlinear spring responds to small, incremental changes like a linear spring with an effective linear spring constant k^e. It is important to note that k^e depends on x_0 and the nonlinear spring constant k'.

2. LINEARIZATION ABOUT A BIAS

The concept in the previous section can be generalized to an electroelastic body [35]. Consider the following three states of an electroelastic body (see Figure 6.2-1):

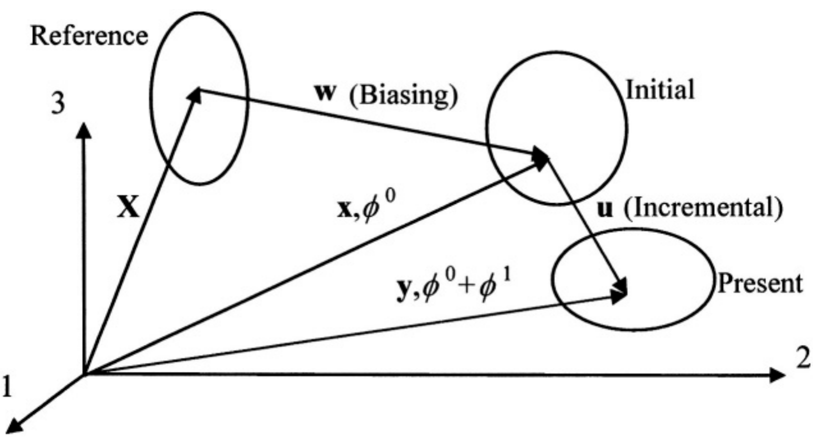

Figure 6.2-1. Reference, initial, and present configurations of an electroelastic body.

(i) The reference state: In this state the body is undeformed and free of electric fields. A generic point at this state is denoted by **X** with Cartesian coordinates X_K. The mass density is ρ_0.

(ii) The initial state: In this state the body is deformed finitely and statically, and carries finite static electric fields. The body is under the action of body force f_α^0, body charge ρ_E^0, prescribed surface position $\bar{x}_\alpha$, surface traction $\bar{T}_\alpha^0$, surface potential $\bar{\phi}^0$ and surface charge $\bar{\sigma}_e^0$. The deformation and fields at this configuration are the initial or biasing fields. The position of the material point associated with **X** is given by **x** = **x**(**X**) or $x_\gamma = x_\gamma(\mathbf{X})$, with strain S_{KL}^0. Greek indices are used for the initial configuration. The electric potential in this state is denoted by $\phi^0(\mathbf{X})$, with electric field $\mathcal{E}_K^0$. **x**(**X**) and $\phi^0(\mathbf{X})$ satisfy the following static equations of nonlinear electroelasticity:

$$
\begin{aligned}
&S_{KL}^0 = (x_{\alpha,K}x_{\alpha,L} - \delta_{KL})/2, \quad \mathcal{E}_K^0 = -\phi_{,K}^0, \quad E_\alpha^0 = -\phi_{,\alpha}^0, \\
&T_{KL}^0 = \rho_0 \left.\frac{\partial \psi}{\partial S_{KL}}\right|_{S_{KL}^0, \mathcal{E}_K^0}, \quad \mathcal{P}_K^0 = -\rho_0 \left.\frac{\partial \psi}{\partial \mathcal{E}_K}\right|_{S_{KL}^0, \mathcal{E}_K^0}, \\
&J^0 = \det(x_{\alpha,K}), \\
&K_{K\alpha}^0 = x_{\alpha,L}T_{KL}^0 + M_{K\alpha}^0, \quad \mathcal{D}_K^0 = \varepsilon_0 J^0 X_{K,\alpha}X_{L,\alpha}\mathcal{E}_L^0 + \mathcal{P}_K^0, \\
&M_{K\alpha}^0 = J^0 X_{K,\beta}\varepsilon_0 (E_\beta^0 E_\alpha^0 - \frac{1}{2}E_\gamma^0 E_\gamma^0 \delta_{\beta\alpha}), \\
&K_{K\alpha,K}^0 + \rho_0 f_\alpha^0 = 0, \quad \mathcal{D}_{K,K}^0 = \rho_E^0.
\end{aligned}
\tag{6.2-1}
$$

(iii) The present state: In this state, time-dependent, small, incremental deformations and electric fields are applied to the deformed body at the initial state. The body is under the action of f_i, ρ_E, $\bar{y}_i$, $\bar{T}_i$, $\bar{\phi}$ and $\bar{\sigma}$. The final position of **X** is given by **y** = **y**(**X**,t), and the final electric potential is $\phi(\mathbf{X},t) = \phi^0(\mathbf{X}) + \phi^1(\mathbf{X},t)$. **y**(**X**,$t$) and $\phi(\mathbf{X},t)$ satisfy the dynamic equations of nonlinear electroelasticity:

$$
\begin{aligned}
&S_{KL} = (y_{i,K}y_{i,L} - \delta_{KL})/2, \quad \mathcal{E}_K = -\phi_{,K}, \quad E_i = -\phi_{,i} \\
&T_{KL}^S = \rho_0 \left.\frac{\partial \psi}{\partial S_{KL}}\right|_{S_{KL}, \mathcal{E}_K}, \quad \mathcal{P}_K = -\rho_0 \left.\frac{\partial \psi}{\partial \mathcal{E}_K}\right|_{S_{KL}, \mathcal{E}_K}, \\
&K_{Lj} = y_{j,K}T_{KL}^S + M_{Lj}, \quad \mathcal{D}_K = \varepsilon_0 J C_{KL}^{-1}\mathcal{E}_L + \mathcal{P}_K, \\
&M_{Lj} = JX_{L,i}\varepsilon_0 (E_i E_j - \frac{1}{2}E_k E_k \delta_{ij}), \\
&K_{Lj,L} + \rho_0 f_j = \rho_0 \dot{v}_j, \quad \mathcal{D}_{K,K} = \rho_E.
\end{aligned}
\tag{6.2-2}
$$

2.1 Linearization of Differential Equations

Let the incremental displacement be $\mathbf{u}(\mathbf{X},t)$ (see Figure 6.2-1). $\mathbf{u}$ and ϕ^1 are assumed to be infinitesimal. We write $\mathbf{y}$ and ϕ as

$$
\begin{aligned}
y_i(\mathbf{X},t) &= \delta_{i\alpha}[x_\alpha(\mathbf{X},t) + \lambda u_\alpha(\mathbf{X},t)], \\
\phi(\mathbf{X},t) &= \phi^0(\mathbf{X},t) + \lambda \phi^1(\mathbf{X},t),
\end{aligned}
\tag{6.2-3}
$$

where a dimensionless parameter λ is introduced to indicate the smallness of the incremental deformations and fields. In the following, terms quadratic in or of higher order of λ will be dropped. Substitution of (6.2-3) into (6.2-2) yields

$$
\begin{aligned}
S_{KL} &= (y_{i,K}y_{i,L} - \delta_{KL})/2 \cong S_{KL}^0 + \lambda S_{KL}^1, \\
\mathcal{E}_K &= -\phi_{,K} = \mathcal{E}_K^0 + \lambda \mathcal{E}_K^1,
\end{aligned}
\tag{6.2-4}
$$

where

$$
\begin{aligned}
S_{KL}^1 &= (x_{\alpha,K}u_{\alpha,L} + x_{\alpha,L}u_{\alpha,K})/2, \\
\mathcal{E}_K^1 &= -\phi_{,K}^1,
\end{aligned}
\tag{6.2-5}
$$

and

$$
\begin{aligned}
T_{KL}^S &\cong T_{KL}^0 + \lambda T_{KL}^1, \\
\mathcal{P}_K &\cong \mathcal{P}_K^0 + \lambda \mathcal{P}_K^1,
\end{aligned}
\tag{6.2-6}
$$

where

$$
\begin{aligned}
T_{KL}^1 &= \rho_0 \left.\frac{\partial^2 \psi}{\partial S_{KL}\partial S_{MN}}\right|_{S_{KL}^0,\mathcal{E}_K^0} S_{MN}^1 + \rho_0 \left.\frac{\partial^2 \psi}{\partial \mathcal{E}_{KL}\partial \mathcal{E}_M}\right|_{S_{KL}^0,\mathcal{E}_K^0} \mathcal{E}_M^1, \\
\mathcal{P}_K^1 &= -\rho_0 \left.\frac{\partial^2 \psi}{\partial \mathcal{E}_K\partial S_{MN}}\right|_{S_{KL}^0,\mathcal{E}_K^0} S_{MN}^1 - \rho_0 \left.\frac{\partial^2 \psi}{\partial \mathcal{E}_K\partial \mathcal{E}_M}\right|_{S_{KL}^0,\mathcal{E}_K^0} \mathcal{E}_M^1.
\end{aligned}
\tag{6.2-7}
$$

To the first order of λ,

$$
\begin{aligned}
X_{K,j} &= X_{K,\alpha}x_{\alpha,j} = X_{K,\alpha}(\delta_{i\alpha}y_i - u_\alpha)_{,j} \\
&= X_{K,\alpha}(\delta_{i\alpha}\delta_{i,j} - \lambda u_{\alpha,j}) = X_{K,\alpha}(\delta_{j\alpha} - \lambda u_{\alpha,L}X_{L,j}) \\
&= X_{K,\alpha}[\delta_{j\alpha} - \lambda u_{\alpha,L}X_{L,\beta}(\delta_{j\beta} - \lambda u_{\beta,M}X_{M,j})] \\
&\cong X_{K,\alpha}[\delta_{j\alpha} - \lambda u_{\alpha,L}X_{L,\beta}\delta_{j\beta}] \\
&= \delta_{j\alpha}X_{K,\alpha} - \lambda X_{K,\alpha}u_{\alpha,L}X_{L,\beta}\delta_{\beta j}.
\end{aligned}
\tag{6.2-8}
$$

From (1.1-22),

$$
\begin{aligned}
JX_{K,j} &= \frac{1}{2}\varepsilon_{KLM}\varepsilon_{jlm}y_{l,L}y_{m,M} \\
&\cong J^0\delta_{j\alpha}X_{K,\alpha} + \lambda\varepsilon_{KLM}\varepsilon_{j\beta\gamma}x_{\beta,L}u_{\gamma,M} \\
&= J^0\delta_{j\alpha}X_{K,\alpha} + \lambda J^0\delta_{j\alpha}(X_{K,\alpha}X_{L,\gamma} - X_{K,\gamma}X_{L,\alpha})u_{\gamma,L}.
\end{aligned}
\tag{6.2-9}
$$

For the electric field

$$
\begin{aligned}
E_i &= -\phi_{,i} = -\phi_{,K}X_{K,i} = \mathcal{E}_K X_{K,i} \\
&\cong \delta_{i\alpha}E^0_\alpha + \lambda\delta_{i\alpha}(\mathcal{E}^1_M X_{M,\alpha} - E^0_\beta u_{\beta,M}X_{M,\alpha}).
\end{aligned}
\tag{6.2-10}
$$

Then the Maxwell stress tensor and $JC^{-1}_{KL}\mathcal{E}_L$ can be expanded as

$$
\begin{aligned}
M_{Ki} &= \delta_{i\alpha}(M^0_{K\alpha} + \lambda M^1_{K\alpha}), \\
JC^{-1}_{KL}\mathcal{E}_L &= (JC^{-1}_{KL}\mathcal{E}_L)^0 + \lambda(JC^{-1}_{KL}\mathcal{E}_L)^1,
\end{aligned}
\tag{6.2-11}
$$

where

$$
\begin{aligned}
M^1_{K\alpha} &= g_{K\alpha L\gamma}u_{\gamma,L} - r_{LK\alpha}\mathcal{E}^1_L, \\
(JC^{-1}_{KL}\mathcal{E}_L)^1 &= r_{KL\gamma}u_{\gamma,L} + l_{KL}\mathcal{E}^1_L,
\end{aligned}
\tag{6.2-12}
$$

and

$$
\begin{aligned}
g_{K\alpha L\gamma} &= \varepsilon_0 J^0[E^0_\alpha E^0_\beta(X_{K,\beta}X_{L,\gamma} - X_{K,\gamma}X_{L,\beta}) \\
&\quad - E^0_\alpha E^0_\gamma X_{K,\beta}X_{L,\beta} \\
&\quad + E^0_\beta E^0_\gamma(X_{K,\alpha}X_{L,\beta} - X_{K,\beta}X_{L,\alpha}) \\
&\quad + \frac{1}{2}E^0_\beta E^0_\beta(X_{K,\gamma}X_{L,\alpha} - X_{K,\alpha}X_{L,\gamma})], \\
r_{KL\gamma} &= \varepsilon_0 J^0(E^0_\alpha X_{K,\alpha}X_{L,\gamma} - E^0_\alpha X_{K,\gamma}X_{L,\alpha} - E^0_\gamma X_{K,\alpha}X_{L,\alpha}), \\
l_{KL} &= \varepsilon_0 J^0 X_{K,\alpha}X_{L,\alpha}.
\end{aligned}
\tag{6.2-13}
$$

Then

$$
\begin{aligned}
K_{Ki} &= y_{i,L}T^S_{KL} + M_{Ki} \cong \delta_{i\alpha}(K^0_{K\alpha} + \lambda K^1_{K\alpha}), \\
\mathcal{D}_K &= \varepsilon_0 JC^{-1}_{KL}\mathcal{E}_L + \mathcal{P}_K \cong \mathcal{D}^0_K + \lambda\mathcal{D}^1_K,
\end{aligned}
\tag{6.2-14}
$$

where

$$
\begin{aligned}
K^1_{K\alpha} &= u_{\alpha,L}T^0_{KL} + x_{\alpha,L}T^1_{KL} + M^1_{K\alpha}, \\
\mathcal{D}^1_K &= \varepsilon_0(JC^{-1}_{KL}E_L)^1 + \mathcal{P}^1_K.
\end{aligned}
\tag{6.2-15}
$$

Equation (6.2-15) can be further written as

$$K^1_{L\gamma} = G_{L\gamma M\alpha} u_{\alpha,M} - R_{ML\gamma} \mathcal{E}^1_M, \qquad (6.2\text{-}16)$$
$$\mathcal{D}^1_K = R_{KL\gamma} u_{\gamma,L} + L_{KL} \mathcal{E}^1_L,$$

which shows that the incremental stress tensor and electric displacement vector depend linearly on the incremental displacement gradient and potential gradient. In (6.2-16),

$$G_{K\alpha L\gamma} = x_{\alpha,M} \rho_0 \left.\frac{\partial^2 \psi}{\partial S_{KM} \partial S_{LN}}\right|_{S^0_{KL}, \mathcal{E}^0_K} x_{\gamma,N}$$
$$+ T^0_{KL} \delta_{\alpha\gamma} + g_{K\alpha L\gamma} = G_{L\gamma K\alpha},$$
$$R_{KL\gamma} = -\rho_0 \left.\frac{\partial^2 \psi}{\partial \mathcal{E}_K \partial S_{ML}}\right|_{S^0_{KL}, \mathcal{E}^0_K} x_{\gamma,M} + r_{KL\gamma}, \qquad (6.2\text{-}17)$$
$$L_{KL} = -\rho_0 \left.\frac{\partial^2 \psi}{\partial \mathcal{E}_K \partial \mathcal{E}_L}\right|_{S^0_{KL}, \mathcal{E}^0_K} + l_{KL} = L_{LK}.$$

$G_{K\alpha L\gamma}$, $R_{KL\gamma}$, and L_{KL} are called the effective or apparent elastic, piezoelectric, and dielectric constants. They depend on the initial deformation $x_\alpha(\mathbf{X})$ and electric potential $\phi^0(\mathbf{X})$. Even when a material is considered linear, i.e., only the second-order material constants need to be considered, the effective material constants still show modifications by the biasing fields. The effective material constants in general have lower symmetry than the fundamental linear elastic, piezoelectric, and dielectric constants. This is called induced anisotropy or symmetry breaking. There can be as many as 45 independent components for $G_{K\alpha L\gamma}$, 27 independent components for $R_{KL\gamma}$, and 6 independent components for L_{KL}. Since the fields in the present configuration satisfy (6.2-2) and the biasing fields satisfy (6.2-1), we have

$$K^1_{K\alpha,K} + \rho_0 f^1_\alpha = \rho_0 \ddot{u}_\alpha, \qquad (6.2\text{-}18)$$
$$\mathcal{D}^1_{K,K} = \rho^1_E,$$

where f^1_α and ρ^1_E are determined from

$$f_i = \delta_{i\alpha}(f^0_\alpha + \lambda f^1_\alpha), \quad \rho_E = \rho^0_E + \lambda \rho^1_E. \qquad (6.2\text{-}19)$$

In the above derivation, λ can be set to 1 everywhere.

The boundary value problem for the incremental fields **u** and ϕ^1 consists of the following equations and boundary conditions:

$$
\begin{aligned}
&K^1_{K\alpha,K} + \rho_0 f^1_\alpha = \rho_0 \ddot{u}_\alpha, \quad \text{in} \quad V,\\
&\mathcal{D}^1_{K,K} = \rho^1_E, \quad \text{in} \quad V,\\
&K^1_{L\gamma} = G_{L\gamma M\alpha} u_{\alpha,M} - R_{ML\gamma} \mathcal{E}^1_M, \quad \text{in} \quad V,\\
&\mathcal{D}^1_K = R_{KL\gamma} u_{\gamma,L} + L_{KL} \mathcal{E}^1_L, \quad \text{in} \quad V,\\
&u_\alpha = \bar{u}_\alpha \quad \text{on} \quad S_y,\\
&\phi^1 = \bar{\phi}^1 \quad \text{on} \quad S_\phi,\\
&K^1_{L\alpha} N_L = \bar{T}^1_\alpha \quad \text{on} \quad S_T,\\
&\mathcal{D}^1_K N_K = -\bar{\sigma}^1_e \quad \text{on} \quad S_D.
\end{aligned}
\qquad (6.2\text{-}20)
$$

Because of the dependence of $G_{K\alpha L\gamma}$, $R_{KL\gamma}$, and L_{KL} on the initial deformations and fields, (6.2-20) in general are equations with variable coefficients.

2.2 A Variational Principle

The symmetries shown in (6.2-17) imply that the differential operators in (6.2-20) are self-adjoint (see Section 6). It can be verified that the stationary condition of the following variational functional under the constraint of the boundary conditions on S_y and S_ϕ yields (6.2-18) and the boundary conditions on S_T and S_D:

$$
\begin{aligned}
\Pi(\mathbf{u},\phi^1) = &\int_{t_0}^{t} dt \int_V \Big(\frac{1}{2}\rho_0 \dot{u}_\alpha \dot{u}_\alpha - \frac{1}{2} G_{K\alpha L\gamma} u_{K,\alpha} u_{L,\gamma}\\
&- R_{KL\gamma} \phi^1_{,K} u_{L,\gamma} + \frac{1}{2} L_{KL} \phi^1_{,K} \phi^1_{,L} + \rho_0 f^1_\alpha u_\alpha - \rho^1_E \phi^1\Big) dV \qquad (6.2\text{-}21)\\
&+ \int_{t_0}^{t} dt \int_{S_T} \bar{T}^1_\alpha u_\alpha dS - \int_{t_0}^{t} dt \int_{S_D} \bar{\sigma}^1_e \phi^1 dS.
\end{aligned}
$$

2.3 Linearization Using the Total Stress Formulation

With the total stress formulation in Section 7 of Chapter 1, the derivation for the equations of the incremental fields can be written in a more compact form as

$$\hat{T}_{KL} = \rho_0 \frac{\partial \hat{\psi}}{\partial S_{KL}} \cong \hat{T}^0_{KL} + \lambda \hat{T}^1_{KL},$$

$$\mathcal{D}_K = -\rho_0 \frac{\partial \hat{\psi}}{\partial \mathcal{E}_K} \cong \mathcal{D}^0_K + \lambda \mathcal{D}^1_K,$$

$$\hat{T}^1_{KL} = \rho_0 \frac{\partial^2 \hat{\psi}}{\partial S_{KL} \partial S_{MN}}\bigg|_{S^0_{KL}, \mathcal{E}^0_K} E^1_{MN} + \rho_0 \frac{\partial^2 \hat{\psi}}{\partial S_{KL} \partial \mathcal{E}_M}\bigg|_{S^0_{KL}, \mathcal{E}^0_K} \mathcal{E}^1_M,$$

$$\mathcal{D}^1_K = -\rho_0 \frac{\partial^2 \hat{\psi}}{\partial \mathcal{E}_K \partial S_{MN}}\bigg|_{S^0_{KL}, \mathcal{E}^0_K} E^1_{MN} - \rho_0 \frac{\partial^2 \hat{\psi}}{\partial \mathcal{E}_K \partial \mathcal{E}_M}\bigg|_{S^0_{KL}, \mathcal{E}^0_K} \mathcal{E}^1_M, \tag{6.2-22}$$

$$K_{Ki} = y_{i,L} \hat{T}_{KL} \cong \delta_{i\alpha} (K^0_{K\alpha} + \lambda K^1_{K\alpha}),$$

$$K^1_{K\alpha} = u_{\alpha,L} \hat{T}^0_{KL} + x_{\alpha,L} \hat{T}^1_{KL},$$

$$K^1_{L\gamma} = G_{L\gamma M\alpha} u_{\alpha,M} - R_{ML\gamma} \mathcal{E}^1_M,$$

$$\mathcal{D}^1_K = R_{KL\gamma} u_{\gamma,L} + L_{KL} \mathcal{E}^1_L,$$

where

$$G_{K\alpha L\gamma} = x_{\alpha,M}\, \rho_0 \frac{\partial^2 \hat{\psi}}{\partial S_{KM} \partial S_{LN}}\bigg|_{S^0_{KL}, \mathcal{E}^0_K} x_{\gamma,N} + \hat{T}^0_{KL} \delta_{\alpha\gamma} = G_{L\gamma K\alpha},$$

$$R_{KL\gamma} = -\rho_0 \frac{\partial^2 \hat{\psi}}{\partial \mathcal{E}_K \partial S_{ML}}\bigg|_{S^0_{KL}, \mathcal{E}^0_K} x_{\gamma,M}, \tag{6.2-23}$$

$$L_{KL} = -\rho_0 \frac{\partial^2 \hat{\psi}}{\partial \mathcal{E}_K \partial \mathcal{E}_L}\bigg|_{S^0_{KL}, \mathcal{E}^0_K} = L_{LK}.$$

Problem

6.2-1. Show (6.2-9).

3. VARIATIONAL APPROACH

The equations for the small incremental fields can also be obtained by making power series expansions in terms of the small incremental fields in the variational functional of nonlinear electroelasticity [36]. Consider the dynamic form of the total energy formulation in (1.8-5). Let

$$y_i = \delta_{i\alpha}(x_\alpha + \lambda u_\alpha) . \tag{6.3-1}$$

Other quantities of the present state can then be written as

$$\begin{aligned}
\phi &= \phi^0 + \lambda\phi^1 + \lambda^2\phi^2 \cdots, \\
f_i &= \delta_{i\alpha}(f_\alpha^0 + \lambda f_\alpha^1 + \lambda^2 f_\alpha^2 \cdots), \\
\rho_E &= \rho_E^0 + \lambda\rho_E^1 + \lambda^2\rho_E^2 \cdots, \\
\bar{y}_i &= \delta_{i\alpha}(\bar{\xi}_\alpha + \lambda\bar{u}_\alpha), \\
\bar{T}_i &= \delta_{i\alpha}(\bar{T}_\alpha^0 + \lambda\bar{T}_\alpha^1 + \lambda^2\bar{T}_\alpha^2 \cdots), \\
\bar{\phi} &= \bar{\phi}^0 + \lambda\bar{\phi}^1 + \lambda^2\bar{\phi}^2 \cdots, \\
\bar{\sigma}_e &= \bar{\sigma}_e^0 + \lambda\bar{\sigma}_e^1 + \lambda^2\bar{\sigma}_e^2 \cdots,
\end{aligned} \tag{6.3-2}$$

where, due to nonlinearity, higher powers of λ may arise. Note that in (6.3-2) the superscripts 0, 1, 2 are for orders of expansions, not for powers except in λ^2. We want to derive equations governing **u** and ϕ^1. From (6.3-1) and (6.3-2), we can further write

$$\begin{aligned}
S_{KL} &= S_{KL}^0 + \lambda S_{KL}^1 + \lambda^2 S_{KL}^2, \\
\mathcal{E}_K &= \mathcal{E}_K^0 + \lambda \mathcal{E}_K^1 + \lambda^2 \mathcal{E}_K^2 \cdots,
\end{aligned} \tag{6.3-3}$$

where

$$\begin{aligned}
S_{KL}^0 &= (\xi_{\alpha,K}\xi_{\alpha,L} - \delta_{KL})/2, \\
S_{KL}^1 &= (\xi_{\alpha,K}u_{\alpha,L} + \xi_{\alpha,L}u_{\alpha,K})/2, \\
S_{KL}^2 &= u_{\alpha,K}u_{\alpha,L}/2, \\
\mathcal{E}_K^0 &= -\phi_{,K}^0, \quad \mathcal{E}_K^1 = -\phi_{,K}^1, \quad \mathcal{E}_K^2 = -\phi_{,K}^2.
\end{aligned} \tag{6.3-4}$$

Substituting (6.3-l)-(6.3-4) into the dynamic form of the Π in (1.8-5), we obtain

$$\Pi = \Pi^0 + \lambda\Pi^1 + \lambda^2\Pi^2 \cdots, \tag{6.3-5}$$

where

$$\begin{aligned}
\Pi^0 = & \int_{t_0}^t dt \int_V [\frac{1}{2}\rho_0\dot{x}_\alpha\dot{x}_\alpha - \rho_0\hat{\psi}(S_{KL}^0, \mathcal{E}_K^0) \\
& + \rho_0 f_\alpha^0 x_\alpha - \rho_E^0\phi^0]dV \\
& + \int_{t_0}^t dt \int_{S_T} \bar{T}_\alpha^0 x_\alpha dS - \int_{t_0}^t dt \int_{S_D} \bar{\sigma}_e^0\phi^0 dS,
\end{aligned} \tag{6.3-6}$$

$$\Pi^1 = \int_{t_0}^{t} dt \int_V \, [\rho_0 \dot{x}_\alpha \dot{u}_\alpha - \rho_0 \frac{\partial \hat{\psi}}{\partial S_{KL}}\bigg|_{S_{KL}^0, \mathcal{E}_K^0} E_{KL}^1 - \rho_0 \frac{\partial \hat{\psi}}{\partial \mathcal{E}_K}\bigg|_{S_{KL}^0, \mathcal{E}_K^0} W_K^1$$

$$+ \rho_0 f_\alpha^0 u_\alpha + \rho_0 f_\alpha^1 x_\alpha - \rho_E^0 \phi^1 - \rho_E^1 \phi^0]dV \tag{6.3-7}$$

$$+ \int_{t_0}^{t} dt \int_{S_T} (\overline{T}_\alpha^0 u_\alpha + \overline{T}_\alpha^1 x_\alpha) dS$$

$$- \int_{t_0}^{t} dt \int_{S_D} \, (\overline{\sigma}_e^0 \phi^1 + \overline{\sigma}_e^1 \phi^0) dS,$$

and

$$\Pi^2 = \int_{t_0}^{t} dt \int_V \, [\frac{1}{2} \rho_0 \dot{u}_\alpha \dot{u}_\alpha - \rho_0 \frac{\partial \hat{\psi}}{\partial S_{KL}}\bigg|_{S_{KL}^0, \mathcal{E}_K^0} S_{KL}^2 - \rho_0 \frac{\partial \hat{\psi}}{\partial \mathcal{E}_K}\bigg|_{S_{KL}^0, \mathcal{E}_K^0} \mathcal{E}_K^2$$

$$- \frac{1}{2} \rho_0 \frac{\partial^2 \hat{\psi}}{\partial S_{KL} \partial S_{MN}}\bigg|_{S_{KL}^0, \mathcal{E}_K^0} S_{KL}^1 S_{MN}^1 - \rho_0 \frac{\partial^2 \hat{\psi}}{\partial \mathcal{E}_M \partial S_{KL}}\bigg|_{S_{KL}^0, \mathcal{E}_K^0} \mathcal{E}_M^1 E_{KL}^1$$

$$- \frac{1}{2} \rho_0 \frac{\partial^2 \hat{\psi}}{\partial \mathcal{E}_M \partial \mathcal{E}_N}\bigg|_{S_{KL}^0, \mathcal{E}_K^0} \mathcal{E}_M^1 \mathcal{E}_N^1 \tag{6.3-8}$$

$$+ \rho_0 f_\alpha^1 u_\alpha + \rho_0 f_\alpha^2 x_\alpha - \rho_E^0 \phi^2 - \rho_E^1 \phi^1 - \rho_E^2 \phi^0]dV$$

$$+ \int_{t_0}^{t} dt \int_{S_T} (\overline{T}_\alpha^1 u_\alpha + \overline{T}_\alpha^2 x_\alpha) dS$$

$$- \int_{t_0}^{t} dt \int_{S_D} \, (\overline{\sigma}_e^0 \phi^2 + \overline{\sigma}_e^1 \phi^1 + \overline{\sigma}_e^2 \phi^0) dS.$$

Comparing (6.3-6) to (1.7-1), we recognize (6.3-6) to be simply the variational functional for the initial deformation which may be dynamic. Since the initial deformation here satisfies the dynamic form of (6.2-1), Π^1 in (6.3-7) can be written into the following much simpler form:

$$\Pi^1 = \int_{t_0}^{t} dt \int_V \, (\rho_0 f_\alpha^1 x_\alpha - \rho_E^1 \phi^0) dV$$
$$+ \int_{t_0}^{t} dt \int_{S_T} \, \overline{T}_\alpha^1 x_\alpha dS - \int_{t_0}^{t} dt \int_{S_D} \, \overline{\sigma}_e^1 \phi^0 dS, \tag{6.3-9}$$

which does not depend on **u** and ϕ^1 anymore. If f_α^0, ρ_E^0, $\overline{T}_\alpha^0$, and $\overline{\sigma}_e^0$ are held constant, or, in other words, $f_\alpha^1 = \rho_E^1 = \overline{T}_\alpha^1 = \overline{\sigma}_e^1 = 0$, then $\Pi^1 = 0$ which simply shows that Π^0 is the variational functional for the initial deformation. Since we are interested in equations for the first-order incremental fields **u** and ϕ^1, we drop all second-order quantities involving ϕ^2, f_α^2, ρ_E^2, $\overline{T}_\alpha^2$ and $\overline{\sigma}_e^2$ in Π^2 and obtain

$$\Pi^2(\mathbf{u},\phi^1) = \int_{t_0}^{t} dt \int_V [\frac{1}{2}\rho_0 \dot{u}_\alpha \dot{u}_\alpha - \frac{1}{2}G_{K\alpha L\gamma} u_{K,\alpha} u_{L,\gamma} - R_{KL\gamma}\phi^1_{,K} u_{L,\gamma}$$

$$+\frac{1}{2}L_{KL}\phi^1_{,K}\phi^1_{,L} + \rho_0 f_\alpha^1 u_\alpha - \rho_E^1 \phi^1]dV \tag{6.3-10}$$

$$+\int_{t_0}^{t} dt \int_{S_T} \overline{T}_\alpha^1 u_\alpha dS - \int_{t_0}^{t} dt \int_{S_D} \overline{\sigma}_e^1 \phi^1 dS,$$

where

$$\begin{aligned} G_{K\alpha L\gamma} &= x_{\alpha,M}\rho_0 \left.\frac{\partial^2 \hat{\psi}}{\partial S_{KM}\partial S_{LN}}\right|_{S_{KL}^0, \mathcal{E}_K^0} x_{\gamma,N} \\ &\quad + \rho_0 \left.\frac{\partial \hat{\psi}}{\partial S_{KL}}\right|_{S_{KL}^0, \mathcal{E}_K^0} \delta_{\alpha\gamma} = G_{L\gamma K\alpha}, \\ R_{KL\gamma} &= -\rho_0 \left.\frac{\partial^2 \hat{\psi}}{\partial \mathcal{E}_K \partial S_{ML}}\right|_{S_{KL}^0, \mathcal{E}_K^0} x_{\gamma,M}, \\ L_{KL} &= -\rho_0 \left.\frac{\partial^2 \hat{\psi}}{\partial \mathcal{E}_K \partial \mathcal{E}_L}\right|_{S_{KL}^0, \mathcal{E}_K^0} = L_{LK}. \end{aligned} \tag{6.3-11}$$

Equation (6.3-11) are the same as (6.2-23). When (1.8-1) is introduced into (6.3-11), with the use of (1.8-2) and (1.8-6)-(1.8-8), (6.2-17) will result.

4. SMALL BIASING FIELDS

In some applications, the biasing deformations and fields are also infinitesimal. In this case, usually only their first-order effects on the incremental fields need to be considered. Then the following energy density of a cubic polynomial is sufficient:

$$\rho_0 \psi(E_{KL}, \mathcal{E}_K) = \frac{1}{2}c_{ABCD}S_{AB}S_{CD} - e_{ABC}\mathcal{E}_A S_{BC} - \frac{1}{2}\chi_{AB}\mathcal{E}_A\mathcal{E}_B$$

$$+\frac{1}{6}c_{ABCDEF}S_{AB}S_{CD}S_{EF} + \frac{1}{2}k_{ABCDE}\mathcal{E}_A S_{BC}S_{DE} \tag{6.4-1}$$

$$-\frac{1}{2}b_{ABCD}\mathcal{E}_A\mathcal{E}_B S_{CD} - \frac{1}{6}\chi_{ABC}\mathcal{E}_A\mathcal{E}_B\mathcal{E}_C,$$

where the subscripts indicating the orders of the material constants have been dropped. For small biasing fields it is convenient to introduce the small displacement vector $\boldsymbol{w}$ of the initial deformation (see Figure 6.2-1), given as

$$x_\alpha = \delta_{\alpha K} X_K + w_\alpha . \tag{6.4-2}$$

Then, neglecting terms quadratic in the gradients of w and ϕ^0, the effective material constants take the following form [35]:

$$G_{K\alpha L\gamma} = c_{K\alpha L\gamma} + \hat{c}_{K\alpha L\gamma}, \quad R_{KL\gamma} = e_{KL\gamma} + \hat{e}_{KL\gamma}, \quad L_{KL} = \varepsilon_{KL} + \hat{\varepsilon}_{KL}, \tag{6.4-3}$$

where

$$\begin{aligned}
\hat{c}_{K\alpha L\gamma} &= T^0_{KL}\delta_{\alpha\gamma} + c_{K\alpha LN} w_{\gamma,N} + c_{KNL\gamma} w_{\alpha,N} \\
&\quad + c_{K\alpha L\gamma AB} S^0_{AB} + k_{AK\alpha L\gamma}\mathcal{E}^0_A, \\
\hat{e}_{KL\gamma} &= e_{KLM} w_{\gamma,M} - k_{KL\gamma AB} S^0_{AB} + b_{AKL\gamma}\mathcal{E}^0_A \\
&\quad + \varepsilon_0(\mathcal{E}^0_K\delta_{L\gamma} - \mathcal{E}^0_L\delta_{K\gamma} - \mathcal{E}^0_M\delta_{M\gamma}\delta_{KL}), \\
\hat{\varepsilon}_{KL} &= b_{KLAB} S^0_{AB} + \chi_{KLA}\mathcal{E}^0_A + \varepsilon_0(S^0_{MM}\delta_{KL} - 2S^0_{KL}), \\
S^0_{AB} &\cong (w_{A,B} + w_{B,A})/2, \\
\mathcal{E}^0_K &= -\phi^0_{,K}.
\end{aligned} \tag{6.4-4}$$

It is important to note that the third-order material constants are necessary for a complete description of the lowest order effects of the biasing fields.

5. THEORY OF INITIAL STRESS

In certain applications, e.g., buckling of thin structures, consideration of initial stresses without initial deformations is sufficient. Such a theory is called the initial stress theory in elasticity. It can be obtained from the theory for incremental fields derived in Section 2. We set $\mathbf{x} = \mathbf{X}$ in the equations for small fields superposed on finite biasing fields. Furthermore, for buckling analysis, a quadratic expression of ψ with second-order material constants only and the corresponding linear constitutive relations are sufficient. The biasing fields can be treated as infinitesimal fields. Then the effective material constants sufficient for describing the buckling phenomenon take the following simple form:

$$\begin{aligned}
G_{K\alpha L\gamma} &= c_{K\alpha L\gamma} + T^0_{KL}\delta_{\alpha\gamma}, \\
R_{KL\gamma} &= e_{KL\gamma} + \varepsilon_0(\mathcal{E}^0_K\delta_{L\gamma} - \mathcal{E}^0_L\delta_{K\gamma} - \mathcal{E}^0_M\delta_{M\gamma}\delta_{KL}), \\
L_{KL} &= \varepsilon_{KL},
\end{aligned} \tag{6.5-1}$$

where T^0_{KL} is the initial stress and $\mathcal{E}^0_K$ is the initial electric field.

Results obtained in buckling analyses of a few thin piezoelectric beams, plates, and shells show that the buckling load of a piezoelectric structure is often related to the corresponding elastic buckling load obtained from an analysis neglecting the piezoelectric coupling in the following manner

$$P_{\text{cr}}^{\text{Piezoelectric}} = (1+\lambda k^2) P_{\text{cr}}^{\text{Elastic}}, \qquad (6.5\text{-}2)$$

where λ is a small, positive number. λ may depend on the material and geometry of the structure. $k^2 = e^2/c\varepsilon > 0$ is an electromechanical coupling factor. When (6.5-2) is true the electromechanical coupling tends to increase the buckling load. In such a case an elastic analysis ignoring the piezoelectric coupling yields a conservative estimate of the buckling load. This is not surprising in view of the piezoelectric stiffening effect. Specific results on buckling of thin piezoelectric structures can be found in the references in a review article [37].

6. FREQUENCY PERTURBATION

Many piezoelectric devices are resonant devices for which frequency consideration is of fundamental importance in design. Analysis based on linear piezoelectricity can provide understanding of the operating principles and basic design tools. This type of analysis is represented by Mindlin's early work on the eigenvalue problem of Section 6 of Chapter 4 [38]. However, devices designed based on linear piezoelectricity are deficient in certain applications. Knowledge of the frequency stability due to environmental effects (e.g., temperature change, force, and acceleration) which cause biasing deformations and frequency shifts is often required for a successful design. For the lowest order effect of the biasing fields, we need to study the eigenvalue problem of an electroelastic body vibrating with the presence of a small bias. From (6.4-4) we have

$$\begin{aligned}
&-[(c_{L\gamma M\alpha} + \varepsilon\hat{c}_{L\gamma M\alpha})u_{\alpha,M} + (e_{ML\gamma} + \varepsilon\hat{e}_{ML\gamma})\phi^1_{,M}]_{,L} = \rho_0\lambda u_\gamma, \quad \text{in} \quad V,\\
&[-(e_{KL\gamma} + \varepsilon\hat{e}_{KL\gamma})u_{\gamma,L} + (\varepsilon_{KL} + \varepsilon\hat{\varepsilon}_{KL})\phi^1_{,L}]_{,K} = 0, \quad \text{in} \quad V,\\
&u_\alpha = 0, \quad \text{on} \quad S_u,\\
&K^1_{L\gamma}N_L = [c_{L\gamma M\alpha} + \varepsilon\hat{c}_{L\gamma M\alpha})u_{\alpha,M} + (e_{ML\gamma} + \varepsilon\hat{e}_{ML\gamma})\phi^1_{,M}]N_L = 0, \quad \text{on} \quad S_T,\\
&\phi^1 = 0, \quad \text{on} \quad S_\phi,\\
&\mathcal{D}^1_K N_K = [(e_{KL\gamma} + \varepsilon\hat{e}_{KL\gamma})u_{\gamma,L} - (\varepsilon_{KL} + \varepsilon\hat{\varepsilon}_{KL})\phi^1_{,L}]N_K = 0, \quad \text{on} \quad S_D,
\end{aligned}$$

(6.6-1)

where $\lambda = \omega^2$, ω and $\{\mathbf{u}, \phi^1\} = \mathbf{U}$ are the resonance frequency and the corresponding mode, respectively, when the biasing fields are present and may be called a perturbed frequency and mode. ε is an artificially introduced dimensionless number to show the smallness of the biasing fields. In terms of the abstract notation in Section 6 of Chapter 4, Equation (6.6-1) can be written as [39]

$$\begin{aligned}
&(\mathbf{A} + \varepsilon \hat{\mathbf{A}})\mathbf{U} = \lambda \mathbf{B}\mathbf{U}, \quad \text{in} \quad V, \\
&u_\alpha = 0, \quad \text{on} \quad S_u, \\
&[(\mathbf{K} + \varepsilon \hat{\mathbf{K}})\mathbf{U}]_{L\gamma} N_L = 0, \quad \text{on} \quad S_T, \\
&\phi^1 = 0, \quad \text{on} \quad S_\phi, \\
&[(\mathbf{D} + \varepsilon \hat{\mathbf{D}})\mathbf{U}]_K N_K = 0, \quad \text{on} \quad S_D,
\end{aligned} \tag{6.6-2}$$

where

$$\begin{aligned}
&\mathbf{AU} = \{-(c_{L\gamma M\alpha} u_{\alpha,M} + e_{ML\gamma} \phi^1_{,M})_{,L}, (-e_{KL\gamma} u_{\gamma,L} + \varepsilon_{KL} \phi^1_{,L})_{,K}\}, \\
&\hat{\mathbf{A}}\mathbf{U} = \{-(\hat{c}_{L\gamma M\alpha} u_{\alpha,M} + \hat{e}_{ML\gamma} \phi^1_{,M})_{,L}, (-\hat{e}_{KL\gamma} u_{\gamma,L} + \hat{\varepsilon}_{KL} \phi^1_{,L})_{,K}\}, \\
&\mathbf{BU} = \{\rho_0 u_\gamma, 0\}, \\
&(\mathbf{KU})_{L\gamma} = c_{L\gamma M\alpha} u_{\alpha,M} + e_{ML\gamma} \phi^1_{,M}, \\
&(\hat{\mathbf{K}}\mathbf{U})_{L\gamma} = \hat{c}_{L\gamma M\alpha} u_{\alpha,M} + \hat{e}_{ML\gamma} \phi^1_{,M}, \\
&(\mathbf{DU})_K = e_{KL\gamma} u_{\gamma,L} - \varepsilon_{KL} \phi^1_{,L}, \\
&(\hat{\mathbf{D}}\mathbf{U})_K = \hat{e}_{KL\gamma} u_{\gamma,L} - \hat{\varepsilon}_{KL} \phi^1_{,L}.
\end{aligned} \tag{6.6-3}$$

We make the following expansions:

$$\begin{aligned}
&\lambda \cong \lambda^{(0)} + \varepsilon \lambda^{(1)}, \\
&\mathbf{U} = \begin{Bmatrix} u_\alpha \\ \phi^1 \end{Bmatrix} \cong \begin{Bmatrix} u_\alpha^{(0)} \\ \phi^{(0)} \end{Bmatrix} + \varepsilon \begin{Bmatrix} u_\alpha^{(1)} \\ \phi^{(1)} \end{Bmatrix} = \mathbf{U}^{(0)} + \varepsilon \mathbf{U}^{(1)}.
\end{aligned} \tag{6.6-4}$$

Substituting (6.6-4) into (6.6-2), collecting terms of equal powers of ε, the following perturbation problems of successive orders can be obtained. Zero-order:

$$
\begin{aligned}
&\mathbf{AU}^{(0)} = \lambda^{(0)}\mathbf{BU}^{(0)}, \quad \text{in} \quad V,\\
&u_\alpha^{(0)} = 0, \quad \text{on} \quad S_u,\\
&(\mathbf{KU}^{(0)})_{L\gamma} N_L = 0, \quad \text{on} \quad S_T,\\
&\phi^{(0)} = 0, \quad \text{on} \quad S_\phi,\\
&(\mathbf{DU}^{(0)})_K N_K = 0, \quad \text{on} \quad S_D,
\end{aligned}
\tag{6.6-5}
$$

which we recognize to be the eigenvalue problem for vibrations of a linear piezoelectric body without biasing fields, treated in Section 6 of Chapter 4. The solution to the zero-order problem, $\lambda^{(0)}$ and $\mathbf{U}^{(0)}$, is assumed known and the first-order problem below is to be solved:

$$
\begin{aligned}
&\mathbf{AU}^{(1)} + \hat{\mathbf{A}}\mathbf{U}^{(0)} = \lambda^{(0)}\mathbf{BU}^{(1)} + \lambda^{(1)}\mathbf{BU}^{(0)}, \quad \text{in} \quad V,\\
&u_\alpha^{(1)} = 0, \quad \text{on} \quad S_u,\\
&(\mathbf{KU}^{(1)} + \hat{\mathbf{K}}\mathbf{U}^{(0)})_{L\gamma} N_L = 0, \quad \text{on} \quad S_T,\\
&\phi^{(1)} = 0, \quad \text{on} \quad S_\phi,\\
&(\mathbf{DU}^{(1)} + \hat{\mathbf{D}}\mathbf{U}^{(0)})_K N_K = 0, \quad \text{on} \quad S_D.
\end{aligned}
\tag{6.6-6}
$$

The equations for the first-order problem can be written as

$$
\mathbf{AU}^{(1)} = \lambda^{(0)}\mathbf{BU}^{(1)} + \lambda^{(1)}\mathbf{BU}^{(0)} - \hat{\mathbf{A}}\mathbf{U}^{(0)}. \tag{6.6-7}
$$

Multiply both sides of (6.6-7) by $\mathbf{U}^{(0)}$

$$
\begin{aligned}
&< \mathbf{AU}^{(1)}; \mathbf{U}^{(0)} >\\
&\quad = \lambda^{(0)} < \mathbf{BU}^{(1)}; \mathbf{U}^{(0)} > + \lambda^{(1)} < \mathbf{BU}^{(0)}; \mathbf{U}^{(0)} >\\
&\quad - < \hat{\mathbf{A}}\mathbf{U}^{(0)}; \mathbf{U}^{(0)} > .
\end{aligned}
\tag{6.6-8}
$$

Similar to (4.6-7), it can be shown that

$$
\begin{aligned}
&< \mathbf{AU}^{(0)}; \mathbf{U}^{(1)} >\\
&\quad = -\int_S [(\mathbf{KU}^{(0)})_{L\gamma} N_L u_\gamma^{(1)} + (\mathbf{DU}^{(0)})_K N_K \phi^{(1)}] dS\\
&\quad + \int_S [(\mathbf{KU}^{(1)})_{M\alpha} N_M u_\alpha^{(0)} + (\mathbf{DU}^{(1)})_L N_L \phi^{(0)}] dS\\
&\quad + < \mathbf{U}^{(0)}; \mathbf{AU}^{(1)} > .
\end{aligned}
\tag{6.6-9}
$$

With (6.6-5) and (6.6-6), Equation (6.6-9) becomes

$$
\begin{aligned}
&< \mathbf{AU}^{(0)}; \mathbf{U}^{(1)} > = -\int_{S_T} (\hat{\mathbf{K}}\mathbf{U}^{(0)})_{M\alpha} N_M u_\alpha^{(0)} dS\\
&\quad - \int_{S_D} (\hat{\mathbf{D}}\mathbf{U}^{(0)})_L N_L \phi^{(0)} dS + < \mathbf{U}^{(0)}; \mathbf{AU}^{(1)} > .
\end{aligned}
\tag{6.6-10}
$$

Substitute (6.6-10) into (6.6-8):

$$\begin{aligned}&\int_{S_T} (\hat{\mathbf{K}}\mathbf{U}^{(0)})_{M\alpha} N_M u_\alpha^{(0)} dS \\ &+ \int_{S_D} (\hat{\mathbf{D}}\mathbf{U}^{(0)})_L N_L \phi^{(0)} dS + <\mathbf{A}\mathbf{U}^{(0)};\mathbf{U}^{(1)}> \\ &= \lambda^{(0)} <\mathbf{U}^{(1)};\mathbf{B}\mathbf{U}^{(0)}> + \lambda^{(1)} <\mathbf{B}\mathbf{U}^{(0)};\mathbf{U}^{(0)}> \\ &- <\hat{\mathbf{A}}\mathbf{U}^{(0)};\mathbf{U}^{(0)}>,\end{aligned} \tag{6.6-11}$$

which can be further written as

$$\begin{aligned}&<\mathbf{A}\mathbf{U}^{(0)} - \lambda^{(0)}\mathbf{B}\mathbf{U}^{(0)};\mathbf{U}^{(1)}> \\ &+ \int_{S_T} (\hat{\mathbf{K}}\mathbf{U}^{(0)})_{M\alpha} N_M u_\alpha^{(0)} dS + \int_{S_D} (\hat{\mathbf{D}}\mathbf{U}^{(0)})_L N_L \phi^{(0)} dS \\ &= \lambda^{(1)} <\mathbf{B}\mathbf{U}^{(0)};\mathbf{U}^{(0)}> - <\hat{\mathbf{A}}\mathbf{U}^{(0)};\mathbf{U}^{(0)}>.\end{aligned} \tag{6.6-12}$$

With Equation $(6.6\text{-}5)_1$, from (6.6-12)

$$\begin{aligned}\lambda^{(1)} &= \frac{1}{<\mathbf{B}\mathbf{U}^{(0)};\mathbf{U}^{(0)}>} \\ &\times\Big[<\hat{\mathbf{A}}\mathbf{U}^{(0)};\mathbf{U}^{(0)}> + \int_{S_T} (\hat{\mathbf{K}}\mathbf{U}^{(0)})_{M\alpha} N_M u_\alpha^{(0)} dS \\ &+ \int_{S_D} (\hat{\mathbf{D}}\mathbf{U}^{(0)})_L N_L \phi^{(0)} dS\Big].\end{aligned} \tag{6.6-13}$$

The above expressions are for the eigenvalue $\lambda = \omega^2$. For ω we make the following expansion:

$$\omega \cong \omega^{(0)} + \varepsilon\omega^{(1)}. \tag{6.6-14}$$

Then

$$\begin{aligned}\lambda &= \omega^2 \cong (\omega^{(0)} + \varepsilon\omega^{(1)})^2 \\ &\cong (\omega^{(0)})^{(2)} + 2\varepsilon\omega^{(0)}\omega^{(1)} \cong \lambda^{(0)} + \varepsilon\lambda^{(1)}.\end{aligned} \tag{6.6-15}$$

Hence

$$\begin{aligned}\frac{\varepsilon\omega^{(1)}}{\omega^{(0)}} &\cong \frac{1}{2(\omega^{(0)})^2}\varepsilon\lambda^{(1)} \\ &= \frac{1}{2(\omega^{(0)})^2}\frac{1}{<\mathbf{B}\mathbf{U}^{(0)};\mathbf{U}^{(0)}>} \\ &\times\Big[<\varepsilon\hat{\mathbf{A}}\mathbf{U}^{(0)};\mathbf{U}^{(0)}> + \int_{S_T} (\varepsilon\hat{\mathbf{K}}\mathbf{U}^{(0)})_{M\alpha} N_M u_\alpha^{(0)} dS \\ &+ \int_{S_D} (\varepsilon\hat{\mathbf{D}}\mathbf{U}^{(0)})_L N_L \phi^{(0)} dS\Big],\end{aligned} \tag{6.6-16}$$

or

$$\frac{\omega-\omega^{(0)}}{\omega^{(0)}} \cong \frac{1}{2(\omega^{(0)})^2} \frac{1}{\int_V \rho_0 u_\alpha^{(0)} u_\alpha^{(0)} dV}$$
$$\times \Big\{ \int_V \Big[-(\varepsilon \hat{c}_{L\gamma M\alpha} u_{\alpha,M}^{(0)} + \varepsilon \hat{e}_{ML\gamma} \phi_{,M}^{(0)})_{,L} u_\gamma^{(0)}$$
$$+ (-\varepsilon \hat{e}_{KL\gamma} u_{\gamma,L}^{(0)} + \varepsilon \hat{\varepsilon}_{KL} \phi_{,L}^{(0)})_{,K} \phi^{(0)} \Big] dV \tag{6.6-17}$$
$$+ \int_{S_T} (\varepsilon \hat{c}_{L\gamma M\alpha} u_{\alpha,M}^{(0)} + \varepsilon \hat{e}_{ML\gamma} \phi_{,M}^{(0)}) N_L u_\gamma^{(0)} dS$$
$$+ \int_{S_D} (\varepsilon \hat{e}_{KL\gamma} u_{\gamma,L}^{(0)} - \varepsilon \hat{\varepsilon}_{KL} \phi_{,L}^{(0)}) N_K \phi^{(0)} dS \Big] \Big\}.$$

With integration by parts, we can write (6.6-17) further as

$$\frac{\omega-\omega^{(0)}}{\omega^{(0)}} \cong \frac{1}{2(\omega^{(0)})^2} \frac{1}{\int_V \rho_0 u_\alpha^{(0)} u_\alpha^{(0)} dV} \tag{6.6-18}$$
$$\times \int_V (\varepsilon \hat{c}_{L\gamma M\alpha} u_{\alpha,M}^{(0)} u_{\gamma,L}^{(0)} + 2\varepsilon \hat{e}_{ML\gamma} \phi_{,M}^{(0)} u_{\gamma,L}^{(0)} - \varepsilon \hat{\varepsilon}_{KL} \phi_{,L}^{(0)} \phi_{,K}^{(0)}) dV.$$

When ε is set to 1, (6.6-18) becomes the well-known first-order perturbation integral for frequency shifts [40].

7. ELECTROSTRICTIVE CERAMICS

As the linear coupling between mechanical and electric fields, piezoelectricity cannot exist in isotropic materials. Mathematically this is the consequence of the fact that a third-rank isotropic tensor with a pair of symmetric indices has to vanish. Electrostriction is a nonlinear electroelastic coupling effect that exists in all dielectrics, isotropic or anisotropic. In the simplest description, electrostriction can be described by the term $b_{ABCD}\mathcal{E}_A \mathcal{E}_B S_{CD}$ in the energy density (6.4-1).

7.1 Nonlinear Theory

Electrostrictive ceramics are macroscopically isotropic due to their polycrystalline structure. For isotropic materials, there are not many independent components of the material tensors, linear or nonlinear. Instead of (6.4-1), it is more convenient to use representations based on tensor invariants of the strain tensor S with components S_{KL} and the material electric field vector $\mathcal{E}$ with components $\mathcal{E}_K$. The invariant representation automatically yields three-dimensional constitutive relations with a few independent material parameters. With the integrity bases for isotropic functions of a symmetric tensor and a vector, it can be determined that for isotropic electroelastic ceramics the energy density function ψ can be written as

$$\psi = \psi(I_1, I_2, I_3, I_4, I_5, I_6), \tag{6.7-1}$$

where the six invariants I_1 through I_6 are given by [1]

$$\begin{aligned} I_1 &= \mathrm{tr}(\mathcal{S}), \quad I_2 = \mathrm{tr}(\mathcal{S}^2), \quad I_3 = \mathrm{tr}(\mathcal{S}^3), \\ I_4 &= \mathcal{E} \cdot \mathcal{E}, \quad I_5 = \mathcal{E} \cdot \mathcal{S} \cdot \mathcal{E}, \quad I_6 = \mathcal{E} \cdot \mathcal{S}^2 \cdot \mathcal{E}. \end{aligned} \tag{6.7-2}$$

In (6.7-2), $\mathcal{S}^2$ stands for $\mathcal{S} \cdot \mathcal{S}$. Equations (6.7-1) and (6.7-2) imply the following constitutive relations for the symmetric stress tensor and the polarization vector:

$$\begin{aligned} \mathbf{T}^S &= \frac{\partial \Sigma}{\partial I_1} \mathbf{1} + 2\frac{\partial \Sigma}{\partial I_2} \mathcal{S} + 3\frac{\partial \Sigma}{\partial I_3} \mathcal{S}^2 + \frac{\partial \Sigma}{\partial I_5} \mathcal{E} \otimes \mathcal{E} \\ &+ \frac{\partial \Sigma}{\partial I_6} [\mathcal{E} \otimes (\mathcal{S} \cdot \mathcal{E}) + (\mathcal{S} \cdot \mathcal{E}) \otimes \mathcal{E}], \end{aligned} \tag{6.7-3}$$

$$\mathcal{P} = -2\frac{\partial \Sigma}{\partial I_4} \mathcal{E} - 2\frac{\partial \Sigma}{\partial I_5} \mathcal{S} \cdot \mathcal{E} - 2\frac{\partial \Sigma}{\partial I_6} \mathcal{S}^2 \cdot \mathcal{E}, \tag{6.7-4}$$

where **1** is the unit tensor of rank two, and $\otimes$ represents tensor or dyadic product. Equation (6.7-3) and (6.7-4) are the most general constitutive relations of isotropic, nonlinear electroelastic materials. Although seemingly simple, they can be complicated functions of $\mathcal{S}_{KL}$ and $\mathcal{E}_K$. Under the inversion of $\mathcal{E} \to -\mathcal{E}$, we have $\mathcal{P} \to -\mathcal{P}$ and $\mathbf{T}^S \to \mathbf{T}^S$, indicating that $\mathcal{P}$ is odd and $\mathbf{T}^S$ is even in $\mathcal{E}$. Therefore linear dependence of $\mathbf{T}^S$ on $\mathcal{E}$ (piezoelectricity) is not allowed, but higher order couplings are possible. In particular, electrostrictive effect can be seen from, e.g., the fourth term on the right-hand side of (6.7-3), which is due to I_5.

7.2 Effects of a Small, Electrical Bias

Electrostrictive ceramics operate under a biasing electric field. If a small biasing electric field $\mathcal{E}^0$ is applied, the small biasing fields are purely electrical because there is no linear electromechanical coupling in the material. In such a case, the effective material constants under the electrical bias are

$$
\begin{aligned}
G_{K\alpha L\gamma} &= c_{K\alpha LN}, \\
R_{KL\gamma} &= b_{AKL\gamma}\mathcal{E}_A^0 + \varepsilon_0(\mathcal{E}_K^0\delta_{L\gamma} - \mathcal{E}_L^0\delta_{K\gamma} - \mathcal{E}_M^0\delta_{M\gamma}\delta_{KL}), \qquad (6.7\text{-}5) \\
L_{KL} &= \varepsilon_{KL}.
\end{aligned}
$$

Thus electrostrictive ceramics appear to be piezoelectric under a biasing electric field, and the effective piezoelectric constants $R_{KL\gamma}$ are tunable by the biasing electric field $\mathcal{E}^0$.

For the simplest model of electrostrictive ceramics, consider the case of infinitesimal deformation. We construct an energy density function as follows:

$$
\rho_0\psi = c_1 I_1^2 + c_2 I_2 - \varepsilon_0\chi I_4/2 + b_1 I_1 I_4 + b_2 I_5, \qquad (6.7\text{-}6)
$$

where c_1 and c_2 are elastic constants, χ is the relative dielectric permittivity, b_1 and b_2 are electrostrictive constants. The constitutive relations generated by (6.7-4) are

$$
\begin{aligned}
T_{KL}^S &= (2c_1 I_1 + b_1 I_4)\delta_{KL} + 2c_2 S_{KL} + b_2\mathcal{E}_K\mathcal{E}_L, \\
\mathcal{P}_K &= (\varepsilon_0\chi - 2b_1 I_1)\mathcal{E}_K - 2b_2 S_{KL}\mathcal{E}_L.
\end{aligned} \qquad (6.7\text{-}7)
$$

Under a biasing electric field $\mathcal{E}^0$ in the x_3 direction, from (6.7-5), the effective piezoelectric constants can be obtained as

$$
\begin{aligned}
&R_{113} = R_{131} = R_{223} = R_{232} = -\mathcal{E}_3^0 b_2 - \varepsilon_0\mathcal{E}_3^0, \\
&R_{311} = R_{322} = -2\mathcal{E}_3^0 b_1 + \varepsilon_0\mathcal{E}_3^0, \\
&R_{333} = -2\mathcal{E}_3^0(b_1 + b_2) - \varepsilon_0\mathcal{E}_3^0, \\
&\text{All other } R_{KL\gamma} = 0.
\end{aligned} \qquad (6.7\text{-}8)
$$

Note that since there are only two electrostrictive material constants, the following relation exists

$$
2R_{113} + R_{311} = R_{333}. \qquad (6.7\text{-}9)
$$

The nonzero tensor components of the electrostrictive constants are related to the material constants in (6.7-6) by

$$
\begin{aligned}
b_{3113} &= b_{3131} = b_{3223} = b_{3232} = -b_2, \\
b_{3311} &= b_{3322} = -2b_1, \\
b_{3333} &= -2(b_1 + b_2).
\end{aligned} \qquad (6.7\text{-}10)
$$