Chapter 4

VIBRATIONS OF FINITE BODIES

This chapter and Chapter 5 are on the linear dynamics of piezoelectrics. In this chapter we discuss time-harmonic vibrations of finite bodies, which are fundamental to device applications. Both free and forced vibrations are examined. Sections 1 to 5 present exact solutions from the three-dimensional equations. Section 6 provides some general results of the eigenvalue problem for the free vibration of a piezoelectric body. Sections 7 to 11 give approximate solutions of a few vibration problems that are very useful but do not allow simple, exact solutions. However, with some very accurate approximations, the problems can be solved very easily. Section 12 presents a special problem, i.e., frequency shifts of a piezoelectric body due to small amounts of mass added to its surface. This problem is particularly useful in sensor applications. It is treated by a perturbation method and a simple formula for frequency shifts is obtained.

1. THICKNESS-STRETCH VIBRATION OF A CERAMIC PLATE (THICKNESS EXCITATION)

Solutions to thickness vibrations of piezoelectric plates can be obtained in a general manner [19]. To simplify the algebra we discuss a few special cases in Sections 1 to 3. Consider a ceramic plate poled along the x_3 axis (see Figure 4.1-1). The plate is bounded by two planes at $x_3 = \pm h$ which are traction-free and electroded. A time-harmonic voltage is applied across the plate thickness.

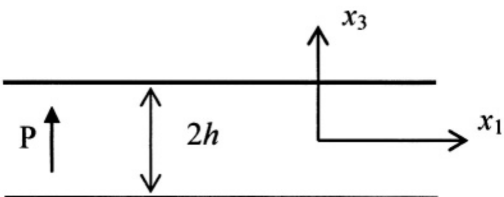

Figure 4.1-1. An electroded ceramic plate with thickness poling.

1.1 Boundary-Value Problem

The boundary-value problem is:

$$T_{ji,j} = \rho \ddot{u}_i, \quad D_{i,i} = 0, \quad \text{in} \quad V,$$
$$T_{ij} = c_{ijkl} S_{kl} - e_{kij} E_k, \quad D_i = e_{ikl} S_{kl} + \varepsilon_{ik} E_k, \quad \text{in} \quad V,$$
$$S_{ij} = (u_{i,j} + u_{j,i})/2, \quad E_i = -\phi_{,i}, \quad \text{in} \quad V, \tag{4.1-1}$$
$$T_{3j} = 0, \quad x_3 = \pm h,$$
$$\phi(x_3 = h) - \phi(x_3 = -h) = V e^{i\omega t}.$$

Consider a possible solution in the following form:

$$u_3 = u_3(x_3) e^{i\omega t}, \quad u_1 = u_2 = 0, \quad \phi = \phi(x_3) e^{i\omega t}. \tag{4.1-2}$$

The nontrivial components of strain and electric field are

$$S_{33} = u_{3,3}, \quad E_3 = -\phi_{,3}, \tag{4.1-3}$$

where the time-harmonic factor has been dropped. The nontrivial stress and electric displacement components are

$$T_{11} = T_{22} = c_{13} u_{3,3} + e_{31} \phi_{,3}$$
$$T_{33} = c_{33} u_{3,3} + e_{33} \phi_{,3}, \tag{4.1-4}$$
$$D_3 = e_{33} u_{3,3} - \varepsilon_{33} \phi_{,3}.$$

The equations to be satisfied are

$$c_{33} u_{3,33} + e_{33} \phi_{,33} = -\rho \omega^2 u_3,$$
$$e_{33} u_{3,33} - \varepsilon_{33} \phi_{,33} = 0. \tag{4.1-5}$$

Equation $(4.1\text{-}5)_2$ can be integrated to yield

$$\phi = \frac{e_{33}}{\varepsilon_{33}} u_3 + B_1 x_3 + B_2, \tag{4.1-6}$$

where B_1 and B_2 are integration constants, and B_2 is immaterial. Substitute Equation (4.1-6) into the expressions for T_{33}, D_3, and $(4.1\text{-}5)_1$:

$$T_{33} = \bar{c}_{33} u_{3,3} + e_{33} B_1, \quad D_3 = -\varepsilon_{33} B_1, \tag{4.1-7}$$

$$\bar{c}_{33} u_{3,33} = -\rho \omega^2 u_3, \tag{4.1-8}$$

where

$$\bar{c}_{33} = c_{33}(1 + k_{33}^2), \quad k_{33}^2 = \frac{e_{33}^2}{\varepsilon_{33} c_{33}}. \tag{4.1-9}$$

The general solution to (4.1-8) and the corresponding expression for the electric potential are

$$u_3 = A_1 \sin \xi x_3 + A_2 \cos \xi x_3,$$
$$\phi = \frac{e_{33}}{\varepsilon_{33}}(A_1 \sin \xi x_3 + A_2 \cos \xi x_3) + B_1 x_3 + B_2, \tag{4.1-10}$$

where A_1 and A_2 are integration constants, and

$$\xi^2 = \frac{\rho}{\bar{c}_{33}}\omega^2. \tag{4.1-11}$$

The expression for stress is then

$$T_{33} = \bar{c}_{33}(A_1\xi \cos \xi x_3 - A_2\xi \sin \xi x_3) + e_{33}B_1. \tag{4.1-12}$$

The boundary conditions require that

$$\bar{c}_{33}A_1\xi \cos \xi h - \bar{c}_{33}A_2\xi \sin \xi h + e_{33}B_1 = 0,$$
$$\bar{c}_{33}A_1\xi \cos \xi h + \bar{c}_{33}A_2\xi \sin \xi h + e_{33}B_1 = 0, \tag{4.1-13}$$
$$2\frac{e_{33}}{\varepsilon_{33}}A_1 \sin \xi h + 2B_1 h = V,$$

or, add the first two, and subtract the first two from each other:

$$\bar{c}_{33}A_1\xi \cos \xi h + e_{33}B_1 = 0,$$
$$\bar{c}_{33}A_2\xi \sin \xi h = 0, \tag{4.1-14}$$
$$2\frac{e_{33}}{\varepsilon_{33}}A_1 \sin \xi h + 2B_1 h = V.$$

1.2 Free Vibration

Consider free vibrations with $V = 0$ first. Equation (4.1-14) decouples into two sets of equations.

1.2.1 Anti-Symmetric Modes

One set is called anti-symmetric modes for which

$$\bar{c}_{33}A_2\xi \sin \xi h = 0. \tag{4.1-15}$$

Nontrivial solutions may exist if

$$\sin \xi h = 0, \tag{4.1-16}$$

or

$$\xi^{(n)}h = \frac{n\pi}{2}, \quad n = 0, 2, 4, 6, \cdots, \tag{4.1-17}$$

which determines the resonance frequencies

$$\omega^{(n)} = \frac{n\pi}{2h}\sqrt{\frac{\bar{c}_{33}}{\rho}}, \quad n = 0, 2, 4, 6, \cdots. \tag{4.1-18}$$

Equation (4.1-16) implies that $B_1 = 0$ and $A_1 = 0$. The corresponding modes are

$$u_3^{(n)} = \cos\xi^{(n)}x_3, \quad \phi^{(n)} = \frac{e_{33}}{\varepsilon_{33}}\cos\xi^{(n)}x_3, \tag{4.1-19}$$

where $n = 0$ is a rigid body mode.

1.2.1 Symmetric Modes

For symmetric modes

$$\begin{aligned} &\bar{c}_{33}A_1\xi\cos\xi h + e_{33}B_1 = 0, \\ &2\frac{e_{33}}{\varepsilon_{33}}A_1\sin\xi h + 2B_1 h = 0. \end{aligned} \tag{4.1-20}$$

The resonance frequencies are determined by

$$\begin{vmatrix} \bar{c}_{33}\xi\cos\xi h & e_{33} \\ \frac{e_{33}}{\varepsilon_{33}}\sin\xi h & h \end{vmatrix} = \bar{c}_{33}\xi h\cos\xi h - \frac{e_{33}^2}{\varepsilon_{33}}\sin\xi h = 0, \tag{4.1-21}$$

or

$$\tan\xi h = \frac{\xi h}{\bar{k}_{33}^2}, \tag{4.1-22}$$

where

$$\bar{k}_{33}^2 = \frac{e_{33}^2}{\varepsilon_{33}\bar{c}_{33}} = \frac{e_{33}^2}{\varepsilon_{33}c_{33}(1+k_{33}^2)} = \frac{k_{33}^2}{1+k_{33}^2} = (k_{33}^t)^2. \tag{4.1-23}$$

Equations (4.1-22) and (4.1-20) determine the resonance frequencies and modes. For symmetric modes, $A_2 = 0$.

1.3 Forced Vibration

Next consider forced vibrations. From Equation (4.1-14), $A_2 = 0$ which means that anti-symmetric modes are not excitable by a thickness electric field, and

$$A_1 = \frac{\begin{vmatrix} 0 & e_{33} \\ V & 2h \end{vmatrix}}{\begin{vmatrix} \bar{c}_{33}\xi\cos\xi h & e_{33} \\ 2\dfrac{e_{33}}{\varepsilon_{33}}\sin\xi h & 2h \end{vmatrix}} = \frac{-e_{33}V}{2\bar{c}_{33}\xi h\cos\xi h - 2\dfrac{e_{33}^2}{\varepsilon_{33}}\sin\xi h}, \tag{4.1-24}$$

$$B_1 = \frac{\begin{vmatrix} \bar{c}_{33}\xi\cos\xi h & 0 \\ 2\dfrac{e_{33}}{\varepsilon_{33}}\sin\xi h & V \end{vmatrix}}{\begin{vmatrix} \bar{c}_{33}\xi\cos\xi h & e_{33} \\ 2\dfrac{e_{33}}{\varepsilon_{33}}\sin\xi h & 2h \end{vmatrix}} = \frac{V\bar{c}_{33}\xi\cos\xi h}{2\bar{c}_{33}\xi h\cos\xi h - 2\dfrac{e_{33}^2}{\varepsilon_{33}}\sin\xi h}. \tag{4.1-25}$$

Hence

$$D_3 = -\varepsilon_{33}B_1 = -\varepsilon_{33}\frac{V}{2h}\frac{\xi h}{\xi h - \bar{k}_{33}^2\tan\xi h} = -\sigma_e, \tag{4.1-26}$$

where σ_e is the surface charge per unit area on the electrode at $x_3 = h$. The capacitance per unit area is

$$C = \frac{\sigma_e}{V} = \frac{\varepsilon_{33}}{2h}\frac{\xi h}{\xi h - \bar{k}_{33}^2\tan\xi h}. \tag{4.1-27}$$

We note the following limits:

$$\lim_{e_{33}\to 0} C = \frac{\varepsilon_{33}}{2h},$$
$$\lim_{\omega\to 0} C = \frac{\varepsilon_{33}}{2h}\frac{1}{1-\dfrac{k_{33}^2}{1+k_{33}^2}} = \frac{\varepsilon_{33}}{2h}(1+k_{33}^2) = C_0, \tag{4.1-28}$$

where C_0 is the static capacitance. The motional capacitance C_m is defined by

$$\begin{aligned} C_m &= C - C_0 = \frac{\varepsilon_{33}}{2h}\left[\frac{\xi h}{\xi h - \bar{k}_{33}^2\tan\xi h} - (1+k_{33}^2)\right] \\ &= \frac{\varepsilon_{33}}{2h}\frac{\xi h - (1+k_{33}^2)(\xi h - \bar{k}_{33}^2\tan\xi h)}{\xi h - \bar{k}_{33}^2\tan\xi h} \\ &= \frac{\varepsilon_{33}}{2h}\frac{-k_{33}^2\xi h + k_{33}^2\tan\xi h}{\xi h - \bar{k}_{33}^2\tan\xi h} = k_{33}^2\frac{\varepsilon_{33}}{2h}\frac{\tan\xi h - \xi h}{\xi h - \bar{k}_{33}^2\tan\xi h}. \end{aligned} \tag{4.1-29}$$

Note that C_m depends on electromechanical coupling.

Problem

4.1-1. Study thickness-shear vibration of a ceramic plate with in-plane poling under thickness excitation. Hint: Consider $u_1 = 0$, $u_2 = 0$, $u_3 = u_3(x_1,t)$, and $\phi = \phi(x_1,t)$.

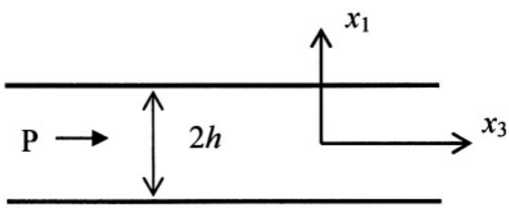

Figure 4.1-2. An electroded ceramic plate with in-plane poling.

2. THICKNESS-STRETCH VIBRATION OF A CERAMIC PLATE (LATERAL EXCITATION)

Consider a ceramic plate poled in the x_3 direction (Figure 4.2-1). The two major surfaces are traction-free and are unelectroded. A voltage is applied across $x_1 = \pm\infty$ and a uniform electric field $E_3(t) = Ee^{i\omega t}$ is produced.

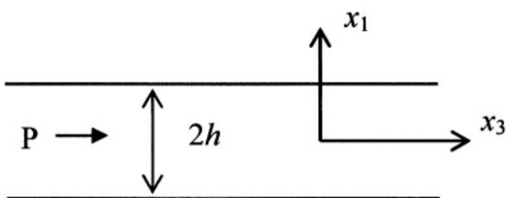

Figure 4.2-1. An unelectroded ceramic plate with in-plane poling.

2.1 Boundary-Value Problem

The boundary-value problem is:

$$
\begin{aligned}
&T_{ji,j} = \rho \ddot{u}_i, \quad D_{i,i} = 0, \quad \text{in} \quad V, \\
&T_{ij} = c_{ijkl} S_{kl} - e_{kij} E_k, \quad D_i = e_{ikl} S_{kl} + \varepsilon_{ik} E_k, \quad \text{in} \quad V, \\
&S_{ij} = (u_{i,j} + u_{j,i})/2, \quad E_i = -\phi_{,i}, \quad \text{in} \quad V, \\
&T_{1j} = 0, \quad D_1 = 0, \quad x_1 = \pm h, \\
&\phi = -x_3 E e^{i\omega t}, \quad \text{in} \quad V.
\end{aligned}
\tag{4.2-1}
$$

Consider the possibility of the following fields:

$$
u_1 = u_1(x_1) e^{i\omega t}, \quad u_2 = u_3 = 0. \tag{4.2-2}
$$

The nontrivial strain and electric field components are

$$
S_{11} = u_{1,1}, \quad E_3 = E, \tag{4.2-3}
$$

where the time-harmonic factor has been dropped. The nontrivial stress and electric displacement components are

$$
\begin{aligned}
T_{11} &= c_{11} u_{1,1} - e_{31} E, \\
T_{22} &= c_{21} u_{1,1} - e_{31} E, \\
T_{33} &= c_{31} u_{1,1} - e_{33} E, \\
D_3 &= e_{31} u_{1,1} + \varepsilon_{33} E.
\end{aligned}
\tag{4.2-4}
$$

The electrical boundary conditions and the charge equation are trivially satisfied. The equation of motion and the mechanical boundary conditions take the following form:

$$
\begin{aligned}
&c_{11} u_{1,11} = -\rho \omega^2 u_1, \quad -h < x_1 < h, \\
&c_{11} u_{1,1} - e_{31} E = 0, \quad x_1 = \pm h,
\end{aligned}
\tag{4.2-5}
$$

which shows that we effectively have an elastic plate driven by a surface traction. The general solution to $(4.2\text{-}5)_1$ is

$$
u_1 = A_1 \sin \xi x_1 + A_2 \cos \xi x_1, \tag{4.2-6}
$$

where A_1 and A_2 are integration constants, and

$$
\xi^2 = \frac{\rho}{c_{11}} \omega^2. \tag{4.2-7}
$$

Then the expression for the stress component relevant to the boundary conditions is

$$
T_{11} = c_{11}(A_1 \xi \cos \xi x_2 - A_2 \xi \sin \xi x_2) - e_{31} E. \tag{4.2-8}
$$

The boundary conditions require that

$$
\begin{aligned}
&c_{11}(A_1 \xi \cos \xi h - A_2 \xi \sin \xi h) - e_{31} E = 0, \\
&c_{11}(A_1 \xi \cos \xi h + A_2 \xi \sin \xi h) - e_{31} E = 0,
\end{aligned}
\tag{4.2-9}
$$

or, add and then subtract

$$c_{11}A_1\xi\cos\xi h = e_{31}E,$$
$$c_{11}A_2\xi\sin\xi h = 0. \quad (4.2\text{-}10)$$

2.2 Free Vibration

First consider free vibrations with $E = 0$. From $(4.2\text{-}10)_2$ nontrivial solutions may exist if

$$\sin\xi h = 0, \quad (4.2\text{-}11)$$

or

$$\xi^{(n)}h = \frac{n\pi}{2}, \quad n = 0, 2, 4, 6, \cdots, \quad (4.2\text{-}12)$$

which determines the following resonance frequencies

$$\omega^{(n)} = \frac{n\pi}{2h}\sqrt{\frac{c_{11}}{\rho}}, \quad n = 0, 2, 4, 6, \cdots. \quad (4.2\text{-}13)$$

Equation (4.2-11) implies that $A_1 = 0$. The corresponding modes are

$$u_1 = \cos\xi^{(n)}x_1, \quad (4.2\text{-}14)$$

which are called anti-symmetric modes, $n = 0$ represents a rigid body mode. For symmetric modes from $(4.2\text{-}10)_1$ ($E = 0$),

$$\cos\xi h = 0, \quad (4.2\text{-}15)$$

or

$$\xi^{(n)}h = \frac{n\pi}{2}, \quad n = 1, 3, 5, \cdots, \quad (4.2\text{-}16)$$

which determines the following resonance frequencies

$$\omega^{(n)} = \frac{n\pi}{2h}\sqrt{\frac{c_{11}}{\rho}}, \quad n = 1, 3, 5, \cdots. \quad (4.2\text{-}17)$$

Equation (4.2-15) implies that $A_2 = 0$. The corresponding modes are

$$u_1 = \sin\xi^{(n)}x_1. \quad (4.2\text{-}18)$$

2.3 Forced Vibration

For forced vibrations $A_2 = 0$ and from $(4.2\text{-}10)_1$,

$$A_1 = \frac{e_{31}}{c_{11}\xi h\cos\xi h}Eh. \quad (4.2\text{-}19)$$

The displacement field is

$$u_1 = \frac{e_{31}}{c_{11}\xi h \cos \xi h} Eh \sin \xi x_1 e^{i\omega t} . \tag{4.2-20}$$

Problem

4.2-1. Study the thickness-shear vibration of a ceramic plate with thickness poling under lateral excitation. Hint: Consider $u_1 = u_1(x_3, t)$, $u_2 = 0$, $u_3 = 0$, and $\phi = -x_1 E e^{i\omega t}$.

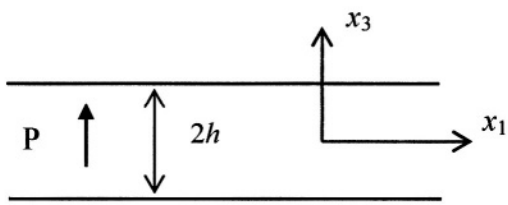

Figure 4.2-2. An unelectroded ceramic plate with thickness poling.

3. THICKNESS-SHEAR VIBRATION OF A QUARTZ PLATE (THICKNESS EXCITATION)

Consider a rotated Y-cut quartz plate. The two major surfaces are traction-free and are electroded, with a driving voltage across the thickness. This structure represents a widely used piezoelectric resonator.

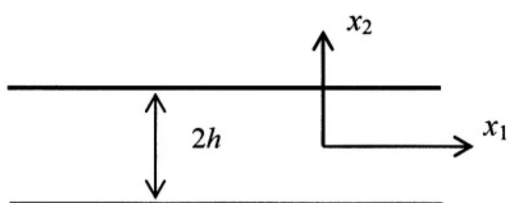

Figure 4.3-1. An electroded quartz plate.

3.1 Boundary-Value Problem

The boundary-value problem is:

$$T_{ji,j} = \rho \ddot{u}_i, \quad D_{i,i} = 0, \quad \text{in} \quad V,$$
$$T_{ij} = c_{ijkl} S_{kl} - e_{kij} E_k, \quad D_i = e_{ikl} S_{kl} + \varepsilon_{ik} E_k, \quad \text{in} \quad V,$$
$$S_{ij} = (u_{i,j} + u_{j,i})/2, \quad E_i = -\phi_{,i}, \quad \text{in} \quad V, \tag{4.3-1}$$
$$T_{2j} = 0, \quad x_2 = \pm h,$$
$$\phi(x_2 = h) - \phi(x_2 = -h) = Ve^{i\omega t}.$$

The problem is mathematically the same as the one in Section 1. Its solution can be obtained from that in Section 1 by changing notation. Because of the importance of this solution in applications, we solve this problem below so that this section can be used independently. Consider the possibility of the following displacement and potential fields:

$$u_1 = u_1(x_2)e^{i\omega t}, \quad u_2 = u_3 = 0, \quad \phi = \phi(x_2)e^{i\omega t}. \tag{4.3-2}$$

The nontrivial components of strain, electric field, stress, and electric displacement are

$$2S_{12} = u_{1,2}, \quad E_2 = -\phi_{,2}, \tag{4.3-3}$$

and

$$T_{31} = c_{56} u_{1,2} + e_{25} \phi_{,2}, \quad T_{12} = c_{66} u_{1,2} + e_{26} \phi_{,2},$$
$$D_2 = e_{26} u_{1,2} - \varepsilon_{22} \phi_{,2}, \quad D_3 = e_{36} u_{1,2} - \varepsilon_{23} \phi_{,2}, \tag{4.3-4}$$

where the time-harmonic factor has been dropped. The equation of motion and the charge equation require that

$$T_{21,2} = c_{66} u_{1,22} + e_{26} \phi_{,22} = -\rho \omega^2 u_1,$$
$$D_{2,2} = e_{26} u_{1,22} - \varepsilon_{22} \phi_{,22} = 0. \tag{4.3-5}$$

Equation $(4.3\text{-}5)_2$ can be integrated to yield

$$\phi = \frac{e_{26}}{\varepsilon_{22}} u_1 + B_1 x_2 + B_2, \tag{4.3-6}$$

where B_1 and B_2 are integration constants, and B_2 is immaterial. Substituting (4.3-6) into the expressions for T_{21}, D_2, and $(4.3\text{-}5)_1$ we obtain

$$T_{21} = \bar{c}_{66} u_{1,2} + e_{26} B_1, \quad D_2 = -\varepsilon_{22} B_1, \tag{4.3-7}$$

$$\bar{c}_{66} u_{1,22} = -\rho \omega^2 u_1, \tag{4.3-8}$$

where

$$\bar{c}_{66} = c_{66}(1 + k_{26}^2), \quad k_{26}^2 = \frac{e_{26}^2}{\varepsilon_{22} c_{66}}. \tag{4.3-9}$$

The general solution to (4.3-8) and the corresponding expression for the electric potential are

$$u_1 = A_1 \sin \xi x_2 + A_2 \cos \xi x_2,$$
$$\phi = \frac{e_{26}}{\varepsilon_{22}}(A_1 \sin \xi x_2 + A_2 \cos \xi x_2) + B_1 x_2 + B_2, \tag{4.3-10}$$

where A_1 and A_2 are integration constants, and

$$\xi^2 = \frac{\rho}{\bar{c}_{66}}\omega^2. \tag{4.3-11}$$

Then the expression for the stress component relevant to boundary conditions is

$$T_{21} = \bar{c}_{66}(A_1\xi \cos \xi x_2 - A_2 \xi \sin \xi x_2) + e_{26} B_1. \tag{4.3-12}$$

The boundary conditions require that

$$\bar{c}_{66} A_1 \xi \cos \xi h - \bar{c}_{66} A_2 \xi \sin \xi h + e_{26} B_1 = 0,$$
$$\bar{c}_{66} A_1 \xi \cos \xi h + \bar{c}_{66} A_2 \xi \sin \xi h + e_{26} B_1 = 0, \tag{4.3-13}$$
$$2\frac{e_{26}}{\varepsilon_{22}} A_1 \sin \xi h + 2 B_1 h = V,$$

or, add the first two, and subtract the first two from each other:

$$\bar{c}_{66} A_1 \xi \cos \xi h + e_{26} B_1 = 0,$$
$$\bar{c}_{66} A_2 \xi \sin \xi h = 0, \tag{4.3-14}$$
$$2\frac{e_{26}}{\varepsilon_{22}} A_1 \sin \xi h + 2 B_1 h = V.$$

3.2 Free Vibration

First we consider free vibrations with $V = 0$. Equation (4.3-14) decouples into two sets of equations. For symmetric modes,

$$\bar{c}_{66} A_2 \xi \sin \xi h = 0. \tag{4.3-15}$$

Nontrivial solutions may exist if

$$\sin \xi h = 0, \tag{4.3-16}$$

or

$$\xi^{(n)} h = \frac{n\pi}{2}, \quad n = 0, 2, 4, 6, \cdots, \tag{4.3-17}$$

which determines the following resonance frequencies

$$\omega^{(n)} = \frac{n\pi}{2h}\sqrt{\frac{\bar{c}_{66}}{\rho}}, \quad n = 0, 2, 4, 6, \cdots. \tag{4.3-18}$$

Equation (4.3-16) implies that $B_1 = 0$ and $A_1 = 0$. The corresponding modes are

$$u_1 = \cos \xi^{(n)} x_2, \quad \phi = \frac{e_{26}}{\varepsilon_{22}} \cos \xi^{(n)} x_2, \tag{4.3-19}$$

where $n = 0$ represents a rigid body mode. For anti-symmetric modes,

$$\begin{aligned} &\bar{c}_{66} A_1 \xi \cos \xi h + e_{26} B_1 = 0, \\ &2 \frac{e_{26}}{\varepsilon_{22}} A_1 \sin \xi h + 2 B_1 h = 0. \end{aligned} \tag{4.3-20}$$

The resonance frequencies are determined by

$$\begin{vmatrix} \bar{c}_{66} \xi \cos \xi h & e_{26} \\ \frac{e_{26}}{\varepsilon_{22}} \sin \xi h & h \end{vmatrix} = \bar{c}_{66} \xi h \cos \xi h - \frac{e_{26}^2}{\varepsilon_{22}} \sin \xi h = 0, \tag{4.3-21}$$

or

$$\tan \xi h = \frac{\xi h}{\bar{k}_{26}^2}, \tag{4.3-22}$$

where

$$\bar{k}_{26}^2 = \frac{e_{26}^2}{\varepsilon_{22} \bar{c}_{66}} = \frac{e_{26}^2}{\varepsilon_{22} c_{66} (1 + k_{26}^2)} = \frac{k_{26}^2}{1 + k_{26}^2}. \tag{4.3-23}$$

Equations (4.3-22) and (4.3-20) determine the resonance frequencies and modes. If the small piezoelectric coupling for quartz is neglected in (4.3-22), a set of frequencies similar to (4.3-17) with n equals odd numbers can be determined for a set of modes with sine dependence on the thickness coordinate. Static thickness-shear deformation and the first few thickness-shear modes in a plate are shown in Figure 4.3-2.

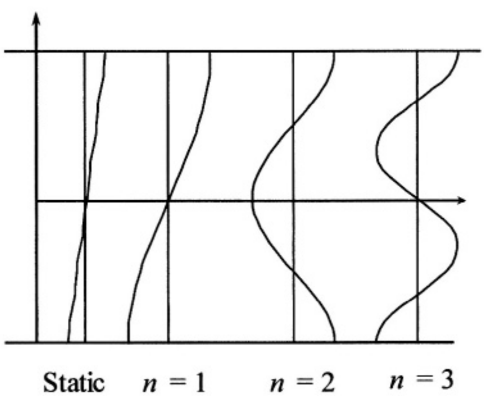

Figure 4.3-2. Thickness-shear deformation and modes in a plate.

3.3 Forced Vibration

For forced vibration we have $A_2 = 0$ and

$$A_1 = \frac{\begin{vmatrix} 0 & e_{26} \\ V & 2h \end{vmatrix}}{\begin{vmatrix} \bar{c}_{66}\xi\cos\xi h & e_{26} \\ 2\dfrac{e_{26}}{\varepsilon_{22}}\sin\xi h & 2h \end{vmatrix}} = \frac{-e_{26}V}{2\bar{c}_{66}\xi h\cos\xi h - 2\dfrac{e_{26}^2}{\varepsilon_{22}}\sin\xi h}, \qquad (4.3\text{-}24)$$

$$B_1 = \frac{\begin{vmatrix} \bar{c}_{66}\xi\cos\xi h & 0 \\ 2\dfrac{e_{26}}{\varepsilon_{22}}\sin\xi h & V \end{vmatrix}}{\begin{vmatrix} \bar{c}_{66}\xi\cos\xi h & e_{26} \\ 2\dfrac{e_{26}}{\varepsilon_{22}}\sin\xi h & 2h \end{vmatrix}} = \frac{V\bar{c}_{66}\xi\cos\xi h}{2\bar{c}_{66}\xi h\cos\xi h - 2\dfrac{e_{26}^2}{\varepsilon_{22}}\sin\xi h}. \qquad (4.3\text{-}25)$$

Hence

$$D_2 = -\varepsilon_{22}B_1 = -\varepsilon_{22}\frac{V}{2h}\frac{\xi h}{\xi h - \bar{k}_{26}^2\tan\xi h} = -\sigma_e, \qquad (4.3\text{-}26)$$

where σ_e is the surface charge per unit area on the electrode at $x_2 = h$. The capacitance per unit area is

$$C = \frac{\sigma_e}{V} = \frac{\varepsilon_{22}}{2h}\frac{\xi h}{\xi h - \bar{k}_{26}^2\tan\xi h}. \qquad (4.3\text{-}27)$$

We note the following limits:

$$\lim_{e_{26}\to 0} C = \frac{\varepsilon_{22}}{2h},$$

$$\lim_{\omega\to 0} C = \frac{\varepsilon_{22}}{2h}\frac{1}{1 - \dfrac{k_{26}^2}{1 + k_{26}^2}} = \frac{\varepsilon_{22}}{2h}(1 + k_{26}^2). \qquad (4.3\text{-}28)$$

3.4 Mechanical Effects of Electrodes

In certain applications, e.g., piezoelectric resonators, the electrodes cannot be treated as a constraint on the electric potential only, and its mechanical effects need to be considered. This may include the inertial effect of the electrode mass and the stiffness of the electrode. Consider a quartz plate with electrodes of unequal thickness on its two major faces as shown in Figure 4.3-3 [20].

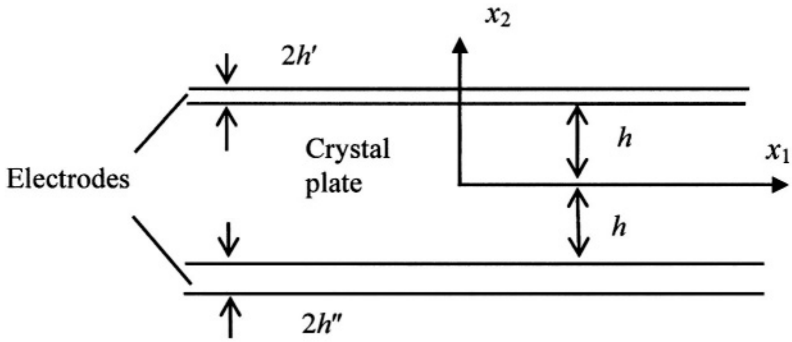

Figure 4.3-3. A quartz plate with electrodes of different thickness.

We are interested in free vibration frequencies. The governing equations are

$$\begin{aligned}
&T_{ji,j} = -\rho\omega^2 u_i, \quad D_{i,i} = 0, \quad -h < x_2 < h, \\
&T_{ij} = c_{ijkl}S_{kl} - e_{kij}E_k, \quad D_i = e_{ikl}S_{kl} + \varepsilon_{ik}E_k, \quad -h < x_2 < h, \\
&S_{ij} = (u_{i,j} + u_{j,i})/2, \quad E_i = -\phi_{,i}, \quad -h < x_2 < h, \\
&T_{ji,j} = -\rho'\omega^2 u_i, \quad -h-h'' < x_2 < -h, \quad h < x_2 < h+h', \\
&T_{ij} = c'_{ijkl}S_{kl}, \quad S_{ij} = (u_{i,j} + u_{j,i})/2, \\
&\quad -h-h'' < x_2 < -h, \quad h < x_2 < h+h',
\end{aligned} \tag{4.3-29}$$

where ρ' and c'_{ijkl} are the mass density and the elastic constants of the electrodes. The two electrodes are of the same isotropic material. The outer surfaces of the electrodes are traction-free. The electrodes are shorted. We have the following boundary and continuity conditions:

$$\begin{aligned}
&T_{2j} = 0, \quad x_2 = h + 2h', \quad x_2 = -h - 2h'', \\
&u_j(x_2 = h^-) = u_j(x_2 = h^+), \\
&T_{2j}(x_2 = h^-) = T_{2j}(x_2 = h^+), \\
&u_j(x_2 = -h^+) = u_j(x_2 = -h^-), \\
&T_{2j}(x_2 = -h^+) = T_{2j}(x_2 = -h^-), \\
&\phi(x_2 = h) = \phi(x_2 = -h).
\end{aligned} \tag{4.3-30}$$

Fields inside the plate are still given by (4.3-10), $(4.3\text{-}7)_2$, and (4.3-12).

For fields inside the electrodes, consider the upper electrode first:

$$T_{21,2} = c'_{66} u_{1,22} = -\rho' \omega^2 u_1 \,, \tag{4.3-31}$$

$$u_1 = A'_1 \sin \xi'(x_2 - h) + A'_2 \cos \xi'(x_2 - h), \tag{4.3-32}$$

$$T_{21} = c'_{66} [A'_1 \xi' \cos \xi'(x_2 - h) - A'_2 \xi' \sin \xi'(x_2 - h)], \tag{4.3-33}$$

where A'_1 and A'_2 are integration constants, and

$$(\xi')^2 = \frac{\rho'}{c'_{66}} \omega^2 . \tag{4.3-34}$$

Similarly, for the lower electrode we have

$$u_1 = A''_1 \sin \xi'(x_2 + h) + A''_2 \cos \xi'(x_2 + h) \tag{4.3-35}$$

$$T_{21} = c'_{66} [A''_1 \xi' \cos \xi'(x_2 + h) - A''_2 \xi' \sin \xi'(x_2 + h)], \tag{4.3-36}$$

where A''_1 and A''_2 are integration constants.

Substituting (4.3-10), (4.3-7)$_2$, (4.3-12), (4.3-32), (4.3-33), (4.3-35), and (4.3-36) into (4.3-30), we obtain

$$\begin{aligned}
& A_1 \sin \xi h + A_2 \cos \xi h = A'_2, \\
& - A_1 \sin \xi h + A_2 \cos \xi h = A''_2, \\
& \bar{c}_{66} \xi (A_1 \cos \xi h - A_2 \sin \xi h) + e_{26} B_1 = c'_{66} \xi' A'_1, \\
& \bar{c}_{66} \xi (A_1 \cos \xi h + A_2 \sin \xi) h + e_{26} B_1 = c'_{66} \xi' A''_1, \\
& A'_1 \cos \xi' 2h' - A'_2 \sin \xi 2h' = 0, \\
& A''_1 \cos \xi 2h'' + A''_2 \sin \xi 2h'' = 0, \\
& \frac{e_{26}}{\varepsilon_{22}} A_1 \sin \xi h + B_1 h = 0.
\end{aligned} \tag{4.3-37}$$

For nontrivial solutions of the undetermined constants, the determinant of the coefficient matrix of (4.3-37) has to vanish. This results in the following frequency equation:

$$\begin{aligned}
& \left(1 - \bar{k}_{26}^2 \frac{\tan \xi h}{\xi h}\right) \left[2 \tan \xi h + \sqrt{\frac{\rho' c'_{66}}{\rho \bar{c}_{66}}} \left(\tan \xi' 2h' + \tan \xi' 2h'' \right) \right] \\
& = \sqrt{\frac{\rho' c'_{66}}{\rho \bar{c}_{66}}} \tan \xi h \left[\tan \xi h \left(\tan \xi' 2h' + \tan \xi' 2h'' \right) \right. \\
& \qquad \left. + 2 \sqrt{\frac{\rho' c'_{66}}{\rho \bar{c}_{66}}} \tan \xi' 2h' \tan \xi' 2h'' \right].
\end{aligned} \tag{4.3-38}$$

We make the following observations from (4.3-38).

(i) In the limit of $h' \to 0$ and $h'' \to 0$, i.e., the mechanical effects of the electrodes are neglected, (4.3-38) reduces to

$$\left(1-\bar{k}_{26}^2\frac{\tan\xi h}{\xi h}\right)\tan\xi h=0\,, \tag{4.3-39}$$

which is the frequency equation of both symmetric and anti-symmetric modes given in (4.3-16) and (4.3-22).

(ii) When $h' = h''$, i.e., the electrodes are of the same thickness, (4.3-38) reduces to

$$\begin{aligned}&\left(1-\bar{k}_{26}^2\frac{\tan\xi h}{\xi h}-\sqrt{\frac{\rho' c'_{66}}{\rho\bar{c}_{66}}}\tan\xi h\tan\xi' 2h'\right)\\&\times\left(\tan\xi h+\sqrt{\frac{\rho' c'_{66}}{\rho\bar{c}_{66}}}\tan\xi' 2h'\right)=0.\end{aligned} \tag{4.3-40}$$

The first factor of (4.3-40) is the frequency equation for the anti-symmetric modes given in [21]. The second factor is for symmetric modes. For small h', i.e., very thin electrodes, we approximately have

$$\tan\xi' 2h'\cong\xi' 2h',\quad \sqrt{\frac{\rho' c'_{66}}{\rho\bar{c}_{66}}}\tan\xi' 2h'\cong R\xi h,\quad R=\frac{\rho' 2h'}{\rho h}. \tag{4.3-41}$$

In this case the first factor of (4.3-40) reduces to

$$\tan\xi h=\frac{\xi h}{\bar{k}_{26}^2+R(\xi h)^2}\,, \tag{4.3-42}$$

which is the result given in [22]. Note that in Equation (4.3-42) the shear stiffness of the electrodes (c'_{66}) has disappeared. Only the mass effect of the electrodes is left and is represented by the mass ratio R.

(iii) For small h' and h'', i.e., thin and unequal electrodes, Equation (4.3-38) reduces to

$$\begin{aligned}&\left(1-\bar{k}_{26}^2\frac{\tan\xi h}{\xi h}\right)\left[2\tan\xi h+(R'+R'')\xi h\right]\\&=\xi h\tan\xi h\left[(R'+R'')\tan\xi h+2R'R''\xi h\right],\end{aligned} \tag{4.3-43}$$

where we have denoted

$$R'=\frac{\rho' 2h'}{\rho h},\quad R''=\frac{\rho' 2h''}{\rho h}. \tag{4.3-44}$$

To the lowest (first) order of the mass effect, the $R'R''$ term on the right-hand side of Equation (4.3-44) can be dropped.

Problem

4.3-1. When the electrodes are very thin, only the inertial effect of the electrode mass needs to be considered; its stiffness can be neglected. The boundary condition on an electroded surface is, according to Newton's 2nd law

$$-T_{ji}n_j = \rho' h' \ddot{u}_i = -\rho' h' \omega^2 u_i \,. \qquad (4.3\text{-}45)$$

Use Equation (4.3-45) to study the anti-symmetric thickness-shear vibration of a quartz plate with electrodes of equal thickness and derive Equation (4.3-42).

4. TANGENTIAL THICKNESS-SHEAR VIBRATION OF A CIRCULAR CYLINDER

Consider an infinite circular cylinder of inner radius a and outer radius b. The cylinder is made of ceramics with tangential poling. We choose (r,θ,z) to correspond to (2,3,1) so that the poling direction corresponds to 3. The inner and outer surfaces are electroded. There is no load applied, and we are interested in free vibrations independent of θ.

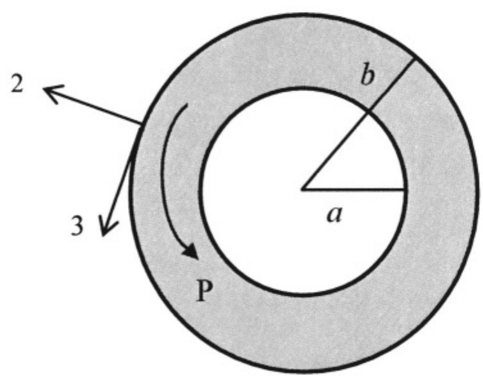

Figure 4.4-1. A circular cylinder with tangential poling.

The boundary-value problem is:

$$\begin{aligned}
&T_{ji,j} = \rho\ddot{u}_i, \quad D_{i,i} = 0, \quad \text{in} \quad V, \\
&T_{ij} = c_{ijkl}S_{kl} - e_{kij}E_k, \quad D_i = e_{ikl}S_{kl} + \varepsilon_{ik}E_k, \quad \text{in} \quad V, \\
&S_{ij} = (u_{i,j} + u_{j,i})/2, \quad E_i = -\phi_{,i}, \quad \text{in} \quad V, \\
&T_{ji}n_j = 0, \quad r = a, b, \\
&\phi(r=a) = \phi(r=b), \quad \text{if the electrodes are shorted,} \\
&\text{or} \quad D_r = 0, \quad r = a, b, \quad \text{if the electrodes are open.}
\end{aligned} \tag{4.4-1}$$

Consider the possibility of the following displacement and potential fields:

$$u_\theta = u_\theta(r)e^{i\omega t}, \quad u_r = u_z = 0, \quad \phi = \phi(r)e^{i\omega t}. \tag{4.4-2}$$

The nontrivial components of strain, electric field, stress, and electric displacement are

$$S_4 = 2S_{r\theta} = \frac{du_\theta}{dr} - \frac{u_\theta}{r}, \quad E_2 = E_r = -\frac{d\phi}{dr}, \tag{4.4-3}$$

$$\begin{aligned}
T_4 = T_{r\theta} &= c_{44}\left(\frac{du_\theta}{dr} - \frac{u_\theta}{r}\right) + e_{15}\frac{d\phi}{dr}, \\
D_2 = D_r &= e_{15}\left(\frac{du_\theta}{dr} - \frac{u_\theta}{r}\right) - \varepsilon_{11}\frac{d\phi}{dr}.
\end{aligned} \tag{4.4-4}$$

Thus on the boundary surfaces at $r = a$ and b there are no tangential electric fields. The electric potential assumes constant values on the electrodes as required. The stress components T_{rr} and T_{rz} vanish everywhere, particularly on the lateral surfaces. The equation of motion and the charge equation to be satisfied are

$$\frac{dT_{r\theta}}{dr} + \frac{2}{r}T_{r\theta} = -\rho\omega^2 u_\theta, \quad \frac{1}{r}(rD_r)_{,r} = 0. \tag{4.4-5}$$

Equation $(4.4\text{-}5)_2$ can be integrated as

$$D_r = e_{15}\frac{C_3}{r}, \tag{4.4-6}$$

where C_3 is an integration constant. Then, from $(4.4\text{-}4)_2$ we have

$$\phi_{,r} = \frac{e_{15}}{\varepsilon_{11}}\left(2S_{r\theta} - \frac{C_3}{r}\right). \tag{4.4-7}$$

Substitution of (4.4-7) into $(4.4\text{-}4)_1$ gives

$$T_{r\theta} = c_{44}\left(\frac{du_\theta}{dr} - \frac{u_\theta}{r}\right) + e_{15}\frac{e_{15}}{\varepsilon_{11}}\left(2S_{r\theta} - \frac{C_3}{r}\right)$$
$$= \bar{c}_{44}\left(\frac{du_\theta}{dr} - \frac{u_\theta}{r}\right) - \frac{e_{15}^2}{\varepsilon_{11}}\frac{C_3}{r}, \tag{4.4-8}$$

where

$$\bar{c}_{44} = c_{44}(1 + k_{15}^2), \quad k_{15}^2 = \frac{e_{15}^2}{\varepsilon_{11}c_{44}}. \tag{4.4-9}$$

Substitute (4.4-8) into $(4.4\text{-}5)_1$:

$$\bar{c}_{44}\left(\frac{d^2u_\theta}{dr^2} - \frac{1}{r}\frac{du_\theta}{dr} + \frac{1}{r^2}u_\theta\right) + \frac{e_{15}^2}{\varepsilon_{11}}\frac{C_3}{r^2}$$
$$+\frac{2}{r}\bar{c}_{44}\left(\frac{du_\theta}{dr} - \frac{u_\theta}{r}\right) - 2\frac{e_{15}^2}{\varepsilon_{11}}\frac{C_3}{r^2} = -\rho\omega^2 u_\theta, \tag{4.4-10}$$

or

$$\bar{c}_{44}\left(\frac{d^2u_\theta}{dr^2} + \frac{1}{r}\frac{du_\theta}{dr} - \frac{1}{r^2}u_\theta\right) + \rho\omega^2 u_\theta = \frac{e_{15}^2}{\varepsilon_{11}}\frac{C_3}{r^2}, \tag{4.4-11}$$

or

$$\frac{d^2u_\theta}{dr^2} + \frac{1}{r}\frac{du_\theta}{dr} - \frac{1}{r^2}u_\theta + \xi^2 u_\theta = \bar{k}_{15}^2\frac{C_3}{r^2}, \tag{4.4-12}$$

where

$$\xi^2 = \frac{\rho\omega^2}{\bar{c}_{44}}, \quad \bar{k}_{15}^2 = \frac{e_{15}^2}{\varepsilon_{11}\bar{c}_{44}} = \frac{e_{15}^2}{\varepsilon_{11}c_{44}(1 + k_{15}^2)} = \frac{k_{15}^2}{1 + k_{15}^2}. \tag{4.4-13}$$

Introduce a dimensionless variable $R = \xi r$. Equation (4.12) can be written as

$$\frac{d^2u_\theta}{dR^2} + \frac{1}{R}\frac{du_\theta}{dR} + \left(1 - \frac{1}{R^2}\right)u_\theta = \bar{k}_{15}^2\frac{C_3}{R^2}, \tag{4.4-14}$$

which is Bessel's equation of order one.

In the following we consider the case when the electrodes at $r = a$ and b are open. The electrical boundary conditions imply, through (4.4-6), that $C_3 = 0$. Then the general solution to (4.4-14) is

$$u_\theta = C_1 J_1(R) + C_2 Y_1(R) = C_1 J_1(\xi r) + C_2 Y_1(\xi r), \tag{4.4-15}$$

where J_1 and Y_1 are the first-order Bessel's functions of the first and second kind, respectively. From (4.4-8) the shear stress is

$$T_{r\theta} = \bar{c}_{44}\left(\frac{du_\theta}{dr} - \frac{u_\theta}{r}\right)$$

$$= \bar{c}_{44}[C_1 J_1'(\xi r)\xi + C_2 Y_1'(\xi r)\xi] - \bar{c}_{44}\frac{1}{r}[C_1 J_1(\xi r) + C_2 Y_1(\xi r)] \quad (4.4\text{-}16)$$

$$= C_1\bar{c}_{44}\xi\left[J_1'(\xi r) - \frac{J_1(\xi r)}{\xi r}\right] + C_2\bar{c}_{44}\xi\left[Y_1'(\xi r) - \frac{Y_1(\xi r)}{\xi r}\right].$$

The traction-free boundary conditions require that

$$C_1\bar{c}_{44}\xi\left[J_1'(\xi a) - \frac{J_1(\xi a)}{\xi a}\right] + C_2\bar{c}_{44}\xi\left[Y_1'(\xi a) - \frac{Y_1(\xi a)}{\xi a}\right] = 0,$$

$$C_1\bar{c}_{44}\xi\left[J_1'(\xi b) - \frac{J_1(\xi b)}{\xi b}\right] + C_2\bar{c}_{44}\xi\left[Y_1'(\xi b) - \frac{Y_1(\xi b)}{\xi b}\right] = 0. \quad (4.4\text{-}17)$$

The frequency equation is given by

$$\begin{vmatrix} J_1'(\xi a) - \dfrac{J_1(\xi a)}{\xi a} & Y_1'(\xi a) - \dfrac{Y_1(\xi a)}{\xi a} \\ J_1'(\xi b) - \dfrac{J_1(\xi b)}{\xi b} & Y_1'(\xi b) - \dfrac{Y_1(\xi b)}{\xi b} \end{vmatrix} = 0. \quad (4.4\text{-}18)$$

Problem

4.4-1. Study the tangential thickness-shear vibration of a circular cylinder of monoclinic crystals [23].

5. AXIAL THICKNESS-SHEAR VIBRATION OF A CIRCULAR CYLINDER

Consider an infinite circular cylinder of inner radius a and outer radius b. The cylinder is made of ceramics with axial poling along the x_3 direction. We choose (r,θ,z) to correspond to (1,2,3) so that the poling direction corresponds to 3. The inner and outer surfaces are electroded. There is no load applied, and we are interested in anti-plane axi-symmetric free vibrations [24].

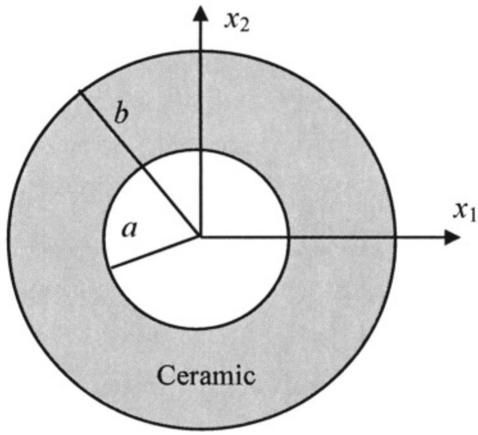

Figure 4.5-1. A circular cylindrical ceramic shell with axial poling.

5.1 Boundary-Value Problem

From Section 6 of Chapter 3, the boundary value problem is:

$$\begin{aligned}
&\bar{c}_{44}\nabla^2 u_z = \rho \ddot{u}_z, \quad a<r<b,\\
&\nabla^2 \psi = 0, \quad a<r<b,\\
&u_z = 0, \quad r=a,b, \quad \text{if the cylindical surfaces are fixed,}\\
&\text{or} \quad T_{rz} = 0, \quad r=a,b, \quad \text{if the cylindical surfaces are free,}\\
&\phi(r=a) = \phi(r=b), \quad \text{if the electrodes are shorted,}\\
&\text{or} \quad D_r = 0, \quad r=a,b, \quad \text{if the electrodes are open,}
\end{aligned} \qquad (4.5\text{-}1)$$

where ϕ and ψ are related by

$$\phi = \psi + \frac{e_{15}}{\varepsilon_{11}} u_z . \qquad (4.5\text{-}2)$$

The stress and electric displacement components are

$$\begin{aligned}
T_{rz} &= \bar{c}_{44} u_{z,r} + e_{15}\psi_{,r},\\
D_r &= -\varepsilon_{11}\psi_{,r}.
\end{aligned} \qquad (4.5\text{-}3)$$

We look for solutions in the following form:

$$\begin{aligned}
u_z(r,t) &= u_z(r)e^{i\omega t},\\
\psi(r,t) &= \psi(r)e^{i\omega t}.
\end{aligned} \qquad (4.5\text{-}4)$$

The equations for u_z and ψ are

$$\nabla^2 u_z = \frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r}\frac{\partial u_z}{\partial r} = -\frac{\rho\omega^2}{\bar{c}_{44}} u_z,$$
$$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} = 0. \tag{4.5-5}$$

The general solution to (4.5-5) is

$$u_z = A_1 J_0(\xi r) + A_2 Y_0(\xi r),$$
$$\psi = A_3 \ln\frac{r}{b} + A_4, \tag{4.5-6}$$

where A_1, A_2, A_3 and A_4 are undetermined constants, J_0 and Y_0 are zero-order Bessel's functions of the first and second kind, and

$$\xi^2 = \frac{\rho\omega^2}{\bar{c}_{44}}. \tag{4.5-7}$$

Hence

$$\phi = \psi + \frac{e_{15}}{\varepsilon_{11}} u_z = \frac{e_{15}}{\varepsilon_{11}}[A_1 J_0(\xi r) + A_2 Y_0(\xi r)] + A_3 \ln\frac{r}{b} + A_4,$$
$$T_{rz} = -\bar{c}_{44}\xi[A_1 J_1(\xi r) + A_2 Y_1(\xi r)] + e_{15}\frac{A_3}{r}, \tag{4.5-8}$$
$$D_r = -\varepsilon_{11}\frac{A_3}{r},$$

where $J_0' = -J_1$ and $Y_0' = -Y_1$ have been used.

5.2 Clamped and Electroded Surfaces

First consider the case when the two cylindrical surfaces are fixed and the two electrodes are shorted. Then we have

$$u = 0, \quad r = a, b,$$
$$\phi = 0, \quad r = a, b, \tag{4.5-9}$$

which implies that

$$\psi = 0, \quad r = a, b. \tag{4.5-10}$$

Hence

$$A_3 = 0, \quad A_4 = 0, \tag{4.5-11}$$

and

$$\begin{vmatrix} J_0(\xi a) & Y_0(\xi a) \\ J_0(\xi b) & Y_0(\xi b) \end{vmatrix} = 0 . \tag{4.5-12}$$

5.3 Free and Unelectroded Surfaces

Next consider the case

$$\begin{aligned} T_{rz} &= 0, \quad r = a, b, \\ D_r &= 0, \quad r = a, b. \end{aligned} \tag{4.5-13}$$

Then $A_3 = 0$ and

$$\begin{vmatrix} J_1(\xi a) & Y_1(\xi a) \\ J_1(\xi b) & Y_1(\xi b) \end{vmatrix} = 0 . \tag{4.5-14}$$

5.4 Free and Electroded Surfaces

Finally, consider

$$\begin{aligned} T_{rz} &= 0, \quad r = a, b, \\ \phi &= 0, \quad r = a, b. \end{aligned} \tag{4.5-15}$$

It can be shown that

$$\begin{vmatrix} J_1(\xi a) & Y_1(\xi a) \\ J_1(\xi b) & Y_1(\xi b) \end{vmatrix} = \frac{\bar{k}_{15}^2}{\xi b \ln \dfrac{a}{b}} \begin{vmatrix} J_0(\xi a) - J_0(\xi b) & J_1(\xi a) - \dfrac{b}{a} J_1(\xi b) \\ Y_0(\xi a) - Y_0(\xi b) & Y_1(\xi a) - \dfrac{b}{a} Y_1(\xi b) \end{vmatrix} . \tag{4.5-16}$$

For large *x*, Bessel functions can be approximated by

$$\begin{aligned} J_\nu(x) &\cong \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4} \right), \\ Y_\nu(x) &\cong \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4} \right). \end{aligned} \tag{4.5-17}$$

Then it can be shown that for large *a* and *b*, (4.5-16) simplifies to

$$\sin \xi(b-a) = \frac{\bar{k}_{15}^2}{\xi b \ln \frac{a}{b}} \left[\left(1 + \frac{b}{a} \right) \cos \xi(b-a) - 2\sqrt{\frac{b}{a}} \right]. \tag{4.5-18}$$

Setting $2h = b\text{-}a$ and allowing $a,\ b \to \infty$, we have

$$\begin{aligned}
&\xi b \ln \frac{a}{b} = \xi b \ln \frac{b-2h}{b} = \xi b \ln\left(1 - \frac{2h}{b} \right) \cong \xi b\left(-\frac{2h}{b} \right) = -\xi 2h, \\
&\sin \xi(b-a) = \sin(\xi 2h) \cong \xi 2h, \\
&\left(1 + \frac{b}{a} \right) \cos \xi(b-a) - 2\sqrt{\frac{b}{a}} \\
&\quad \cong 2\cos(\xi 2h) - 2 = -2[1 - \cos(\xi 2h)] = -4\sin^2 \xi h.
\end{aligned} \tag{4.5-19}$$

Then Equation (4.5-18) reduces to

$$\tan \xi h = \frac{\xi h}{\bar{k}_{15}^2}, \tag{4.5-20}$$

which is the frequency equation for the thickness-shear vibration of a ceramic plate with in-plane poling (see Problem 4.1-1).

Problems

4.5-1. Show (4.5-16).
4.5-2. Show (4.5-18).
4.5-3. Study the case of $u = 0$, $r = a, b$ and $D_r = 0$, $r = a, b$.
4.5-4. Study the axial thickness-shear vibration of a circular cylinder of monoclinic crystals [23].
4.5-5. Study vibrations of a ceramic wedge.

6. SOME GENERAL RESULTS

In this section we prove a few general properties of the eigenvalue problem for the free vibration of a piezoelectric body [25]. The free vibration of a piezoelectric body with frequency ω is governed by the differential equations

$$\begin{aligned}
&-c_{jikl}u_{k,lj} - e_{kji}\phi_{,kj} = \rho\omega^2 u_i, \\
&-e_{ikl}u_{k,li} + \varepsilon_{ik}\phi_{,ki} = 0.
\end{aligned} \tag{4.6-1}$$

6.1 Abstract Formulation

We introduce the following notation:

$$\begin{aligned}
&\lambda=\omega^2, \quad \mathbf{U}=\{u_i,\phi\}, \quad \mathbf{V}=\{v_i,\psi\},\\
&\mathbf{AU}=\{-c_{jikl}u_{k,lj}-e_{kji}\phi_{,kj},-e_{ikl}u_{k,li}+\varepsilon_{ik}\phi_{,ki}\},\\
&\mathbf{BU}=\{\rho u_i,0\},
\end{aligned} \tag{4.6-2}$$

where **U** and **V** are four-vectors and **A** and **B** are operators. Then the eigenvalue problem for the free vibration of a piezoelectric body can be written as

$$\begin{aligned}
&\mathbf{AU}=\lambda\mathbf{BU}, \quad \text{in} \quad V,\\
&u_i=0, \quad \text{on} \quad S_u,\\
&T_{ji}(\mathbf{U})n_j=(c_{jikl}u_{k,l}+e_{kji}\phi_{,k})n_j=0, \quad \text{on} \quad S_T,\\
&\phi=0, \quad \text{on} \quad S_\phi,\\
&D_i(\mathbf{U})n_i=(e_{ikl}u_{k,l}-\varepsilon_{ik}\phi_{,k})n_i=0, \quad \text{on} \quad S_D,
\end{aligned} \tag{4.6-3}$$

which is a homogeneous system. We are interested in nontrivial solutions of **U**. **A** and **B** are real but λ and **U** may be complex at this point. We note that for a nontrivial **U**, its first three components u_i have to be nontrivial, because $u_i = 0$ implies, through (4.6-3), that $\phi = 0$. For convenience we denote the collection of all **U** that are smooth enough and satisfy the boundary conditions in (4.6-3) by

$$H=\{\mathbf{U}\mid \mathbf{U}\text{ satisfies all boundary conditions in }(4.6\text{-}3)_{2\text{-}5}\}. \tag{4.6-4}$$

A scalar product over H is defined by

$$<\mathbf{U};\mathbf{V}>=\int_V (u_i v_i+\phi\psi)dV, \tag{4.6-5}$$

which has the following properties:

$$\begin{aligned}
&<\mathbf{U};\mathbf{V}>=<\mathbf{V};\mathbf{U}>,\\
&<\mathbf{U};\alpha\mathbf{V}+\beta\mathbf{W}>=\alpha<\mathbf{U};\mathbf{V}>+\beta<\mathbf{U};\mathbf{W}>,
\end{aligned} \tag{4.6-6}$$

where α and β are scalars.

6.2 Self-Adjointness

For any **U**, **V** $\in H$

$$
\begin{aligned}
<\mathbf{AU};\mathbf{V}> &=< \{-c_{jikl}u_{k,lj} - e_{kji}\phi_{,kj}, -e_{ikl}u_{k,li} + \varepsilon_{ik}\phi_{,ki}\};\{v_i,\psi\}> \\
&= \int_V [(-c_{jikl}u_{k,lj} - e_{kji}\phi_{,kj})v_i + (-e_{ikl}u_{k,li} + \varepsilon_{ik}\phi_{,ki})\psi]dV \\
&= \int_S [-T_{ji}(\mathbf{U})n_j v_i - D_i(\mathbf{U})n_i\psi]dS \\
&\quad + \int_V [(c_{jikl}u_{k,l}v_{i,j} + e_{kji}\phi_{,k}v_{i,j} + e_{ikl}u_{k,l}\psi_{,i} - \varepsilon_{ik}\phi_{,k}\psi_{,i})dV \\
&= -\int_S [T_{ji}(\mathbf{U})n_j v_i + D_i(\mathbf{U})n_i\psi]dS \\
&\quad + \int_S [T_{kl}(\mathbf{V})n_l u_k + D_k(\mathbf{V})n_k\phi]dS \\
&\quad + \int_V [(-c_{klij}v_{i,jl} - e_{ikl}\psi_{,il})u_k + (-e_{kji}v_{i,jk} + \varepsilon_{ik}\psi_{,ki})\phi]dV \\
&= -\int_S [T_{ji}(\mathbf{U})n_j v_i + D_i(\mathbf{U})n_i\psi]dS \\
&\quad + \int_S [T_{kl}(\mathbf{V})n_l u_k + D_k(\mathbf{V})n_k\phi]dS \\
&\quad + <\mathbf{U};\mathbf{AV}> =< \mathbf{U};\mathbf{AV}>,
\end{aligned}
\tag{4.6-7}
$$

and

$$<\mathbf{BU};\mathbf{V}> =< \mathbf{U};\mathbf{BV}>. \tag{4.6-8}$$

Hence both **A** and **B** are self-adjoint on *H*. Equation (4.6-7) is called the reciprocal theorem in elasticity and Green's identity in mathematics.

6.3 Reality

Let λ be an eigenvalue and **U** the corresponding eigenvector. Hence

$$\mathbf{AU} = \lambda\mathbf{BU}. \tag{4.6-9}$$

Take complex conjugate

$$\mathbf{AU}^* = \lambda^*\mathbf{BU}^*, \tag{4.6-10}$$

where an asterisk means complex conjugate, and we have made use of the fact that **A** and **B** are real. Multiply (4.6-9) by $\mathbf{U}^*$ and (4.6-10) by **U** through the scalar product, and subtract the resulting equations:

$$0 = (\lambda - \lambda^*) < \mathbf{BU};\mathbf{U}^* >. \tag{4.6-11}$$

Since $< \mathbf{BU};\mathbf{U}^* >$ is strictly positive, we have

$$\lambda - \lambda^* = 0, \tag{4.6-12}$$

or λ is real. Then let the real and imaginary parts of **U** be $\mathbf{U}^R$ and $\mathbf{U}^I$. Equation (4.6-9) can be written as

$$\mathbf{A}(\mathbf{U}^R + i\mathbf{U}^I) = \lambda\mathbf{B}(\mathbf{U}^R + i\mathbf{U}^I), \tag{4.6-13}$$

which implies that

$$\mathbf{AU}^R = \lambda\mathbf{BU}^R, \quad \mathbf{AU}^I = \lambda\mathbf{BU}^I. \tag{4.6-14}$$

Equation (4.6-14) shows that $\mathbf{U}^R$ and $\mathbf{U}^I$ are also eigenvectors of λ. In the rest of this section, we will assume that the eigenvectors have been chosen as real.

6.4 Orthogonality

Let $\mathbf{U}^{(m)}$ and $\mathbf{U}^{(n)}$ be two eigenvectors corresponding to two distinct eigenvalues $\lambda^{(m)}$ and $\lambda^{(n)}$, respectively. Then

$$\begin{aligned} \mathbf{AU}^{(m)} &= \lambda^{(m)}\mathbf{BU}^{(m)}, \\ \mathbf{AU}^{(n)} &= \lambda^{(n)}\mathbf{BU}^{(n)}. \end{aligned} \tag{4.6-15}$$

Multiply $(4.6\text{-}15)_1$ by $\mathbf{U}^{(n)}$ and $(4.6\text{-}15)_2$ by $\mathbf{U}^{(m)}$ through the scalar product, and subtract the resulting equations

$$0 = (\lambda^{(m)} - \lambda^{(n)}) < \mathbf{BU}^{(m)}; \mathbf{U}^{(n)} >, \tag{4.6-16}$$

which implies that

$$< \mathbf{BU}^{(m)}; \mathbf{U}^{(n)} > = 0. \tag{4.6-17}$$

The multiplication of $(4.6\text{-}15)_1$ by $\mathbf{U}^{(n)}$ leads to

$$< \mathbf{AU}^{(m)}; \mathbf{U}^{(n)} > = 0. \tag{4.6-18}$$

Equations (4.6-17) and (4.6-18) are called the orthogonality conditions. In unabbreviated form they become

$$\begin{aligned} &< \mathbf{AU}^{(m)}; \mathbf{U}^{(n)} > \\ &= \int_V (c_{ijkl} u_{k,l}^{(m)} u_{i,j}^{(n)} + e_{kji} \phi_{,k}^{(m)} u_{i,j}^{(n)} \\ &\quad + e_{ikl} u_{k,l}^{(m)} \phi_{,i}^{(n)} - \varepsilon_{ik} \phi_{,k}^{(m)} \phi_{,i}^{(n)}) dV = 0, \\ &< \mathbf{BU}^{(m)}; \mathbf{U}^{(n)} > = \int_V \rho_0 u_i^{(m)} u_i^{(n)} dV = 0. \end{aligned} \tag{4.6-19}$$

6.5 Positivity

A subset of H consisting of $\mathbf{U}$ that also satisfies the charge equation is denoted by

$$H^* = \{\mathbf{U} \in H \mid \mathbf{U} \text{ real}, \ -e_{ikl} u_{k,li} + \varepsilon_{ik} \phi_{,ki} = 0 \ \text{in} \ V\}. \tag{4.6-20}$$

For any $\mathbf{U} \in H^*$

$$
\begin{aligned}
&< \mathbf{AU};\mathbf{U} >=< \{-c_{jikl}u_{k,lj} - e_{kji}\phi_{,kj}, -e_{ikl}u_{k,li} + \varepsilon_{ik}\phi_{,ki}\};\{u_i,\phi\} > \\
&= \int_V [(-c_{jikl}u_{k,lj} - e_{kji}\phi_{,kj})u_i + (-e_{ikl}u_{k,li} + \varepsilon_{ik}\phi_{,ki})\phi]dV \\
&= \int_S [-T_{ji}(\mathbf{U})n_j u_i - D_i(\mathbf{U})n_i\phi]dS \\
&\quad + \int_V [(c_{jikl}u_{k,l}u_{i,j} + e_{kji}\phi_{,k}u_{i,j} + e_{ikl}u_{k,l}\phi_{,i} - \varepsilon_{ik}\phi_{,k}\phi_{,i})dV \\
&= \int_V [c_{jikl}u_{k,l}u_{i,j} + \varepsilon_{ik}\phi_{,k}\phi_{,i} + 2(e_{ikl}u_{k,l}\phi_{,i} - \varepsilon_{ik}\phi_{,k}\phi_{,i})]dV \\
&= \int_V [c_{jikl}u_{k,l}u_{i,j} + \varepsilon_{ik}\phi_{,k}\phi_{,i} - 2(e_{ikl}u_{k,li} - \varepsilon_{ik}\phi_{,ki})\phi]dV \\
&\quad + \int_S 2(e_{ikl}u_{k,l} - \varepsilon_{ik}\phi_{,k})n_i\phi dS \\
&= \int_V (c_{jikl}u_{k,l}u_{i,j} + \varepsilon_{ik}\phi_{,k}\phi_{,i})dV \geq 0,
\end{aligned}
\tag{4.6-21}
$$

and

$$< \mathbf{BU};\mathbf{U} > \quad \geq 0. \tag{4.6-22}$$

Multiply (4.6-9) by $\mathbf{U}$

$$< \mathbf{AU};\mathbf{U} >= \lambda < \mathbf{BU};\mathbf{U} >, \tag{4.6-23}$$

which shows that λ is nonnegative.

6.6 Variational Formulation

Consider the following functional (Rayleigh quotient) of $\mathbf{U} \in H$

$$
\begin{aligned}
&\Pi(\mathbf{U}) = \frac{\Lambda(\mathbf{U})}{\Gamma(\mathbf{U})}, \\
&\Lambda(\mathbf{U}) =< \mathbf{AU};\mathbf{U} >, \quad \Gamma(\mathbf{U}) =< \mathbf{BU};\mathbf{U} >.
\end{aligned}
\tag{4.6-24}
$$

The first variation of Π is

$$\delta\Pi = \frac{\Gamma\delta\Lambda - \Lambda\delta\Gamma}{\Gamma^2} = \frac{\delta\Lambda - \Pi\delta\Gamma}{\Gamma}. \tag{4.6-25}$$

Therefore $\delta\Pi = 0$ implies that

$$\delta\Lambda - \Pi\delta\Gamma = 0. \tag{4.6-26}$$

From (4.6-24) we have

$$
\begin{aligned}
\delta\Lambda - \Pi\delta\Gamma &= <\mathbf{AU};\delta\mathbf{U}> + <\delta\mathbf{AU};\mathbf{U}> \\
&\quad -\Pi <\mathbf{BU};\delta\mathbf{U}> -\Pi <\delta\mathbf{BU};\mathbf{U}> \\
&= <\mathbf{AU};\delta\mathbf{U}> + <\mathbf{A}\delta\mathbf{U};\mathbf{U}> \\
&\quad -\Pi <\mathbf{BU};\delta\mathbf{U}> -\Pi <\mathbf{B}\delta\mathbf{U};\mathbf{U}> \\
&= <\mathbf{AU};\delta\mathbf{U}> + <\delta\mathbf{U};\mathbf{AU}> \\
&\quad -\Pi <\mathbf{BU};\delta\mathbf{U}> -\Pi <\delta\mathbf{U};\mathbf{BU}> \\
&= 2 <\mathbf{AU} - \Pi\mathbf{BU};\delta\mathbf{U}>,
\end{aligned}
\tag{4.6-27}
$$

where the small variation $\delta\mathbf{U} \in H$. (4.6-27) implies that

$$
\mathbf{AU} - \Pi\mathbf{BU} = 0\,. \tag{4.6-28}
$$

Hence the $\mathbf{U}$ that makes $\delta\Pi = 0$ is an eigenvector of the eigenvalue Π.

6.7 Perturbation Based on Variational Formulation

Next we consider the case when $\mathbf{A}$ and $\mathbf{B}$ are slightly perturbed but are still self-adjoint, which causes small perturbations in λ and $\mathbf{U}$:

$$
(\mathbf{A} + \Delta\mathbf{A})(\mathbf{U} + \Delta\mathbf{U}) = (\lambda + \Delta\lambda)(\mathbf{B} + \Delta\mathbf{B})(\mathbf{U} + \Delta\mathbf{U})\,. \tag{4.6-29}
$$

We are interested in an expression of $\Delta\lambda$ linear in $\Delta\mathbf{A}$ and $\Delta\mathbf{B}$. From (4.6-24),

$$
\begin{aligned}
\lambda + \Delta\lambda &= \frac{<(\mathbf{A} + \Delta\mathbf{A})(\mathbf{U} + \Delta\mathbf{U});\mathbf{U} + \Delta\mathbf{U}>}{<(\mathbf{B} + \Delta\mathbf{B})(\mathbf{U} + \Delta\mathbf{U});\mathbf{U} + \Delta\mathbf{U}>} \\
&\cong \frac{<\mathbf{AU} + (\Delta\mathbf{A})\mathbf{U} + \mathbf{A}(\Delta\mathbf{U});\mathbf{U} + \Delta\mathbf{U}>}{<\mathbf{BU} + (\Delta\mathbf{B})\mathbf{U} + \mathbf{B}(\Delta\mathbf{U});\mathbf{U} + \Delta\mathbf{U}>} \\
&\cong \frac{<\mathbf{AU};\mathbf{U}> + <\mathbf{AU};\Delta\mathbf{U}> + <(\Delta\mathbf{A})\mathbf{U} + \mathbf{A}(\Delta\mathbf{U});\mathbf{U}>}{<\mathbf{BU};\mathbf{U}> + <\mathbf{BU};\Delta\mathbf{U}> + <(\Delta\mathbf{B})\mathbf{U} + \mathbf{B}(\Delta\mathbf{U});\mathbf{U}>} \\
&= \frac{<\mathbf{AU};\mathbf{U}> + 2<\mathbf{AU};\Delta\mathbf{U}> + <(\Delta\mathbf{A})\mathbf{U};\mathbf{U}>}{<\mathbf{BU};\mathbf{U}> + 2<\mathbf{BU};\Delta\mathbf{U}> + <(\Delta\mathbf{B})\mathbf{U};\mathbf{U}>} \\
&\cong \frac{<\mathbf{AU};\mathbf{U}>}{<\mathbf{BU};\mathbf{U}>}\left(1 + \frac{2<\mathbf{AU};\Delta\mathbf{U}> + <(\Delta\mathbf{A})\mathbf{U};\mathbf{U}>}{<\mathbf{AU};\mathbf{U}>}\right) \\
&\quad \times\left(1 - \frac{2<\mathbf{BU};\Delta\mathbf{U}> + <(\Delta\mathbf{B})\mathbf{U};\mathbf{U}>}{<\mathbf{BU};\mathbf{U}>}\right) \cong
\end{aligned}
$$

$$
\begin{aligned}
&\cong \frac{<\mathbf{AU};\mathbf{U}>}{<\mathbf{BU};\mathbf{U}>}\left(1+\frac{2<\mathbf{AU};\Delta\mathbf{U}>+<(\Delta\mathbf{A})\mathbf{U};\mathbf{U}>}{<\mathbf{AU};\mathbf{U}>}\right. \\
&\qquad \left. -\frac{2<\mathbf{BU};\Delta\mathbf{U}>+<(\Delta\mathbf{B})\mathbf{U};\mathbf{U}>}{<\mathbf{BU};\mathbf{U}>}\right) \\
&= \frac{<\mathbf{AU};\mathbf{U}>}{<\mathbf{BU};\mathbf{U}>}+\frac{2<\mathbf{AU};\Delta\mathbf{U}>+<(\Delta\mathbf{A})\mathbf{U};\mathbf{U}>}{<\mathbf{BU};\mathbf{U}>} \\
&\qquad -\frac{<\mathbf{AU};\mathbf{U}>}{<\mathbf{BU};\mathbf{U}>}\frac{2<\mathbf{BU};\Delta\mathbf{U}>+<(\Delta\mathbf{B})\mathbf{U};\mathbf{U}>}{<\mathbf{BU};\mathbf{U}>} \\
&= \lambda+\frac{2<\mathbf{AU}-\lambda\mathbf{BU};\Delta\mathbf{U}>+<(\Delta\mathbf{A})\mathbf{U};\mathbf{U}>-\lambda<(\Delta\mathbf{B})\mathbf{U};\mathbf{U}>}{<\mathbf{BU};\mathbf{U}>} \\
&= \lambda+\frac{<(\Delta\mathbf{A})\mathbf{U};\mathbf{U}>-\lambda<(\Delta\mathbf{B})\mathbf{U};\mathbf{U}>}{<\mathbf{BU};\mathbf{U}>},
\end{aligned}
\tag{4.6-30}
$$

hence

$$
\Delta\lambda=\frac{<(\Delta\mathbf{A})\mathbf{U};\mathbf{U}>-\lambda<(\Delta\mathbf{B})\mathbf{U};\mathbf{U}>}{<\mathbf{BU};\mathbf{U}>}. \tag{4.6-31}
$$

6.8 Perturbation Based on Abstract Formulation

Equation (4.6-31) can also be obtained from the following perturbation procedure. Expand both sides of (4.6-29). The zero-order terms represent the unperturbed eigenvalue problem. The first-order terms are

$$
\mathbf{A}(\Delta\mathbf{U})+(\Delta\mathbf{A})\mathbf{U}=\Delta\lambda\mathbf{BU}+\lambda(\Delta\mathbf{B})\mathbf{U}+\lambda\mathbf{B}(\Delta\mathbf{U}). \tag{4.6-32}
$$

Multiply both sides by $\mathbf{U}$:

$$
\begin{aligned}
&<\mathbf{A}(\Delta\mathbf{U});\mathbf{U}>+<(\Delta\mathbf{A})\mathbf{U};\mathbf{U}> \\
&\quad =\Delta\lambda<\mathbf{BU};\mathbf{U}>+\lambda<(\Delta\mathbf{B})\mathbf{U};\mathbf{U}>+\lambda<\mathbf{B}(\Delta\mathbf{U});\mathbf{U}>,
\end{aligned}
\tag{4.6-33}
$$

or

$$
\begin{aligned}
&<\Delta\mathbf{U};\mathbf{AU}>+<(\Delta\mathbf{A})\mathbf{U};\mathbf{U}> \\
&\quad =\Delta\lambda<\mathbf{BU};\mathbf{U}>+\lambda<(\Delta\mathbf{B})\mathbf{U};\mathbf{U}>+\lambda<\Delta\mathbf{U};\mathbf{BU}>.
\end{aligned}
\tag{4.6-34}
$$

The first term and the last term in (4.6-34) cancel, and what is left is

$$
\Delta\lambda=\frac{<(\Delta\mathbf{A})\mathbf{U};\mathbf{U}>-\lambda<(\Delta\mathbf{B})\mathbf{U};\mathbf{U}>}{<\mathbf{BU};\mathbf{U}>}, \tag{4.6-35}
$$

which is the same as (4.6-31). Note that in this perturbation procedure, no assumption regarding the self-adjointness of $\Delta\mathbf{A}$ and $\Delta\mathbf{B}$ was made.

7. EXTENSIONAL VIBRATION OF A THIN ROD

Consider a rectangular rod of length l, width w, and thickness t as shown in Figure 4.7-1, where $l >> w >> t$. We are interested in the low frequency extensional vibration of the rod [11]. By low frequency we mean that the wavelength of the vibration modes is much longer than the width and thickness of the rod.

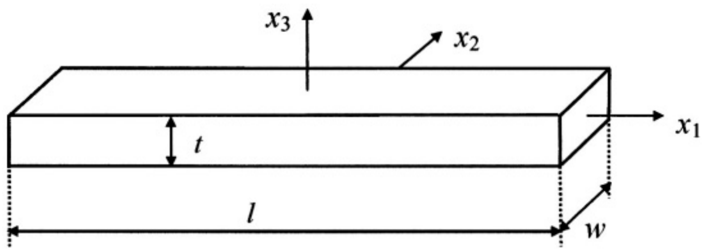

Figure 4.7-1. A piezoelectric rod with rectangular cross section.

As an approximation, it is appropriate to take the vanishing boundary stresses on the surfaces bounding the two small dimensions to vanish everywhere. Consequently

$$T_{11} = T_1(x_1, t), \text{ and all other } T_{ij} = 0. \qquad (4.7\text{-}1)$$

If the surfaces of the area lw are fully electroded with a driving voltage V across the electrodes, the appropriate electrical conditions are

$$E_1 = E_2 = 0, \quad E_3 = -\frac{V}{t}. \qquad (4.7\text{-}2)$$

The pertinent constitutive relations are

$$\begin{aligned} S_1 &= s_{11}T_1 + d_{31}E_3, \\ D_3 &= d_{31}T_1 + \varepsilon_{33}E_3. \end{aligned} \qquad (4.7\text{-}3)$$

Equation $(4.7\text{-}3)_1$ can be inverted to give

$$T_1 = \frac{1}{s_{11}}S_1 - \frac{d_{31}}{s_{11}}E_3. \qquad (4.7\text{-}4)$$

Then the differential equation of motion and boundary conditions are

$$\begin{aligned} &\frac{1}{s_{11}}u_{1,11} = \rho\ddot{u}_1, \quad -l/2 < x_1 < l/2, \\ &T_1 = \frac{1}{s_{11}}u_{1,1} + \frac{d_{31}}{s_{11}}\frac{V}{t} = 0, \quad x_1 = \pm l/2. \end{aligned} \qquad (4.7\text{-}5)$$

Equations (4.7-5) show that the applied voltage effectively acts like two extensional end forces on the rod. For free vibrations, $V = 0$ and the electrodes are shorted. We look for free vibration solution in the form

$$u_1(x_1,t) = u_1(x_1)e^{i\omega t}. \tag{4.7-6}$$

Then the eigenvalue problem is

$$\begin{aligned} &u_{1,11} + \rho s_{11}\omega^2 u_1, \quad -l/2 < x_1 < l/2, \\ &u_{1,1} = 0, \quad x_1 = \pm l/2. \end{aligned} \tag{4.7-7}$$

The solution of $\omega = 0$ and u_1 = constant represents a rigid body mode. For the rest of the modes we try $u_1 = \sin kx_1$. Then, from $(4.7\text{-}7)_1$, $k = \omega\sqrt{\rho s_{11}}$. To satisfy $(4.7\text{-}7)_2$ we must have

$$\cos k\frac{l}{2} = 0, \quad \Rightarrow \quad k_{(n)}\frac{l}{2} = \frac{n\pi}{2}, \quad n = 1,3,5,\cdots, \tag{4.7-8}$$

or

$$\omega_{(n)}\sqrt{\rho s_{11}}\,\frac{l}{2} = \frac{n\pi}{2}, \quad \omega_{(n)} = \frac{n\pi}{l\sqrt{\rho s_{11}}}, \quad n = 1,3,5,\cdots. \tag{4.7-9}$$

Similarly, by considering $u_1 = \cos kx_1$, the following frequencies can be determined:

$$\omega_{(n)} = \frac{n\pi}{l\sqrt{\rho s_{11}}}, \quad n = 2,4,6,\cdots. \tag{4.7-10}$$

The frequencies in (4.7-9) and (4.7-10) are integral multiples of $\omega_{(1)}$ and are called harmonics. $\omega_{(1)}$ is called the fundamental and the rest are called the overtones.

If the surfaces of the area *lt* are fully electroded with a driving voltage V across the electrodes, the appropriate electrical conditions are

$$E_1 = 0, \quad D_3 = 0, \quad E_2 = -\frac{V}{w}. \tag{4.7-11}$$

The pertinent constitutive relations are

$$\begin{aligned} S_1 &= s_{11}T_1 + d_{21}E_2 + d_{31}E_3, \\ D_2 &= d_{21}T_1 + \varepsilon_{22}E_2 + \varepsilon_{23}E_3. \end{aligned} \tag{4.7-12}$$

From the boundary conditions on the areas of *lw,* we take the following to be approximately true everywhere:

$$D_3 = d_{31}T_1 + \varepsilon_{32}E_2 + \varepsilon_{33}E_3 = 0. \tag{4.7-13}$$

With (4.7-13), (4.7-12) can be written as

$$S_1 = \tilde{s}_{11} T_1 + \tilde{d}_{21} E_2, \quad D_2 = \tilde{d}_{21} T_1 + \tilde{\varepsilon}_{22} E_2, \tag{4.7-14}$$

where

$$\tilde{s}_{11} = s_{11} - d_{31}^2 / \varepsilon_{33}, \quad \tilde{d}_{21} = d_{21} - d_{31}\varepsilon_{23} / \varepsilon_{33}, \quad \tilde{\varepsilon}_{22} = \varepsilon_{22} - \varepsilon_{23}^2 / \varepsilon_{33}. \tag{4.7-15}$$

If the surfaces of cross-sectional areas *lw* and *lt* are not electroded, the appropriate electrical conditions are

$$D_2 = D_3 = 0. \tag{4.7-16}$$

The pertinent constitutive relations are

$$S_1 = s_{11} T_1 + g_{11} D_1, \quad E_1 = -g_{11} T_1 + \beta_{11} D_1. \tag{4.7-17}$$

8. RADIAL VIBRATION OF A THIN RING

Axi-symmetric radial vibration can be set up in a thin ceramic ring (see Figure 4.8-1) with radial poling, electroded on its inner and outer surfaces [1].

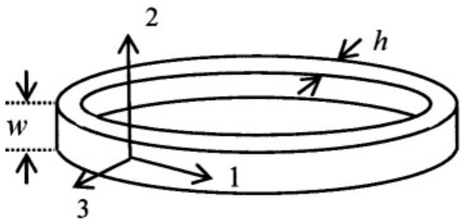

Figure 4.8-1. A ceramic ring with radial poling.

Let R be the mean radius, w the width and h the thickness of the ring. We assume $R >> w >> h$. In cylindrical coordinates, from the boundary conditions, we make the approximation that the following is true throughout the ring:

$$T_{\theta\theta} \neq 0, \quad \text{all other} \quad T_{ij} = 0, \quad E_\theta = E_z = 0. \tag{4.8-1}$$

Let (θ,z,r) correspond to (1,2,3). The radial electric field and the tangential strain are given by

$$S_1 = S_{\theta\theta} = \frac{u_r}{R}, \quad E_3 = E_r = -\frac{V}{h}. \tag{4.8-2}$$

The relevant constitutive relations are

$$\begin{aligned} S_1 &= S_{\theta\theta} = s_{11}T_{\theta\theta} + d_{31}E_r, \\ D_3 &= D_r = d_{31}T_{\theta\theta} + \varepsilon_{33}E_r, \end{aligned} \tag{4.8-3}$$

which can be solved to give

$$\begin{aligned} T_{\theta\theta} &= \frac{1}{s_{11}}\frac{u_r}{R} - \frac{d_{31}}{s_{11}}E_r, \\ D_r &= \frac{d_{31}}{s_{11}}\frac{u_r}{R} + \bar{\varepsilon}_{33}E_r, \end{aligned} \tag{4.8-4}$$

where

$$\bar{\varepsilon}_{33} = \varepsilon_{33} - d_{31}^2 / s_{11}. \tag{4.8-5}$$

The equation of motion takes the following form

$$-\frac{T_{\theta\theta}}{R} = \rho\ddot{u}_r. \tag{4.8-6}$$

Substitution of $(4.8\text{-}4)_1$ into (4.8-6) yields

$$-\frac{1}{s_{11}}\frac{u_r}{R^2} + \frac{d_{31}}{s_{11}R}E_r = \rho\ddot{u}_r. \tag{4.8-7}$$

For free vibrations $V = 0$ and

$$-\frac{1}{s_{11}}\frac{u_r}{R^2} = \rho\ddot{u}_r. \tag{4.8-8}$$

The resonance frequency is

$$\omega^2 = \frac{1}{\rho s_{11}R^2}. \tag{4.8-9}$$

9. RADIAL VIBRATION OF A THIN PLATE

A circular disk of a piezoelectric ceramic poled in the thickness direction is positioned in a coordinate system as shown in Figure 4.9-1. We consider axi-symmetric radial modes [26].

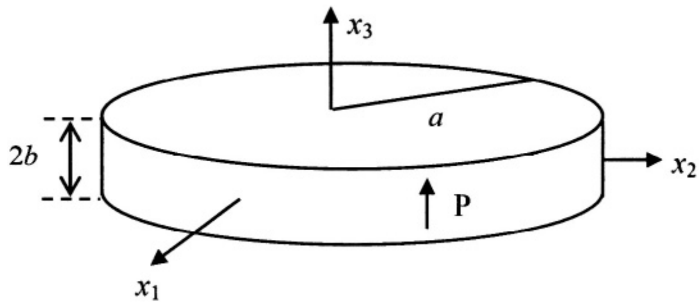

Figure 4.9-1. A circular ceramic plate with thickness poling.

The faces of the disk are traction-free and are completely coated with electrodes. The electrodes are connected to a voltage source of potential $Ve^{i\omega t}$. Under these circumstances, the boundary conditions at $x_3 = \pm b$ are

$$T_{3j} = 0, \quad x_3 = \pm b,$$
$$\phi = \pm \frac{V}{2} e^{i\omega t}, \quad x_3 = \pm b. \qquad (4.9\text{-}1)$$

Since T_3, T_4, and T_5 vanish on both major surfaces of the plate and the plate is thin, these stresses cannot depart much from zero. Consequently they are assumed to vanish throughout. Thus we assume that

$$T_3 = T_4 = T_5 = 0. \qquad (4.9\text{-}2)$$

Furthermore, since the plate is thin and has conducting surfaces,

$$E_1 = 0, \quad E_2 = 0, \quad E_3 = -\frac{V}{2b} e^{i\omega t}. \qquad (4.9\text{-}3)$$

We consider radial modes with

$$u_\theta = 0, \quad \frac{\partial}{\partial \theta} = 0. \qquad (4.9\text{-}4)$$

The constitutive relations are

$$T_{rr} = c_{11}^p u_{r,r} + c_{12}^p u_r / r + e_{31}^p \phi_{,3},$$
$$T_{\theta\theta} = c_{11}^p u_r / r + c_{12}^p u_{r,r} + e_{31}^p \phi_{,3},$$
$$T_{r\theta} = 0, \qquad (4.9\text{-}5)$$
$$D_3 = e_{31}^p (u_{r,r} + u_r / r) - \varepsilon_{33}^p \phi_{,3},$$

where

$$c_{11}^p = c_{11} - c_{13}^2 / c_{33},$$
$$c_{12}^p = c_{12} - c_{13}^2 / c_{33},$$
$$e_{31}^p = e_{31} - e_{33}c_{13} / c_{33}, \tag{4.9-6}$$
$$\varepsilon_{33}^p = \varepsilon_{33} + e_{33}^2 / c_{33}$$

are the effective material constants for a thin plate after the relaxation of the normal stress in the thickness direction. The one remaining equation of motion in cylindrical coordinates is

$$\frac{\partial T_{rr}}{\partial r} + \frac{T_{rr} - T_{\theta\theta}}{r} + \rho f_r = \rho \ddot{u}_r . \tag{4.9-7}$$

Substitution from (4.9-5) for the stress components, we obtain

$$c_{11}^p (u_{,rr} + u_{r,r} / r - u_r / r^2) = \rho \ddot{u}_r , \tag{4.9-8}$$

which, since we are assuming a steady-state problem with frequency ω, becomes

$$u_{r,rr} + \frac{u_{r,r}}{r} + \left(\xi^2 - \frac{1}{r^2} \right) u_r = 0 , \tag{4.9-9}$$

where

$$\xi^2 = \frac{\omega^2}{(v^p)^2}, \quad (v^p)^2 = c_{11}^p / \rho . \tag{4.9-10}$$

Equation (4.9-9) can be written as Bessel's equation of order one. For a solid disk, the motion at the origin is zero and the general solution is

$$u_r = B J_1 (\xi r) e^{i\omega t} , \tag{4.9-11}$$

where J_1 is the first kind Bessel function of the first order. Equation (4.9-11) is subject to the boundary condition

$$T_{rr} = 0, \quad r = a , \tag{4.9-12}$$

hence (4.9-12) requires that

$$c_{11}^p B \left. \frac{dJ_1}{dr} \right|_{r=a} + c_{12}^p B \frac{J_1}{a} = -e_{31}^p \frac{V}{2b} , \tag{4.9-13}$$

where, for convenience, the argument of the Bessel function is not written. From (4.9-13) B can be expressed in terms of V as follows:

$$B = \left[(1 - \sigma^p) \frac{J_1(\xi a)}{a} - \xi J_0(\xi a) \right]^{-1} \frac{e_{31}^p}{c_{11}^p} \frac{V}{2b} , \tag{4.9-14}$$

where

$$\frac{dJ_1(x)}{dx} = J_0(x) - \frac{J_1(x)}{x} \tag{4.9-15}$$

has been used and

$$\sigma^p = c_{12}^p / c_{11}^p, \tag{4.9-16}$$

which may be interpreted as a planar Poisson's ratio, since the material is isotropic in the plane normal to x_3. The total charge on the electrode at the bottom of the plate is given by

$$Q_e = \int_A D_3 dA = 2\pi \int_0^a D_3 r dr\,. \tag{4.9-17}$$

Substitution of (4.9-11) into $(4.9\text{-}5)_4$ and then into (4.9-17) yields

$$Q_e = 2\pi e_{31}^p aBJ_1(\xi a) - \pi\varepsilon_{33}^p Va^2/2b\,. \tag{4.9-18}$$

Hence we obtain for the current that flows to the resonator

$$I = \frac{dQ_e}{dt} = i\omega\left[\frac{2(k_{31}^p)^2 J_1(\xi a)}{(1-\sigma^p)J_1(\xi a) - \xi a J_0(\xi a)} - 1\right]\frac{\varepsilon_{33}^p \pi a^2 V}{2b}, \tag{4.9-19}$$

where

$$(k_{31}^p)^2 = \frac{(e_{31}^p)^2}{\varepsilon_{33}^p c_{11}^p}\,. \tag{4.9-20}$$

At mechanical resonance, the applied voltage can be zero, and from (4.9-13),

$$\left.\frac{dJ_1}{dr}\right|_{r=a} + \sigma^p \frac{J_1}{a} = 0\,. \tag{4.9-21}$$

Or, at the resonance frequency, the current goes to infinity. This condition is determined by setting the square bracketed factor in the denominator of (4.9-14) equal to zero. The resulting equation is

$$\frac{\xi a J_0(\xi a)}{J_1(\xi a)} = 1 - \sigma^p, \tag{4.9-22}$$

which can be brought into the same form as (4.9-21). The antiresonance frequency results when the current goes to zero. The resulting equation is

$$\frac{\xi a J_0(\xi a)}{J_1(\xi a)} = 1 - \sigma^p - 2(k_{31}^p)^2\,. \tag{4.9-23}$$

10. RADIAL VIBRATION OF A THIN CYLINDRICAL SHELL

In this section we analyze the axi-symmetric radial vibration of an unbounded thin ceramic circular cylindrical shell with radial poling, electroded on its inner and outer surfaces (see Figure 4.10-1). A voltage V is applied across the thickness. Let R be the mean radius, and h the thickness of the shell. By a thin shell we mean $R >> h$.

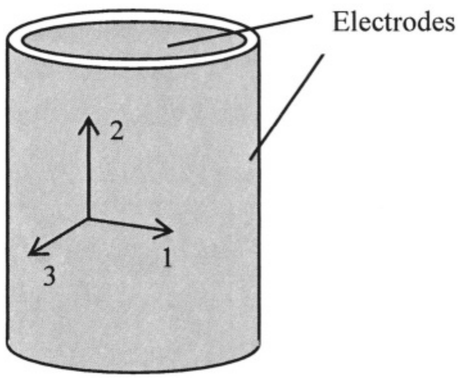

Figure 4.10-1. A thin ceramic circular cylindrical shell.

In cylindrical coordinates, the boundary conditions give

$$T_{rr} = T_{r\theta} = T_{rz} = 0, \quad E_\theta = E_z = 0, \tag{4.10-1}$$

which are taken to be approximately true throughout the shell. We consider motions independent of θ and z. By symmetry

$$T_{\theta z} = 0. \tag{4.10-2}$$

The tangential strain and radial electric field are given by

$$S_1 = S_{\theta\theta} = \frac{u_r}{R}, \quad E_3 = E_r = -\frac{V}{h}. \tag{4.10-3}$$

Let (θ, z, r) correspond to (1,2,3). From

$$S_2 = S_{zz} = s_{11}T_{zz} + s_{21}T_{\theta\theta} + d_{31}E_r = 0, \tag{4.10-4}$$

we solve for

$$T_{zz} = -\frac{s_{12}}{s_{11}}T_{\theta\theta} - \frac{d_{31}}{s_{11}}E_r. \tag{4.10-5}$$

Substituting (4.10-5) into the following constitutive relations

$$S_1 = S_{\theta\theta} = s_{11}T_{\theta\theta} + s_{12}T_{zz} + d_{31}E_r,$$
$$D_3 = D_r = d_{31}(T_{\theta\theta} + T_{zz}) + \varepsilon_{33}E_r, \quad (4.10\text{-}6)$$

we obtain

$$S_{\theta\theta} = \bar{s}_{11}T_{\theta\theta} + \bar{d}_{31}E_r,$$
$$D_r = \bar{d}_{31}T_{\theta\theta} + \bar{\varepsilon}_{33}E_r, \quad (4.10\text{-}7)$$

where

$$\bar{s}_{11} = s_{11} - \frac{s_{12}^2}{s_{11}}, \quad \bar{d}_{31} = d_{31} - \frac{d_{31}s_{12}}{s_{11}}, \quad \bar{\varepsilon}_{33} = \varepsilon_{33} - \frac{d_{31}^2}{s_{11}}. \quad (4.10\text{-}8)$$

Equation (4.10-7) can be inverted to give

$$T_{\theta\theta} = \frac{1}{\bar{s}_{11}}\frac{u_r}{R} - \frac{\bar{d}_{31}}{\bar{s}_{11}}E_r,$$
$$D_r = \frac{\bar{d}_{31}}{\bar{s}_{11}}\frac{u_r}{R} + (\bar{\varepsilon}_{33} - \frac{\bar{d}_{31}^2}{\bar{s}_{11}})E_r. \quad (4.10\text{-}9)$$

Substitution of $(4.10\text{-}9)_1$ into the following equation of motion

$$-\frac{T_{\theta\theta}}{R} = \rho\ddot{u}_r \quad (4.10\text{-}10)$$

yields

$$-\frac{1}{\bar{s}_{11}}\frac{u_r}{R^2} + \frac{\bar{d}_{31}}{\bar{s}_{11}R}E_r = \rho\ddot{u}_r. \quad (4.10\text{-}11)$$

For free vibrations, $V = 0$ and the resonance frequency is

$$\omega^2 = \frac{1}{\rho\bar{s}_{11}R^2}. \quad (4.10\text{-}12)$$

Problem

4.10-1. Study the forced vibration.

11. RADIAL VIBRATION OF A THIN SPHERICAL SHELL

Consider a thin spherical ceramic shell of mean radius R and thickness h with $R >> h$ (see Figure 4.11-1). The ceramic is poled in the thickness direction, with fully electroded inner and outer surfaces. Consider radial vibration of the shell [1].

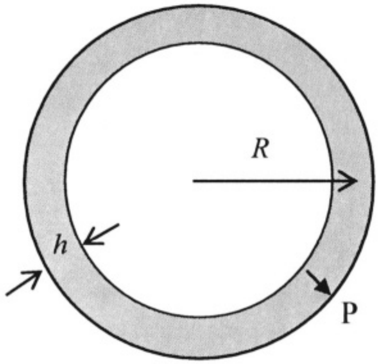

Figure 4.11-1. A spherical ceramic shell with radial poling.

In spherical coordinates the boundary conditions give

$$T_{rr} = T_{r\theta} = T_{r\varphi} = 0, \quad E_\theta = E_\varphi = 0, \tag{4.11-1}$$

which are taken to be valid approximately throughout the shell. For radial motions independent of θ and φ, by symmetry

$$T_{\theta\theta} = T_{\varphi\varphi}, \quad T_{\theta\varphi} = 0. \tag{4.11-2}$$

The relevant strain and electric field components are

$$S_{\theta\theta} = S_{\varphi\varphi} = \frac{u_r}{R}, \quad E_r = -\frac{V}{h}. \tag{4.11-3}$$

Let (r,θ,φ) correspond to (3,1,2) so that poling is along 3. The pertinent constitutive relations are

$$\begin{aligned} S_{\theta\theta} = S_{\varphi\varphi} &= s_{11}T_{\theta\theta} + s_{12}T_{\varphi\varphi} + d_{31}E_r, \\ D_r &= d_{31}(T_{\theta\theta} + T_{\varphi\varphi}) + \varepsilon_{33}E_r, \end{aligned} \tag{4.11-4}$$

which can be inverted to yield

$$\begin{aligned} T_{\theta\theta} = T_{\varphi\varphi} &= \frac{1}{s_{11}+s_{12}}\frac{u_r}{R} - \frac{d_{31}}{s_{11}+s_{12}}E_r, \\ D_r &= \frac{2d_{31}}{s_{11}+s_{12}}\frac{u_r}{R} + \hat{\varepsilon}_{33}E_r, \end{aligned} \tag{4.11-5}$$

where

$$\hat{\varepsilon}_{33} = \varepsilon_{33} - 2d_{31}^2/(s_{11}+s_{12}). \tag{4.11-6}$$

The relevant equation of motion is

$$\frac{-T_{\theta\theta}-T_{\varphi\varphi}}{R}=\rho\ddot{u}_r. \tag{4.11-7}$$

Substitute from (4.11-5)

$$-\frac{2}{s_{11}+s_{12}}\frac{u_r}{R^2}+\frac{2d_{31}}{s_{11}+s_{12}R}E_r=\rho\ddot{u}_r. \tag{4.11-8}$$

For free vibration, $V=0$ and the resonance frequency is

$$\omega^2=\frac{2}{\rho(s_{11}+s_{12})R^2}. \tag{4.11-9}$$

Problem

4.11-1. Study the forced vibration.

12. FREQUENCY SHIFTS DUE TO SURFACE ADDITIONAL MASS

In certain applications, we need to study shifts of resonance frequencies due to a small amount of mass added to the surface of a crystal. One example is the mass effect of a thin surface electrode on resonance frequencies. In addition, many chemical and biological acoustic wave sensors detect certain substances through the mass-frequency effect of the substances accumulated on the crystal surface by chemically or biologically active films. These situations can be modeled by a crystal with a thin film of thickness h' and mass density ρ' on part of the crystal surface (see Figure 4.12-1).

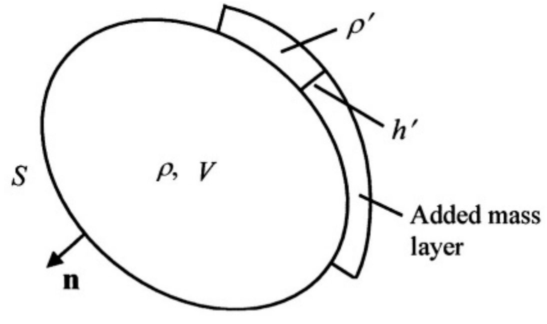

Figure 4.12-1. A crystal with a thin layer of additional mass on part of its surface.

The mass layer is assumed to be very thin. Only the inertial effect of the layer needs to be considered; its stiffness can be neglected. The boundary condition on the surface area with added mass is

$$-T_{ji}n_j = \rho' h' \ddot{u}_i = -\rho' h' \omega^2 u_i \,. \tag{4.12-1}$$

Then the eigenvalue problem for the resonance frequencies and modes of a crystal with surface added mass is

$$\begin{aligned}
&-c_{jikl}u_{k,lj} - e_{kji}\phi_{,kj} = \rho\lambda u_i, \quad \text{in} \quad V,\\
&-e_{ikl}u_{k,li} + \varepsilon_{ik}\phi_{,ki} = 0, \quad \text{in} \quad V,\\
&u_i = 0, \quad \text{on} \quad S_u,\\
&T_{ji}n_j = (c_{jikl}u_{k,l} + e_{kji}\phi_{,k})n_j = \varepsilon\lambda\rho' h' u_i, \quad \text{on} \quad S_T,\\
&\phi = 0, \quad \text{on} \quad S_\phi,\\
&D_i n_i = (e_{ikl}u_{k,l} - \varepsilon_{ik}\phi_{,k})n_i = 0, \quad \text{on} \quad S_D,
\end{aligned} \tag{4.12-2}$$

where we have denoted

$$\lambda = \omega^2 , \tag{4.12-3}$$

and we have artificially introduced a dimensionless number ε to show the smallness of the added mass. When $\varepsilon = 1$, $(4.12\text{-}2)_4$ becomes (4.12-1). In terms of the abstract notation in Section 6, Equation (4.12-2) can be written as

$$\begin{aligned}
&\mathbf{AU} = \lambda\mathbf{BU}, \quad \text{in} \quad V,\\
&u_i = 0, \quad \text{on} \quad S_u,\\
&T_{ji}(\mathbf{U})n_j = \varepsilon\lambda\rho' h' u_i, \quad \text{on} \quad S_T,\\
&\phi = 0, \quad \text{on} \quad S_\phi,\\
&D_i(\mathbf{U})n_i = 0, \quad \text{on} \quad S_D.
\end{aligned} \tag{4.12-4}$$

We make the following perturbation expansion [27]:

$$\begin{aligned}
&\lambda \cong \lambda^{(0)} + \varepsilon\,\lambda^{(1)},\\
&\mathbf{U} = \begin{Bmatrix} u_i \\ \phi \end{Bmatrix} \cong \begin{Bmatrix} u_i^{(0)} \\ \phi^{(0)} \end{Bmatrix} + \varepsilon \begin{Bmatrix} u_i^{(1)} \\ \phi^{(1)} \end{Bmatrix} = \mathbf{U}^{(0)} + \varepsilon\mathbf{U}^{(1)}.
\end{aligned} \tag{4.12-5}$$

Substituting (4.12-5) into (4.12-4), collecting terms of equal powers of ε, the following perturbation problems of successive orders can be obtained. Zero-order:

$$
\begin{aligned}
&-c_{jikl}u_{k,lj}^{(0)} - e_{kji}\phi_{,kj}^{(0)} = \rho\lambda^{(0)}u_i^{(0)}, \quad \text{in} \quad V, \\
&-e_{ikl}u_{k,li}^{(0)} + \varepsilon_{ik}\phi_{,ki}^{(0)} = 0, \quad \text{in} \quad V, \\
&u_i^{(0)} = 0, \quad \text{on} \quad S_u, \\
&(c_{jikl}u_{k,l}^{(0)} + e_{kji}\phi_{,k}^{(0)})n_j = 0, \quad \text{on} \quad S_T, \\
&\phi^{(0)} = 0, \quad \text{on} \quad S_\phi, \\
&(e_{ikl}u_{k,l}^{(0)} - \varepsilon_{ik}\phi_{,k}^{(0)})n_i = 0, \quad \text{on} \quad S_D.
\end{aligned}
\tag{4.12-6}
$$

The solution to the zero-order problem, $\lambda^{(0)}$ and $\mathbf{U}^{(0)}$, is assumed known. The first-order problem below is to be solved:

$$
\begin{aligned}
&-c_{jikl}u_{k,lj}^{(1)} - e_{kji}\phi_{,kj}^{(1)} = \rho\lambda^{(1)}u_i^{(0)} + \rho\lambda^{(0)}u_i^{(1)}, \quad \text{in} \quad V, \\
&-e_{ikl}u_{k,li}^{(1)} + \varepsilon_{ik}\phi_{,ki}^{(1)} = 0, \quad \text{in} \quad V, \\
&u_i^{(1)} = 0, \quad \text{on} \quad S_u, \\
&(c_{jikl}u_{k,l}^{(1)} + e_{kji}\phi_{,k}^{(1)})n_j = \rho'h'\lambda^{(0)}u_i^{(0)}, \quad \text{on} \quad S_T, \\
&\phi^{(1)} = 0, \quad \text{on} \quad S_\phi, \\
&(e_{ikl}u_{k,l}^{(1)} - \varepsilon_{ik}\phi_{,k}^{(1)})n_i = 0, \quad \text{on} \quad S_D.
\end{aligned}
\tag{4.12-7}
$$

The equations for the first-order problem can be written as

$$
\mathbf{AU}^{(1)} = \lambda^{(0)}\mathbf{BU}^{(1)} + \lambda^{(1)}\mathbf{BU}^{(0)}. \tag{4.12-8}
$$

Multiplying both sides of (4.12-8) by $\mathbf{U}^{(0)}$ gives

$$
< \mathbf{AU}^{(1)};\mathbf{U}^{(0)} >= \lambda^{(0)} < \mathbf{BU}^{(1)};\mathbf{U}^{(0)} > +\lambda^{(1)} < \mathbf{BU}^{(0)};\mathbf{U}^{(0)} >. \tag{4.12-9}
$$

From (4.6-7),

$$
\begin{aligned}
&< \mathbf{AU}^{(0)};\mathbf{U}^{(1)} > \\
&= -\int_S [T_{ji}(\mathbf{U}^{(0)})n_j u_i^{(1)} + D_i(\mathbf{U}^{(0)})n_i\phi^{(1)}]dS \\
&+ \int_S [T_{kl}(\mathbf{U}^{(1)})n_l u_k^{(0)} + D_k(\mathbf{U}^{(1)})n_k\phi^{(0)}]dS + < \mathbf{U}^{(0)};\mathbf{AU}^{(1)} >.
\end{aligned}
\tag{4.12-10}
$$

With (4.12-6) and (4.12-7), (4.12-10) becomes

$$
\begin{aligned}
&< \mathbf{AU}^{(0)};\mathbf{U}^{(1)} > \\
&= \int_{S_T} \rho'h'\lambda^{(0)}u_k^{(0)}u_k^{(0)}dS + < \mathbf{U}^{(0)};\mathbf{AU}^{(1)} >.
\end{aligned}
\tag{4.12-11}
$$

Substituting (4.12-11) into (4.12-9) yields

$$< \mathbf{AU}^{(0)};\mathbf{U}^{(1)} > -\int_{S_T} \rho'h'\lambda^{(0)}u_k^{(0)}u_k^{(0)}dS \\ = \lambda^{(0)} < \mathbf{BU}^{(1)};\mathbf{U}^{(0)} > +\lambda^{(1)} < \mathbf{BU}^{(0)};\mathbf{U}^{(0)} >, \tag{4.12-12}$$

which can be further written as

$$< \mathbf{AU}^{(0)} - \lambda^{(0)}\mathbf{BU}^{(0)};\mathbf{U}^{(1)} > -\int_{S_T} \rho'h'\lambda^{(0)}u_k^{(0)}u_k^{(0)}dS \\ = \lambda^{(1)} < \mathbf{BU}^{(0)};\mathbf{U}^{(0)} >. \tag{4.12-13}$$

With (4.12-6), from (4.12-13)

$$\lambda^{(1)} = -\frac{\int_{S_T} \rho'h'\lambda^{(0)}u_k^{(0)}u_k^{(0)}dS}{< \mathbf{BU}^{(0)};\mathbf{U}^{(0)} >} = -\lambda^{(0)} \frac{\int_{S_T} \rho'h'u_k^{(0)}u_k^{(0)}dS}{\int_V \rho u_i^{(0)}u_i^{(0)}dV}. \tag{4.12-14}$$

The above expressions are for the eigenvalue $\lambda = \omega^2$. For ω we make the following expansion:

$$\omega \cong \omega^{(0)} + \varepsilon\omega^{(1)}. \tag{4.12-15}$$

Then

$$\lambda = \omega^2 \cong (\omega^{(0)} + \varepsilon\omega^{(1)})^2 \\ \cong (\omega^{(0)})^2 + 2\varepsilon\omega^{(0)}\omega^{(1)} \cong \lambda^{(0)} + \varepsilon\lambda^{(1)}. \tag{4.12-16}$$

Hence

$$\frac{\varepsilon\omega^{(1)}}{\omega^{(0)}} \cong \frac{1}{2(\omega^{(0)})^2}\varepsilon\lambda^{(1)} \\ = -\frac{1}{2(\omega^{(0)})^2}\varepsilon\lambda^{(0)} \frac{\int_{S_T} \rho'h'u_k^{(0)}u_k^{(0)}dS}{\int_V \rho u_i^{(0)}u_i^{(0)}dV}. \tag{4.12-17}$$

Finally, setting $\varepsilon = 1$ in (4.12-7), we obtain

$$\frac{\omega - \omega^{(0)}}{\omega^{(0)}} \cong -\frac{1}{2}\frac{\int_{S_T} \rho'h'u_k^{(0)}u_k^{(0)}dS}{\int_V \rho u_i^{(0)}u_i^{(0)}dV}. \tag{4.12-18}$$

We make the following observations from (4.2-18):

(i) Clearly, we have $\omega - \omega^{(0)} \le 0$. This shows that a small amount of mass added to the surface tends to lower the resonance frequencies, as expected. On the other hand, if a thin layer of material is removed from the surface, resonance frequencies increase.

(ii) Larger $\rho'h'$ causes more frequency shifts.

(iii) In an area where the surface displacement is large, the added mass has a larger effect on resonance frequencies.

(iv) If the additional mass is essentially a concentrated mass m at a point with Cartesian coordinates y_k on the surface (e.g., a local contamination), then (4.2-18) reduces to

$$\frac{\omega - \omega^{(0)}}{\omega^{(0)}} \cong -\frac{1}{2}\frac{m u_k^{(0)}(\mathbf{y}) u_k^{(0)}(\mathbf{y})}{\int_V \rho u_i^{(0)} u_i^{(0)} dV}. \tag{4.2-19}$$

(v) Obviously, S_T can be several disadjoint areas.

Problem

4.12-1. Use (4.12-18) to analyze Problem 4.3-1 [27].