Chapter 8

FRACTIONAL PROGRAMMING

Johannes B.G. Frenk
Econometric Institute,
Erasmus University,
3000 DR Rotterdam, The Netherlands
frenk@few.eur.nl

Siegfried Schaible
A.G. Anderson Graduate School of Management
University of California,
Riverside, CA 92521-0203, USA
siegfried.schaible@ucr.edu

Abstract Single-ratio and multi-ratio fractional programs in applications are often generalized convex programs. We begin with a survey of applications of single-ratio fractional programs, min-max fractional programs and sum-of-ratios fractional programs. Given the limited advances for the latter class of problems, we focus on an analysis of min-max fractional programs. A parametric approach is employed to develop both theoretical and algorithmic results.

Keywords: Single-ratio fractional programs, min-max fractional programs, sum-of-ratios fractional programs, parametric approach.

1. Introduction.

In various applications of nonlinear programming a ratio of two functions is to be maximized or minimized. In other applications the objective function involves more than one ratio of functions. Ratio optimization problems are commonly called fractional programs. One of the earliest fractional programs (though not called so) is an equilibrium model for an expanding economy introduced by von Neumann (cf. [74])

in 1937. The model determines the growthrate of an economy as the maximum of the smallest of several output-input ratios. At a time when linear programming hardly existed, the author already proposed a duality theory for this nonconvex program. However, apart from a few isolated papers like von Neumann's, a systematic study of fractional programming began much later.

In 1962 Charnes and Cooper (cf. [18]) published their classical paper in which they show that a linear fractional program with one ratio can be reduced to a linear program using a nonlinear variable transformation. Separately, Martos [49] in 1964 (from his Ph.D. dissertation in Hungarian in 1960) showed that linear fractional programs can be solved with an adjacent vertex-following procedure, the same way as linear programs are solved with the simplex method. He recognized that generalized convexity properties (pseudolinearity) of linear ratios enables such a technique which is successfully used in linear programming.

The study of fractional programs with only one ratio has largely dominated the literature in this field until about 1980. Many of the results known then are presented in the first monograph on fractional programming (cf. [65]) which the second author published in 1978. Since then two other monographs solely devoted to fractional programming appeared, one in 1988 authored by Craven (cf. [21]) and one in 1997 by Stancu-Minasian (cf. [71]). An overview of solution methods for single-ratio and multi-ratio fractional location problems appeared in the monograph by Barros (cf. [5]).

Fractional programs with one or more ratios have often been studied in the broader context of generalized convex programming (cf. [4]). Ratios of convex and concave functions as well as composites of such ratios are not convex in general, even in the case of linear ratios. But often they are generalized convex in some sense. From the beginning, fractional programming has benefited from advances in generalized convexity, and vice versa (cf. [50]).

Fractional programming also overlaps with global optimization. Several types of ratio optimization problems have local, nonglobal optima. An extensive survey of fractional programming with one or more ratios appeared in the Handbook of Global Optimization [61]. The survey also contains the largest bibliography on fractional programming with one or multiple ratios so far. It has almost twelve-hundred entries. For a separate, rich bibliography [71] may be consulted.

Very recently two surveys have appeared updating some of the developments reviewed in [61]. The single-ratio and min-max case is treated in [59] and the sum-of-ratios case in [60].

2. Classification of Fractional Programs.

To start with single-ratio fractional programs, let $B \subseteq \mathbb{R}^n$ be a nonempty closed set and $f, g : \mathbb{R}^n \to [-\infty, \infty]$ be extended real-valued functions which are finite-valued on B. Assuming $g(x) > 0$ for every $x \in B$, consider the nonlinear program

$$\inf_{x \in B} \frac{f(x)}{g(x)}. \qquad (P_1)$$

The problem (P_1) is called a *single-ratio fractional program.* In most applications the nonempty feasible region B has more structure and is given by

$$B = \{x \in C : h_k(x) \leq 0, k = 1, ..., l\} \qquad (8.1)$$

with $C \subseteq \mathbb{R}^n$ and $h_k : \mathbb{R}^n \to \mathbb{R}$, $1 \leq k \leq l$ some set of real-valued continuous functions. So far, the functions in the numerator and denominator were not specified. If f, g and $h_k, 1 \leq k \leq l$ are affine functions (linear plus a constant) and $C = \mathbb{R}^n_+$ denotes the nonnegative orthant of $\mathbb{R}^n$, then the optimization problem (P_1) is called a *single-ratio linear fractional program.* Moreover, we call (P_1) a *single-ratio quadratic fractional program* if $C = \mathbb{R}^n_+$, the functions f and g are quadratic and the functions $h_k, 1 \leq k \leq l$ are affine. The minimization problem (P_1) is called a *single-ratio convex fractional program* if C is a convex set, $h_k, 1 \leq k \leq l$ and f are convex functions and g is a positive concave function on B. In addition it is assumed that f is nonnegative on B if g is not affine. In case of a maximization problem the single-ratio fractional program is called a *single-ratio concave fractional program* if f is concave and g is convex. Under these restrictive convexity\concavity assumptions the minimization problem (P_1) is in general a nonconvex problem.

In some applications more than one ratio appears in the objective function. One form of such an optimization problem is the nonlinear programming problem

$$\inf_{x \in B} \sup_{1 \leq i \leq m} \frac{f_i(x)}{g_i(x)} \qquad (P_2)$$

with extended real-valued functions $f_i, g_i : \mathbb{R}^n \to [-\infty, \infty], 1 \leq i \leq m$ which are finite-valued on B with $g_i(x) > 0$ for every $1 \leq i \leq m$ and $x \in B$. The problem (P_2) is often called a *generalized fractional program.* As for single-ratio fractional programs we can specify the functions and make a distinction between multi-ratio linear fractional programs and multi-ratio convex fractional programs. If one g_i is not affine, we need to assume that all functions f_i are nonnegative. Clearly both problems (P_1) and (P_2) are special cases of the following problem.

Let $A \subseteq \mathbb{R}^m$ and $B \subseteq \mathbb{R}^n$ be nonempty closed sets and $f : \mathbb{R}^{m+n} \to [-\infty, \infty]$ be a finite-valued function on $A \times B$. In case $g : \mathbb{R}^{m+n} \to [-\infty, \infty]$ is a finite-valued positive function on $A \times B$, consider the min-max nonlinear programming problem

$$\inf_{x \in B} \sup_{y \in A} \frac{f(y,x)}{g(y,x)}. \qquad (P)$$

Problem (P) is called a *(primal) min-max fractional program.* In order to unify the theory for single-ratio and multi-ratio fractional programs, we consider in Section 6 the so-called parametric approach applied to problem (P) and derive from this approach duality results and algorithmic procedures for problem (P). This yields immediately duality results and algorithmic procedures for problems (P_1) and (P_2).

Another multi-ratio fractional program we encounter in applications is the so-called *sum-of-ratios fractional program* given by

$$\inf_{x \in B} \sum_{i=1}^{m} \frac{f_i(x)}{g_i(x)} \qquad (P_3)$$

with $g_i(x) > 0$ for every $x \in B$ and $1 \leq i \leq m$. It is a more challenging problem than (P_2) as recent studies have shown. We also encounter in applications the so-called *multi-objective fractional program*

$$\inf_{x \in B} \left(\frac{f_1(x)}{g_1(x)}, \ldots, \frac{f_m(x)}{g_m(x)}\right) \qquad (P_4)$$

which is related to (P_2) and (P_3).

In Sections 3 and 4 we will review applications of fractional programs (P_1) and (P_2), respectively. Section 5 focuses on applications of the fractional program (P_3). In addition we review here some of the solution procedures for this rather challenging problem. Finally in Section 6 we return to problems (P_1) and (P_2). In a joint treatment of both involving the more general problem (P) a parametric approach is used for the analysis and development of solution procedures of (P).

3. Applications of Single-Ratio Fractional Programs (P_1).

Single-ratio fractional programs (P_1) arise in management decision making as well as outside of it. They also occur sometimes indirectly in modelling where initially no ratio is involved. The purpose of the following overview is to demonstrate the diversity of problems which can be cast in the form of a single-ratio fractional program. A more comprehensive coverage which also includes additional references for the models below is contained in [61]. For other surveys of applications of a single-ratio fractional program see [21, 59, 62, 64, 65, 71].

Economic Applications.

The efficiency of a system is sometimes characterized by a ratio of technical and/or economical terms. Maximizing the efficiency then leads to a fractional program. Some applications are given below.

- *Maximization of Productivity.*
  Gilmore and Gomory [37] discuss a stock cutting problem in the paper industry for which under the given circumstances it is more appropriate to minimize the ratio of wasted and used amount of raw material rather than just minimizing the amount of wasted material. This stock cutting problem is formulated as a linear fractional program. In a case study, Hoskins and Blom [43] use fractional programming to optimize the allocation of warehouse personnel. The objective is to minimize the ratio of labor cost to the volume entering and leaving the warehouse.

- *Maximization of Return on Investment.*
  In some resource allocation problems the ratio profit/capital or profit/revenue is to be maximized. A related objective is return per cost maximization. Resource allocation problems with this objective are discussed in more detail by Mjelde in [53]. In these models the term 'cost' may either be related to actual expenditure or may stand, for example, for the amount of pollution or the probability of disaster in nuclear energy production. Depending on the nature of the functions describing return, profit, cost or capital, different types of fractional programs are encountered. For example, if the price per unit depends linearly on the output and cost and capital are affine functions, then maximization of the return on investment gives rise to a concave quadratic fractional program (assuming linear constraints). In location analysis maximizing the profitability index (rate of return) is in certain situations preferred to maximizing the net present value, according to [5] and [6] and the cited references.

- *Maximization of Return/Risk.*
  Some portfolio selection problems give rise to a concave nonquadratic fractional program of the form (8.3) below which expresses the maximization of the ratio of expected return and risk. For related concave and nonconcave fractional programs arising in financial planning see [61]. Markov decision processes may also lead to the

maximization of the ratio of mean and standard deviation. A very recent application of fractional programming in portfolio theory is given in [48]. The authors argue that the ratio of two variances gives sophisticated forecasting models with significant predictive power.

- *Minimization of Cost/Time.*
  In several routing problems a cycle in a network is to be determined which minimizes the cost-to-time ratio or maximizes the profit-to-time ratio. Some of these models are combinatorial fractional programs (cf. [56]). Also the average cost objective used within the theory of stochastic regenerative processes (cf. [2]) leads to the minimization of cost per unit time. A particular example occurring within this framework is the determination of the optimal ordering policy of the classical periodic and continuous review single item inventory control models (cf. [12, 13, 34]). Other examples of this framework are maintenance and replacement models. Here the ratio of the expected cost for inspection, maintenance and replacement and the expected time between two inspections is to be minimized (cf. [7, 35]).

- *Maximization of Output/Input.*
  Charnes and Cooper use a linear fractional program as a model to evaluate the efficiency of decision making units (Data Envelopment Analysis (DEA)). Given a collection of decision making units, the efficiency of each unit is obtained from the maximization of a ratio of weighted outputs and inputs subject to the condition that similar ratios for every decision making unit are less than or equal to unity. The variable weights are then the efficiency of each member relative to that of the others. For an extensive recent treatment of DEA see [17]. In the management literature there has been an increasing interest in optimizing relative terms such as relative profit. No longer are these terms merely used to monitor past economic behavior. Instead the optimization of rates is receiving more attention in decision making processes for future projects (cf. [5, 42]). We mention here a case study in which the effectiveness of medical institutions in the area of trauma and burned management was analyzed with help of linear fractional programming (cf. [24]).

Non-Economic Applications.
In information theory the capacity of a communication channel can be defined as the maximal transmission rate over all probabilities. This is a

concave nonquadratic fractional program. Also the eigenvalue problem in numerical mathematics can be reduced to the maximization of the Rayleigh quotient, and hence gives rise to a quadratic fractional program which is generally not concave. An example of a fractional program in physics is given by Falk (cf. [25]). He maximizes the signal-to-noise ratio of an optical filter which is a concave quadratic fractional program.

Indirect Applications.

There are a number of management science problems that indirectly give rise to a concave fractional program. We begin with a recent study which shows that the sensitivity analysis of general decision systems leads to linear fractional programs (cf. [52]). The developed software was used in the appraisal of Hungarian hotels. A concave quadratic fractional program arises in location theory as the dual of a Euclidean multifacility min-max problem. In large scale mathematical programming, decomposition methods reduce the given linear program to a sequence of smaller problems. In some of these methods the subproblems are linear fractional programs. The ratio originates in the minimum-ratio rule of the simplex method.

Fractional programs are also met indirectly in stochastic programming, as first shown by Charnes and Cooper [19] and by Bereanu [14]. This will be illustrated by two models below (cf. [65, 71]).

Consider the following stochastic mathematical program

$$\max\{a^{\mathsf{T}}x : x \in B\} \tag{8.2}$$

where the coefficient vector a is a random vector with a multivariate normal distribution and B is a (deterministic) convex feasible region. It is assumed that the decision maker replaces the above optimization problem by the *maximum probability model*

$$\max\{P(a^{\mathsf{T}}x \geq k) : x \in B\},$$

i.e., he wants to maximize the probability that the random variable $a^{\mathsf{T}}x$ attains at least a value equal to a prescribed level k. Then the optimization problem listed in (8.2) reduces to

$$\max\{\frac{e^{\mathsf{T}}x - k}{\sqrt{x^{\mathsf{T}}Vx}} : x \in B\} \tag{8.3}$$

where e is the mean vector of the random vector a and V its variance-covariance matrix. Hence the maximum probability model of the concave program (8.2) gives rise to a fractional program. If in problem (8.2) the linear objective function is replaced by other types of nonlinear functions, then the maximum probability model leads to various other fractional programs as demonstrated in [65] and [71].

Consider a second stochastic program

$$\max\{f_0(x) + \theta f_1(x) : x \in B\} \tag{8.4}$$

where f_0, f_1 are concave functions on the convex feasible region B, $f_1 > 0$ and θ is a random variable with a continuous cumulative distribution function. Then the maximum probability model for (8.4) gives rise to the fractional program

$$\max\{\frac{f_0(x) - k}{f_1(x)} : x \in B\}. \tag{8.5}$$

For a linear program (8.4) the deterministic equivalent (8.5) becomes a linear fractional program. If f_0 is concave and f_1 linear, then (8.5) is still a concave fractional program. However, if f_1 is also a (nonlinear) concave function, then (8.5) is no longer a concave fractional program. Obviously a quadratic program (8.4) reduces to a quadratic fractional program. For more details on (8.4) and (8.5) see [65, 71].

Stochastic programs (8.2) and (8.4) are met in a wide variety of planning problems. Whenever the maximum probability model is used as a deterministic equivalent, such decision problems lead to a fractional program of one type or another. Hence fractional programs are encountered indirectly in many different applications of mathematical programming, although initially the objective function is not a ratio.

4. Applications of Min-Max Fractional Programs (P_2).

In mathematical economics the multi-ratio fractional program (P_2) arises when the growthrate of an expanding economy is defined as follows (cf. [74]):

$$\text{growthrate} = \max_x \left(\min_{1 \le i \le m} \frac{\text{output}_i(x)}{\text{input}_i(x)} \right) \tag{8.6}$$

where x denotes a feasible production plan of the economy.

In management science simultaneous maximization of rates such as those discussed in the previous section can also lead to a multi-ratio fractional program. This is the case if either in a worst-case approach the model

$$\min_{1 \le i \le m} \frac{f_i(x)}{g_i(x)} \to \sup \tag{8.7}$$

is used or with the help of prescribed ratio goals r_i the model

$$\max_{1\le i\le m} \left|\frac{f_i(x)}{g_i(x)} - r_i\right| \to \inf \tag{8.8}$$

is employed. Examples of the second approach are found in financial planning with different fractional goals or in the allocation of funds under equity considerations. Financial planning with fractional goals is discussed in [38]. Furthermore, multi-facility location-queueing problems giving rise to (P_2) are introduced in [5].

A third area of application of min-max fractional programs is numerical mathematics (cf. [41]). Given the values F_i of a function $F(t)$ in finitely many points t_i of an interval for which an approximating ratio of two polynomials $N(t,x_1)$ and $D(t,x_2)$ with coefficient vectors x_1, x_2 is sought. If the best approximation is defined in the sense of the L_∞-norm, then the following problem is to be solved:

$$\max_i \left|\frac{N(t_i,x_1)}{D(t_i,x_2)} - F_i\right| \to \inf \tag{8.9}$$

with variables x_1, x_2.

At the end of this section on applications of (P_2) we point out that in case of infinitely many ratios (P_2) is related to a fractional semi-infinite program (cf. [41]). Several applications in engineering give rise to such a problem when a lower bound for the smallest eigenvalue of an elliptical differential operator is to be determined (cf. [40]).

For further applications of (P_2) we refer to the very recent survey [59].

5. Sum-of-Ratios Fractional Programs (P_3).

Problem (P_3) arises naturally in decision making when several rates are to be optimized simultaneously and a compromise is sought which optimizes a weighted sum of these rates. In light of the applications of single-ratio fractional programming numerators and denominators may be representing output, input, profit, cost, capital, risk or time, for example. A multitude of applications of the sum-of-ratios problem can be envisioned in this way. Included is the case where some of the ratios are not proper quotients. This describes situations where a compromise is sought between absolute and relative terms like profit and return on investment (profit/capital) or return and return/risk, for example.

Almogy and Levin (cf. [1]) analyze a multistage stochastic shipping problem. A deterministic equivalent of this stochastic problem is formulated which turns out to be a sum-of-ratios problem.

Rao (cf. [57]) discusses various models in cluster analysis. The problem of optimal partitioning of a given set of entities into a number of mutually exclusive and exhaustive groups (clusters) gives rise to various mathematical programming problems depending on which optimality criterion is used. If the objective is to minimize the sum of the squared distances within groups, then a minimum of a sum of ratios is to be determined.

The minimization of the mean response time in queueing-location problems gives rise to (P_3) as well, as shown by Drezner et al. (cf. [23]); see also [75].

Furthermore we mention an inventory model analyzed in [63] which is designed to determine simultaneously optimal lot sizes and an optimal storage allocation in a warehouse. The total cost to be minimized is the sum of fixed cost per unit, storage cost per unit and material handling cost per unit.

In [46] Konno and Inori formulate a bond portfolio optimization problem as a sum-of-ratios problem.

More recently other applications of the sum-of-ratios problem have been identified. Mathis and Mathis [51] formulate a hospital fee optimization problem in this way. The model is used by hospital administrators in the State of Texas to decide on relative increases of charges for different medical procedures in various departments.

According to [20] a number of geometric optimization problems give rise to the sum-of-ratios problem. These often occur in layered manufacturing, for instance in material layout and cloth manufacturing. Quite in contrast to other applications of the sum-of-ratios problem mentioned before, the number of variables is very small (one, two or three), but the number of ratios is large; often there are hundreds or even thousands of ratios involved.

Our current understanding of the structural properties of the sum-of-ratios problem is rather limited. In [36] Freund and Jarre showed that this problem is essentially NP-hard, even in the case of one concave ratio and a concave function. Hence (P_3) is a global optimization problem in contrast to (P_1) and (P_2).

Given the small theoretical basis, it is not surprising that algorithmic advances have been rather limited too. However in recent years some progress has been made. Some of the proposed algorithms have been computationally tested. Typically execution times grow very rapidly in the number of ratios. At this time problems up to about ten ratios can be handled. We refer to the algorithms by Konno and Fukaishi (cf. [45]) (see also [44]) and by Kuno (cf. [47]). The former is superior to several earlier methods (cf. [45]) while the latter is seemingly faster than the former. Clearly a more thorough testing of various proposed algorithms is needed before further conclusions can be drawn. Also some of the applications call for methods which can handle a large number of ratios; e.g., fifty (cf. [1]). Currently such methods are not available.

For a special class of sum-of-ratios problems with up to about one thousand ratios, but only very few variables an algorithm is given in [20]. This method by Chen et al. is superior to the other algorithms on the particular class of problems in manufacturing These are geometric optimization problems arising in layered manufacturing. In contrast to general-purpose algorithms for (P_3), the method in [20] is rather robust with regard to the number of ratios.

Focus of the remainder of this review of fractional programming will be the min-max fractional program (P). It includes as special cases (P_1) and (P_2). For a very recent survey of applications, theoretical results and solution methods for (P_1) and (P_2) since [61] was published we refer to [59]. A corresponding survey for (P_3) since [61] appeared is given in [60]. For a survey of some recent developments for multi-objective fractional programs (P_4) we refer to [33].

6. Analysis of Min-Max Fractional Programs.

In this section we will analyze min-max fractional programs by means of a parametric approach. Although other approaches are also available, this one makes it possible to derive duality results for the (primal) min-max fractional program (P) and at the same time to construct an algorithm which solves problem (P). As already mentioned in Section 2, let $B \subseteq \mathbb{R}^n$ and $A \subseteq \mathbb{R}^m$ be some nonempty closed sets and $f : \mathbb{R}^{m+n} \to [-\infty, \infty]$ a finite-valued function on $A \times B$. Moreover, consider the function $g : \mathbb{R}^{m+n} \to [-\infty, \infty]$ which is a finite-valued positive function on $A \times B$. For the related functions $g_{\text{inf}} : \mathbb{R}^n \to [-\infty, \infty]$ and $g_{\text{sup}} : \mathbb{R}^n \to [-\infty, \infty]$ given by

$$g_{\text{inf}}(x) := \inf_{y \in A} g(y, x) \text{ and } g_{\text{sup}}(x) := \sup_{y \in A} g(y, x)$$

we assume, unless stated otherwise, that the following condition holds.

Condition 8.1 *For every $x \in B$ we have $0 < g_{\text{inf}}(x) \leq g_{\text{sup}}(x) < \infty$.*

For every $x \in B$ we now consider the single-ratio fractional program

$$\lambda_*(x) := \sup_{y \in A} \frac{f(y, x)}{g(y, x)}. \qquad (P^x)$$

This optimization problem is well-defined and its objective function value satisfies $-\infty < \lambda_*(x) \leq \infty$. A more complicated optimization problem is given by the already introduced (primal) min-max fractional program

$$\lambda_* := \inf_{x \in B} \sup_{y \in A} \frac{f(y,x)}{g(y,x)}. \tag{P}$$

Clearly $-\infty \leq \lambda_* = \inf_{x \in B} \lambda_*(x) \leq \infty$. It is not assumed beforehand that the optimization problems (P) and (P^x) have an optimal solution. Therefore we cannot replace sup by max or inf by min. The simpler optimization problem (P^x) is introduced since it will be part of the so-called primal Dinkelbach-type approach discussed in subsection 6.2 to solve the (primal) min-max fractional program (P).

Another optimization problem is to consider for every $y \in A$ the single-ratio fractional program

$$\mu_*(y) := \inf_{x \in B} \frac{f(y,x)}{g(y,x)}. \tag{D^y}$$

Also this problem is well-defined and it satisfies $-\infty \leq \mu_*(y) < \infty$. Clearly for every $y \in A$ we obtain $\mu_*(y) \leq \lambda_*$. Similarly as for the (primal) min-max fractional program we introduce the more complicated optimization problem

$$\mu_* := \sup_{y \in A} \inf_{x \in B} \frac{f(y,x)}{g(y,x)}. \tag{D}$$

This problem is called a *(dual) max-min fractional program.* Clearly its optimal objective function value μ_* satisfies $\mu_* \leq \lambda_*$. Like for the (primal) min-max fractional program we introduce the functions $\underline{g}_{\text{inf}} : \mathbb{R}^m \to [-\infty, \infty]$ and $\overline{g}_{\text{sup}} : \mathbb{R}^m \to [-\infty, \infty]$ given by

$$\underline{g}_{\text{inf}}(y) := \inf_{x \in B} g(y,x) \text{ and } \overline{g}_{\text{sup}}(y) := \sup_{x \in B} g(y,x).$$

Analyzing the so-called dual Dinkelbach-type approach to solve problem (D), we need the following counterpart of Condition 8.1.

Condition 8.2 *For every $y \in A$ we have $0 < \underline{g}_{\text{inf}}(y) \leq \overline{g}_{\text{sup}}(y) < \infty$.*

The simpler optimization problem (D^y) is introduced since it will be part of the dual Dinkelbach-type approach discussed in subsection 6.4 to solve the (dual) max-min fractional program (D). If we consider a single-ratio fractional program, A consists of one element and the optimization problems (P) and (D) are identical. For a classical multi-ratio fractional program (generalized fractional program) A is a finite set consisting of more than one element; hence optimization problems (P) and (D) are different from each other. If programs (P) and (D) are different and additionally $\mu_* = \lambda_*$, both the primal and dual Dinkelbach-type approach can be used to solve optimization problem (P). As already observed before, many results (cf. [4, 5, 21]) were derived for generalized fractional programs. In this section we will consider the more general (primal) min-max and (dual) max-min fractional program and derive similar structural properties for this problem as it was done for the more specialized primal and dual generalized fractional program before.

We selected these more general optimization problems not often considered in the fractional programming literature since one can use similar parametric techniques as for generalized fractional programs and at the same time unify the existing theory for single-ratio and multi-ratio fractional programs. Using the parametric approach one can reduce the max-min and min-max fractional program to so-called (semi-infinite) max-min and min-max programs. Unfortunately, solving these semi-infinite optimization problems efficiently on a computer is very difficult. For an extensive discussion of some of the used procedures the reader should consult [55]. However for special cases there is still room for improvement, and this seems to be a new area of research (cf. [15]). In the theoretical analysis of max-min and min-max fractional programs it will turn out that convexity plays a major role, not only in establishing the equality $\lambda_* = \mu_*$ (a so-called strong duality result), but also in the rate of convergence analysis for the primal and dual Dinkelbach-type parametric approach. Due to symmetry arguments similar type of convergence results hold for these two algorithms.

In case we analyze the primal Dinkelbach-type approach, not all the results are valid under Condition 8.1, and we sometimes need the following stronger condition.

Condition 8.3 *The set $A \subseteq \mathbb{R}^m$ is compact, the function g is positive on $A \times B$ and for every $x \in B$ the functions $y \to f(y,x)$ and $y \to g(y,x)$ are finite-valued and continuous on some open set $U \subseteq \mathbb{R}^m$ containing A.*

If Condition 8.3 holds, then it follows from Corollary 1.2 of [3] that

$$0 < g_{\inf}(x) \leq g_{\sup}(x) < \infty$$

for every $x \in B$, and so this condition implies Condition 8.1. Moreover, the single-ratio fractional program (P^x) has an optimal solution and $\lambda_*(x)$ is finite for every $x \in B$.

In case we also analyze the dual Dinkelbach-type approach, not all results are valid under Condition 8.2, and so we sometimes need the following counterpart of Condition 8.3.

Condition 8.4 *The set $B \subseteq \mathbb{R}^n$ is compact, the function g is positive on $A \times B$ and for every $y \in A$ the functions $x \to f(y,x)$ and $x \to g(y,x)$ are finite-valued and continuous on some open set $V \subseteq \mathbb{R}^n$ containing B.*

Again, if Condition 8.4 holds, it follows from Corollary 1.2 of [3] that

$$0 < \underline{g}_{\inf}(y) \leq \overline{g}_{\sup}(y) < \infty$$

for every $y \in A$, and so this condition implies Condition 8.2. Moreover, the single-ratio fractional program (D^y) has an optimal solution, and $\mu_*(y)$ is finite for every $y \in A$.

Before analyzing in the next subsection the parametric approach applied to (P), we will derive an alternative representation of a generalized fractional program. This alternative representation satisfies automatically Condition 8.3. For a generalized fractional program the set A is given by $\{1, ..., m\}, m < \infty$, and the functions f and g are replaced by the functions $f_i : B \to \mathbb{R}, i \in A$ and $g_i : B \to \mathbb{R}, i \in A$. This means

$$\sup_{y \in A} \frac{f(y,x)}{g(y,x)} = \max_{1 \leq i \leq m} \frac{f_i(x)}{g_i(x)} = \lambda_*(x).$$

In this case the optimization problem (P^x) can be solved trivially.

To obtain a different representation of a generalized fractional program, we introduce the unit simplex

$$\Delta_m := \{y \in \mathbb{R}^m : \sum_{i=1}^m y_i = 1, y_i \geq 0, 1 \leq i \leq m\}.$$

If the vector b belongs to $\mathbb{R}^m_{++}$, the strictly positive orthant of $\mathbb{R}^m$, it is well-known (cf. [4]) that the function $h : \Delta_m \to \mathbb{R}$ given by $h(y) := (y^\top b)^{-1} y^\top a$ is quasiconvex on Δ_m for every $a \in \mathbb{R}^m$. By Condition 8.1 it follows for $g : \mathbb{R}^n \to \mathbb{R}^m$ given by $g(x) := (g_1(x), ..., g_m(x))^\top$ that $g(x) \in \mathbb{R}^m_{++}$ for every $x \in B$. Then for $f : \mathbb{R}^n \to \mathbb{R}^m$ given by $f(x) = (f_1(x), .., f_m(x))^\top$ we have

$$\max_{i \in A} \frac{f_i(x)}{g_i(x)} = \max_{y \in \Delta_m} \frac{y^\top f(x)}{y^\top g(x)} \tag{8.10}$$

for every $x \in B$. Applying relation (8.10) yields

$$\inf_{x \in B} \max_{1 \leq i \leq m} \frac{f_i(x)}{g_i(x)} = \inf_{x \in B} \max_{y \in \Delta_m} \frac{y^\top f(x)}{y^\top g(x)}. \tag{8.11}$$

With this we have found another representation of a generalized fractional program. Using this representation, the corresponding (dual) generalized fractional program is given by

$$\sup_{y \in \Delta_m} \inf_{x \in B} \frac{y^\top f(x)}{y^\top g(x)}.$$

In subsection 6.3 we will give sufficient conditions to guarantee that the primal and dual optimal objective function values coincide. However before discussing this, we will first consider in the next subsection the so-called primal parametric approach for solving the (primal) min-max fractional program (P).

6.1 The Primal Parametric Approach.

To analyze the properties of the (primal) min-max fractional program (P) and at the same time construct some generic algorithm to solve this problem we introduce the function $p : \mathbb{R} \times A \times B \to \mathbb{R}$ given by

$$p(\lambda, y, x) := f(y, x) - \lambda g(y, x)$$

and consider for every $(\lambda, x) \in \mathbb{R} \times B$ the optimization problem

$$p_1(\lambda, x) := \sup_{y \in A} p(\lambda, y, x). \tag{P_λ^x}$$

For every $x \in B$ the function $p_{1,x} : \mathbb{R} \to (-\infty, \infty]$ is now given by

$$p_{1,x}(\lambda) := p_1(\lambda, x). \tag{8.12}$$

Since $g > 0$ on $A \times B$ and $p_{1,x}$ is the supremum of affine functions, it is obvious that $p_{1,x}$ is a decreasing lower semicontinuous convex function. Its so-called effective domain $dom(p_{1,x})$ is defined by (cf. [58])

$$dom(p_{1,x}) := \{\lambda \in \mathbb{R} : p_{1,x}(\lambda) < \infty\} \subseteq \mathbb{R}.$$

By the finiteness of p on $\mathbb{R} \times A \times B$ it is obvious that for every $x \in B$ $dom(p_{1,x}) = \{\lambda \in \mathbb{R} : p_{1,x}(\lambda) \text{ finite}\}$. A more difficult optimization problem than (P_λ^x) is the parametric min-max optimization problem

$$p_2(\lambda) := \inf_{x \in B} p_1(\lambda, x). \tag{P_λ}$$

For this function it holds that $-\infty \leq p_2(\lambda) \leq \infty$ for every $\lambda \in \mathbb{R}$. For the function p_2 the so-called effective domain $dom(p_2)$ is given by

$$dom(p_2) := \{\lambda \in \mathbb{R} : p_2(\lambda) < \infty\} \subseteq \mathbb{R}.$$

By the definition of the functions p_2 and $p_{1,x}$ it is easy to verify that

$$dom(p_2) = \cup_{x \in B} dom(p_{1,x})$$

In the next result we identify for $\lambda_* < \infty$ and $\lambda_*(x) < \infty$ the effective domains of the functions p_2 and $p_{1,x}$.

Lemma 8.1 *Assume Condition 8.1 holds. Then $\lambda_* < \infty$ if and only if $dom(p_2) = \mathbb{R}$, and $\lambda_*(x)$ is finite if and only $dom(p_{1,x}) = \mathbb{R}$.*

Proof. Assume $\lambda_* < \infty$. Suppose by contradiction that there exists some $\lambda \in \mathbb{R}$ satisfying $p_2(\lambda) = \infty$. This implies for every $x \in B$ that $p_1(\lambda, x) = \infty$. Hence for a given $x \in B$ one can find some sequence $\{y_i : i \in \mathbb{N}\} \subseteq A$ satisfying

$$i \leq (\frac{f(y_i, x)}{g(y_i, x)} - \lambda)g(y_i, x) \leq (\frac{f(y_i, x)}{g(y_i, x)} - \lambda)g_{\sup}(x). \tag{8.13}$$

Since $g_{\sup}(x) < \infty$ and λ is finite, we obtain by relation (8.13) that $\lambda_*(x) = \infty$ for every $x \in B$ yielding $\lambda_* = \infty$ which contradicts our assumption.

Conversely, if $dom(p_2) = \mathbb{R}$, then clearly $0 \in dom(p_2)$, and so there exists some $x_0 \in B$ satisfying $\sup_{y \in A} f(y, x_0) < \infty$. Due to $g_{\inf}(x_0) > 0$ it is easy to see that $\lambda_*(x_0) < \infty$, and so $\lambda_* < \infty$ which completes the proof of the first part. By identifying B with $\{x\}$, the second part follows immediately from the first part. □

Using similar algebraic manipulations as in [22] applied to a generalized fractional program one can show the following important result for the optimal value function p_2 of a parametric min-max problem (P_λ). The validity of the so-called parametric approach to solve problem (P) is based on this result.

Theorem 8.1 *Assume Condition 8.1 holds and $\lambda_* < \infty$. Then $\lambda_* < \lambda < \infty$ if and only if $p_2(\lambda) < 0$. Moreover, if $\lambda_*(x) < \infty$, then $\lambda_*(x) < \lambda < \infty$ if and only if $p_1(\lambda, x) < 0$.*

Proof. If $\lambda_* < \infty$ and $\lambda > \lambda_* = \inf_{x \in B} \lambda_*(x)$, then there exist some $x_0 \in B$ and $\epsilon > 0$ satisfying

$$\lambda > \lambda_*(x_0) + \epsilon \geq \frac{f(y, x_0)}{g(y, x_0)} + \epsilon$$

for every $y \in A$. Since $g_{\inf}(x_0) > 0$, this yields

$$f(y, x_0) - \lambda g(y, x_0) \leq -\epsilon g(y, x_0) \leq -\epsilon g_{\inf}(x_0)$$

for every $y \in A$. It follows that

$$p_2(\lambda) \leq p_1(\lambda, x_0) \leq -\epsilon g_{\inf}(x_0) < 0.$$

Conversely, if $p_2(\lambda) < 0$, then there exist some $\epsilon > 0$ and $x_0 \in B$ satisfying $p_1(\lambda, x_0) \leq -\epsilon$. This implies $f(y, x_0) - \lambda g(y, x_0) \leq -\epsilon$ for every $y \in A$, and we obtain for every $y \in A$ that

$$\frac{f(y,x_0)}{g(y,x_0)} \leq \lambda - \frac{\epsilon}{g(y,x_0)} \leq \lambda - \frac{\epsilon}{g_{\text{sup}}(x_0)}. \tag{8.14}$$

Since $g_{\text{sup}}(x_0) < \infty$, it follows from relation (8.14) that $\lambda_* \leq \lambda_*(x_0) < \lambda$, and the proof of the first part is completed. By identifying B with $\{x\}$ the second part follows from the first part. □

A useful implication of Theorem 8.1 is given by the following result.

Lemma 8.2 *Assume Condition 8.1 holds and* $\lambda_*(x) < \infty$ *for some* $x \in B$. *Then* $p_1(\lambda_*(x), x) = 0$.

Proof. By the definition of $\lambda_*(x)$ we obtain $f(y,x) - \lambda_*(x)g(y,x) \leq 0$ for every $y \in A$. This implies $p_1(\lambda_*(x), x) \leq 0$. From Theorem 8.1 it follows that $p_1(\lambda_*(x), x) \geq 0$, and this shows the desired result. □

If Condition 8.1 holds and $\lambda_* < \infty$, we obtain from Theorem 8.1 and Lemma 8.1 that $p_2(\lambda_*) \geq 0$, and $p_2(\lambda)$ is finite for every $\lambda \leq \lambda_*$. In case we only assume that g is positive on $A \times B$ it is easy to verify that $p_2(\lambda) \leq 0$ for every $\lambda > \lambda_*$, and $p_2(\lambda) < 0$ implies $\lambda > \lambda_*$. However as shown by the following single-ratio fractional program satisfying Condition 8.1 and $\lambda_* = 1$, it may happen that $p_2(\lambda) = -\infty$ for every $\lambda > \lambda_*$ and $p_2(\lambda_*) \neq 0$ (cf. [22]).

Example 8.1 For $A = \{1\}$, $f_1(x) = x + 1$, $g_1(x) = x$ and $B = \{x \in \mathbb{R} : x \geq 1\}$ it follows that the optimization problem *(P)* reduces to $inf_{x \in B} \frac{x+1}{x}$, and so $\lambda_* = 1$. Also $0 < g_{\text{inf}}(x) = g_{\text{sup}}(x) = x < \infty$ for every $x \in B$ and $p_2(\lambda_*) = \inf_{x \in B}\{x + 1 - x\} = 1$. Moreover, the optimal solution set of the optimization problem (P_{λ_*}) equals B, and $p_2(\lambda) = -\infty$ for every $\lambda > 1$.

To derive some other properties of the so-called parametric approach we need to investigate in detail the functions p_2 and $p_{1,x}$. We first observe that the positivity of the function g on $A \times B$ implies that the functions p_2 and $p_{1,x}, x \in B$, are decreasing. In the next result it is shown that the decreasing function p_2 is upper semicontinuous.

Theorem 8.2 *Assume Condition 8.1 holds. Then the function* $p_2 : \mathbb{R} \to [-\infty, \infty]$ *is upper semicontinuous.*

Proof. To prove that the function p_2 is upper semicontinuous, let $\alpha \in \mathbb{R}$ and consider the upper level set $U(p_2, \alpha) := \{\lambda \in \mathbb{R} : p_2(\lambda) \geq \alpha\}$. If $U(p_2, \alpha) = \emptyset$, then this set is closed. So we assume that $U(p_2, \alpha) \neq$

$\emptyset$. To show that this set is closed consider some accumulation point $\lambda_\infty \in \mathbb{R}$ of the set $U(p_2, \alpha)$. Hence there exists some sequence $\{\lambda_i : i \in \mathbb{N}\} \subseteq U(p_2, \alpha)$ satisfying $\lim_{i\uparrow\infty} \lambda_i = \lambda_\infty$. If for some $i \in \mathbb{N}$ it holds that $\lambda_i \geq \lambda_\infty$, then by the monotonicity of the function p_2 we obtain $p_2(\lambda_\infty) \geq p_2(\lambda_i) \geq \alpha$, and so $\lambda_\infty \in U(p_2, \alpha)$. Therefore we may assume without loss of generality that $\lambda_i < \lambda_\infty$ for every $i \in \mathbb{N}$. Observe now for every $x \in B$ and $i \in \mathbb{N}$ that

$$p_1(\lambda_\infty, x) \geq p(\lambda_i, y, x) + (\lambda_i - \lambda_\infty) g(y, x)$$

for every $y \in A$. This implies using $\lambda_i < \lambda_\infty$ and $g > 0$ that

$$p_1(\lambda_\infty, x) \geq p(\lambda_i, y, x) + (\lambda_i - \lambda_\infty) g_{\text{sup}}(x)$$

for every $y \in A$, and hence

$$p_1(\lambda_\infty, x) \geq p_1(\lambda_i, x) + (\lambda_i - \lambda_\infty) g_{\text{sup}}(x). \tag{8.15}$$

Since $\lambda_i \in U(p_2, \alpha)$, we obtain for every $x \in B$ that $p_1(\lambda_i, x) \geq \alpha$. By relation (8.15), $\lim_{i\uparrow\infty} \lambda_i = \lambda_\infty$ and $0 < g_{\text{sup}}(x) < \infty$ this yields for every $x \in B$ that $p_1(\lambda_\infty, x) \geq \alpha$. Hence $p_2(\lambda_\infty) \geq \alpha$, and so $\lambda_\infty \in U(p_2, \alpha)$. Applying Theorem 1.7 of [29] yields that p_2 is upper semicontinuous. □

By Theorem 8.2 and Lemma 1.30 of [29] we obtain

$$\lim_{s\uparrow\lambda} p_2(s) = \limsup_{s\uparrow\lambda} p_2(s) \leq p_2(\lambda).$$

Since for every $s < \lambda$ we know that $p_2(s) \geq p_2(\lambda)$, this yields

$$\lim_{s\uparrow\lambda} p_2(s) = p_2(\lambda).$$

Again by the monotonicity of p_2 it follows that $\lim_{s\downarrow\lambda} p_2(s)$ exists. But this limit might not be equal to $p_2(\lambda)$. Therefore the function p_2 is left-continuous with right-hand limits.

An important consequence of Theorem 8.2 is given by the next result. To show this result we first introduce a so-called set-valued mapping $S : X \to 2^Y$ (cf. [3]) with 2^Y denoting the set of all subsets of the nonempty set $Y \subseteq \mathbb{R}^m$ and X a nonempty closed subset of $\mathbb{R}^n$. If $S : X \to 2^Y$ is a set-valued mapping, it is always assumed that $S(x) \subseteq Y$ is nonempty for every $x \in X$. The graph of a set-valued mapping $S : X \to 2^Y$ is given by

$$\text{graph}(S) = \{(x, y) \in X \times Y : y \in S(x)\}.$$

An important subclass of set-valued mappings is introduced in the next definition (cf. [11]).

Definition 8.1 *The set-valued mapping $S : X \to 2^Y$ where X is a closed set is called closed if its graph is a closed set.*

By the definition of a closed set it is immediately clear that the set-valued mapping $S : X \to 2^Y$ is closed if and only if for any sequence $\{x_k : k \in \mathbb{N}\} \subseteq X$ and $y_k \in S(x_k)$, $k \in \mathbb{N}$ it follows that

$$\lim_{k\uparrow\infty} x_k = x_\infty \text{ and } \lim_{k\uparrow\infty} y_k = y_\infty \Rightarrow y_\infty \in S(x_\infty).$$

Examples of set-valued mappings occurring in min-max optimization are the set-valued mappings $S_{p_1} : \mathbb{R} \times B \to 2^A$ and $S_{p_2} : \mathbb{R} \to 2^B$ given by

$$S_{p_1}(\lambda, x) := \{y \in A : p_1(\lambda, x) = p(\lambda, y, x)\} \tag{8.16}$$

and

$$S_{p_2}(\lambda) := \{x \in B : p_2(\lambda) = p_1(\lambda, x)\}. \tag{8.17}$$

The set $S_{p_1}(\lambda, x)$ represents the set of optimal solutions of the optimization problem (P_λ^x), while the set $S_{p_2}(\lambda)$ denotes the set of optimal solutions in B of the optimization problem (P_λ). Also we consider the set-valued mapping $S_p : \mathbb{R} \to 2^{A\times B}$ given by

$$S_p(\lambda) := \{(y, x) \in A \times B : p_2(\lambda) = p_1(\lambda, x) = p(\lambda, y, x)\}. \tag{8.18}$$

This set represents the set of optimal solutions of the optimization problem (P_λ). For the above set-valued mappings one can show the following result. It is always assumed in the next result that the sets $S_{p_1}(\lambda, x), S_{p_2}(\lambda)$ and $S_p(\lambda)$ are nonempty on their domain.

Lemma 8.3 *Assume Condition 8.1 holds and the functions f and g are finite-valued and continuous on some open set $W \subseteq \mathbb{R}^{m+n}$ containing $A \times B$. Then the set-valued mappings S_{p_1}, S_{p_2} and S_p are closed.*

Proof. We first show that the set-valued mapping S_{p_1} is closed. To start with this, consider some sequence $\{(\lambda_k, y_k, x_k) : y_k \in S_{p_1}(\lambda_k, x_k)\}_{k\in\mathbb{N}}$ satisfying $\lim_{k\uparrow\infty} \lambda_k = \lambda_\infty \in \mathbb{R}$, $\lim_{k\uparrow\infty} x_k = x_\infty$ and $\lim_{k\uparrow\infty} y_k = y_\infty$. Since A and B are closed sets, this yields $x_\infty \in B$ and $y_\infty \in A$ and by the definition of p_1 we obtain

$$p(\lambda_\infty, y_\infty, x_\infty) \leq p_1(\lambda_\infty, x_\infty). \tag{8.19}$$

Since the function p is continuous on $\mathbb{R} \times A \times B$, it is easy to verify using Theorem 1.7 of [29] that the function p_1 is lower semicontinuous on $\mathbb{R} \times B$. Using this together with Lemma 1.30 of [29] and $p_1(\lambda_k, x_k) = p(\lambda_k, y_k, x_k)$ we obtain

$$p(\lambda_\infty, y_\infty, x_\infty) = \liminf_{k\uparrow\infty} p_1(\lambda_k, x_k) \geq p_1(\lambda_\infty, x_\infty).$$

Then by relation (8.19) it follows that $y_\infty \in S_{p_1}(\lambda_\infty, x_\infty)$. This shows that the set S_{p_1} is closed. To prove that the set-valued mapping S_{p_2} is closed we consider some sequence $\{(\lambda_k, x_k) : x_k \in S_{p_2}(x_k)\}_{k\in\mathbb{N}}$ satisfying $\lim_{k\uparrow\infty} \lambda_k = \lambda_\infty \in \mathbb{R}$ and $\lim_{k\uparrow\infty} x_k = x_\infty$. By Theorem 8.2 and Lemma 1.30 of [29] we obtain

$$p_2(\lambda_\infty) \geq \limsup_{k\uparrow\infty} p_2(\lambda_k). \tag{8.20}$$

Since p_1 is lower semicontinuous on $\mathbb{R} \times B$, it follows that

$$\limsup_{k\uparrow\infty} p_2(\lambda_k) \geq \liminf_{k\uparrow\infty} p_1(\lambda_k, x_k) \geq p_1(\lambda_\infty, x_\infty).$$

Hence by relation (8.20) we obtain

$$p_2(\lambda_\infty) \geq p_1(\lambda_\infty, x_\infty).$$

Using $x_\infty \in B$ this shows that $x_\infty \in S_{p_2}(\lambda_\infty)$. Hence we have verified that S_{p_2} is closed.

Finally, to show that S_p is closed, consider a sequence $\{(\lambda_k, y_k, x_k) : (y_k, x_k) \in S_p(\lambda_k)\}_{k\in\mathbb{N}}$ satisfying $\lim_{k\uparrow\infty} \lambda_k = \lambda_\infty \in \mathbb{R}$, $\lim_{k\uparrow\infty} x_k = x_\infty$ and $\lim_{k\uparrow\infty} y_k = y_\infty$. Since $y_k \in S_{p_1}(\lambda_k, x_k)$, it follows that $y_\infty \in S_{p_1}(\lambda_\infty, x_\infty)$ using the fact that S_{p_1} is closed. This shows $p(\lambda_\infty, y_\infty, x_\infty) = p_1(\lambda_\infty, x_\infty)$. Moreover, since $x_k \in S_{p_2}(\lambda_k)$, we obtain $x_\infty \in S_{p_2}(\lambda_\infty)$ using the fact that S_{p_2} is closed. Hence $p_1(\lambda_\infty, x_\infty) = p_2(\lambda_\infty)$. Therefore (y_∞, x_∞) is an optimal solution of the min-max fractional program *(P)*. This completes the proof. □

We will now consider for every $x \in B$ the decreasing convex function $p_{1,x} : \mathbb{R} \to \mathbb{R}$, introduced in relation (8.12). In the next result it is shown for $\lambda_*(x)$ finite that this function is Lipschitz continuous with Lipschitz constant $g_{\sup}(x)$.

Lemma 8.4 *Assume Condition 8.1 holds and $\lambda_*(x)$ is finite for $x \in B$. Then the function $p_{1,x} : \mathbb{R} \to (-\infty, \infty)$ is strictly decreasing and Lipschitz continuous with Lipschitz constant $g_{\sup}(x)$ and this function satisfies $\lim_{\lambda\uparrow\infty} p_{1,x}(\lambda) = -\infty$ and $\lim_{\lambda\downarrow-\infty} p_{1,x}(\lambda) = \infty$.*

Proof. If $\lambda_*(x)$ is finite for some $x \in B$, then we know by Lemma 8.1 that $p_{1,x}(\lambda)$ is finite for every $\lambda \in \mathbb{R}$. Selecting some $\mu \in \mathbb{R}$, using $g_{\sup}(x) < \infty$ and the fact that $p_{1,x}(\mu)$ is finite, it is easy to verify that

$$|p_{1,x}(\lambda) - p_{1,x}(\mu)| \leq g_{\sup}(x)|\lambda - \mu| \tag{8.21}$$

for every $\lambda \in \mathbb{R}$. Hence $p_{1,x}$ is a Lipschitz continuous convex function with Lipschitz constant $g_{\sup}(x) < \infty$. Also it is easy to verify using $g_{\inf}(x) > 0$ that

$$p_{1,x}(\lambda) - p_{1,x}(\mu) \geq (\mu - \lambda) g_{\inf}(x) \tag{8.22}$$

for every $\lambda < \mu$. This shows that $p_{1,x}$ is strictly decreasing on $\mathbb{R}$. Again by relation (8.22) we obtain for a given μ and $\lambda \downarrow -\infty$ that $\lim_{\lambda \downarrow -\infty} p_{1,x}(\lambda) = \infty$ and for a given λ and $\mu \uparrow \infty$ that $\lim_{\mu \uparrow \infty} p_{1,x}(\mu) = -\infty$. □

If $\lambda_*(x)$ is finite, it follows from Lemma 8.4 and Theorem 1.13 of [29] that the finite-valued convex function $p_{1,x}$ has a nonempty subgradient set $\partial p_{1,x}(\lambda)$ for every $\lambda \in \mathbb{R}$. Hence for every $a \in \partial p_{1,x}(\lambda)$ and $\mu, \lambda \in \mathbb{R}$ the subgradient inequality

$$p_{1,x}(\mu) \geq p_{1,x}(\lambda) + a(\mu - \lambda)$$

holds. Applying relation (8.21) and the fact that $p_{1,x}$ is strictly decreasing we obtain

$$g_{\sup}(x) \geq p_{1,x}(\lambda - 1) - p_{1,x}(\lambda) \geq -a \tag{8.23}$$

for every $a \in \partial p_{1,x}(\lambda)$. Furthermore, applying relation (8.22) yields

$$-g_{\inf}(x) \geq p_{1,x}(\lambda + 1) - p_{1,x}(\lambda) \geq a \tag{8.24}$$

for every $a \in \partial p_{1,x}(\lambda)$. Hence by relations (8.23) and (8.24) it follows that

$$\partial p_{1,x}(\lambda) \subseteq [-g_{\sup}(x), -g_{\inf}(x)]. \tag{8.25}$$

To give a more detailed representation of the subgradient set $\partial p_{1,x}(\mu)$ it is convenient to assume that the set $S_{p_1}(\lambda, x)$ introduced in relation (8.16) is nonempty. As already observed, this set represents the set of optimal solutions of the parametric problem (P^x_λ). It is easy to see that $-g(y,x) \in \partial p_{1,x}(\lambda)$ for every $y \in S_{p_1}(\lambda, x)$. Since $\partial p_{1,x}(\lambda)$ is a closed convex set, this implies

$$[-\sup_{y \in S_{p_1}(\lambda,x)} g(y,x), -\inf_{y \in S_{p_1}(\lambda,x)} g(y,x)] \subseteq \partial p_{1,x}(\lambda). \tag{8.26}$$

Although it is possible for a finite $\lambda_*(x)$ to give a complete representation of the subgradient set $\partial p_{1,x}(\lambda)$ for every $\lambda \in \mathbb{R}$, we only consider the following important subcase.

Lemma 8.5 *Assume Condition 8.3 holds. Then it follows for every $x \in B$ that $\lambda_*(x)$ is finite, $S_{p_1}(\lambda, x)$ is a nonempty compact set for every $(\lambda, x) \in \mathbb{R} \times B$ and*

$$\partial p_{1,x}(\lambda) = [-\max_{y \in S_{p_1}(\lambda,x)} g(y,x), -\min_{y \in S_{p_1}(\lambda,x)} g(y,x)].$$

Also for every $a_\lambda \in \partial p_{1,x}(\lambda)$ and $a_\mu \in \partial p_{1,x}(\mu)$ and $\lambda > \mu$ it holds that $0 > a_\lambda \geq a_\mu$.

Proof. Since the functions $y \to f(y,x)$ and $y \to g(y,x)$ are continuous, $g > 0$ on $A \times B$ and A is compact, we obtain that $\lambda_*(x)$ is finite. By the same argument it also follows that $S_{p_1}(\lambda, x)$ is nonempty for every $\lambda \in \mathbb{R}$. Also by the continuity of the function $y \to f(y,x) - \lambda g(y,x)$ the set $S_{p_1}(\lambda, x) \subseteq A$ is closed and hence compact. Using now the proof of Lemma 3.2 in [8] and the fact that $S_{p_1}(\lambda, x)$ is a compact set yields the desired representation of the subgradient set $\partial p_{1,x}(\lambda)$. To show the last part we observe by the subgradient inequality that $p_{1,x}(\mu) \geq p_{1,x}(\lambda) + a_\lambda(\mu - \lambda)$. Moreover, applying the same argument it follows that $p_{1,x}(\lambda) \geq p_{1,x}(\mu) + a_\mu(\lambda - \mu)$. Adding these two inequalities yields

$$p_{1,x}(\mu) + p_{1,x}(\lambda) \geq p_{1,x}(\lambda) + p_{1,x}(\mu) + (a_\mu - a_\lambda)(\lambda - \mu),$$

and since $\lambda > \mu$, it follows that $a_\mu - a_\lambda \leq 0$. □

Looking at the proof of the last inequality it is only needed that the subgradient sets $\partial p_{1,x}(\lambda)$ and $\partial p_{1,x}(\mu)$ are nonempty. In view of Lemma 8.1 this is true if $\lambda_*(x)$ is finite and Condition 8.1 holds. By relation (8.11) the above conditions are clearly satisfied for a generalized fractional program.

In the next lemma we show the following important improvement of Lemma 8.1 and Lemma 8.2.

Lemma 8.6 *Assume Condition 8.1 holds. Then the set* $\{\lambda \in \mathbb{R} : p_1(\lambda, x) = 0\}$ *is nonempty if and only if* $\lambda_*(x) < \infty$. *Moreover, if this set is nonempty, it only contains the finite value* $\lambda_*(x)$.

Proof. If the set $\{\lambda \in \mathbb{R} : p_1(\lambda, x) = 0\}$ is nonempty, then it follows for any λ belonging to this set that $f(y,x) \leq \lambda g(y,x)$ for every $y \in A$. This shows by the positivity of g on $A \times B$ that $\lambda_*(x) \leq \lambda < \infty$. Also by Lemma 8.2 we obtain for $\lambda_*(x)$ finite that $p_1(\lambda_*(x), x) = 0$. This proves the first part of the above result. To prove the second part, we observe that by Lemma 8.4 the function $p_{1,x}$ is strictly decreasing. This completes the proof. □

Up to now we did not assume that there exists some $x \in B$ satisfying $\lambda_* = \lambda_*(x) < \infty$, i.e., that the min-max fractional program (P) has an optimal solution in B. In the next theorem we show the implications of this assumption. To do so, consider the (possibly empty) set $D_0 \subseteq \mathbb{R}$ given by

$$D_0 := \{\lambda \in \mathbb{R} : p_2(\lambda) = 0 \text{ and } S_{p_2}(\lambda) \text{ is nonempty}\}. \tag{8.27}$$

It is now possible to prove the following theorem.

Theorem 8.3 *If Condition 8.1 holds, then* $\lambda_* = \lambda_*(x_0) < \infty$ *for some* $x_0 \in B$ *if and only if* $D_0 = \{\lambda_*\}$. *Moreover, if* $\lambda_* = \lambda_*(x_0) < \infty$ *for some* $x_0 \in B$, *then*

$$S_{p_2}(\lambda_*) = \{x \in B : \lambda_* = \lambda_*(x)\}.$$

Proof. By Lemma 8.2 it follows for $\lambda_* = \lambda_*(x) < \infty$ that $p_1(\lambda_*, x) = 0$. Since $p_1(\lambda_*, x) \geq p_2(\lambda_*) \geq 0$, this shows that

$$0 = p_1(\lambda_*, x) = p_2(\lambda_*). \tag{8.28}$$

Using relation (8.28) with x replaced by x_0 it follows for $\lambda_* = \lambda_*(x_0) < \infty$ that λ_* belongs to D_0. Hence we still need to show that D_0 only contains λ_*. Consider therefore an arbitrary λ belonging to D_0. By the definition of D_0 in relation (8.27) one can find some $x_0 \in B$ satisfying $0 = p_2(\lambda) = p_1(\lambda, x_0)$, and this implies by Lemma 8.6 that $\lambda_*(x_0) = \lambda$. Since $p_2(\lambda) = 0$, it follows by Theorem 8.1 that $\lambda \leq \lambda_*$, and this shows that $\lambda_*(x_0) = \lambda \leq \lambda_* \leq \lambda_*(x_0)$. Hence $\lambda = \lambda_*$, and we have verified that D_0 only contains λ_*.

To prove the converse we obtain for $\lambda_* \in D_0$ that $0 = p_2(\lambda_*) = p_1(\lambda_*, x_0)$ for some $x_0 \in B$. Applying Lemma 8.6 yields $\lambda_*(x_0)$ is finite and $\lambda_* = \lambda_*(x_0)$ which proves the "only if" implication. To verify the second part it follows by relation (8.28) that x belongs to $S_{p_2}(\lambda_*)$ for every $x \in B$ satisfying $\lambda_* = \lambda_*(x)$, and so

$$\{x \in B : \lambda_* = \lambda_*(x)\} \subseteq S_{p_2}(\lambda_*).$$

To prove the reverse inclusion, let $x \in S_{p_2}(\lambda_*)$. Since $\lambda_* = \lambda_*(x_0) < \infty$ for some $x_0 \in B$, it follows from relation (8.28) with x replaced by x_0 that $0 = p_2(\lambda_*)$. Since $x \in S_{p_2}(\lambda_*)$, this implies $p_1(\lambda_*, x) = p_2(\lambda_*) = 0$. Applying now Lemma 8.6 yields $\lambda_* = \lambda_*(x)$. □

If we introduce the (possibly empty) set $D_1 \subseteq \mathbb{R}$ given by

$$D_1 := \{\lambda \in \mathbb{R} : p_2(\lambda) = 0 \text{ and } (P_\lambda) \text{ has an optimal solution}\},$$

then without Condition 8.1 one can show, using similar techniques as before, the following result. Note the vector (y, x) is an optimal solution of the (primal) min-max problem *(P)* if and only if $(y, x) \in A \times B$ and $\lambda_* = \lambda_*(x) = f(y, x)(g(y, x))^{-1}$.

Theorem 8.4 *The (primal) min-max fractional program (P) has an optimal solution if and only if* $D_1 = \{\lambda_*\}$. *Moreover, if (P) has an optimal solution, then the set* $S_p(\lambda_*)$ *listed in relation (8.18) is nonempty and*

$$S_p(\lambda_*) = \{(y, x) \in A \times B : \lambda_* = \lambda_*(x) = \frac{f(y, x)}{g(y, x)}\}.$$

For the moment this concludes our discussion of some of the theoretical properties related to the parametric approach. In the next subsection we will consider the (primal) Dinkelbach-type algorithm and use the previously derived properties to show its convergence.

6.2 The Primal Dinkelbach-Type Algorithm.

In this section we will introduce the so-called primal Dinkelbach-type algorithm to solve the (primal) min-max fractional program (P). A similar approach for a slightly different min-max fractional program satisfying some compactness assumptions on the feasible sets A and B was considered by Ţigan (cf. [72, 73]). Contrary to [72] the feasible set A in this section does not depend on y. Due to this our assumptions are less restrictive. Using Lemma 8.1 and the fact that the (primal) Dinkelbach-type algorithm is based on solving a sequence of parametric optimization problems (P_λ) for $\lambda \geq \lambda_*$ it is natural to assume that the (primal) min-max fractional program *(P)* satisfies the next condition.

Condition 8.5

- Condition 8.1 holds and $\lambda_*(x)$ is finite for every $x \in B$.
- If λ_* is finite, then for every $\lambda \geq \lambda_*$ the set $S_{p_2}(\lambda)$ is nonempty while for $\lambda_* = -\infty$ the set $S_{p_2}(\lambda)$ is nonempty for every $\lambda \in \mathbb{R}$.

Contrary to the analysis in [22] for generalized fractional programs we do not assume that the min-max fractional program (P) has an optimal solution. Also for generalized fractional programs the first part of Condition 8.5 is automatically satisfied. If Condition 8.5 holds, then one can execute the following so-called primal Dinkelbach-type algorithm. The geometrical interpretation of this algorithm is as follows. By Theorem 8.3 we need to find the zero point λ_* of the optimal value function p_2. Starting at a given point $\lambda > \lambda_*$ it follows by Theorem 8.1 that $p_2(\lambda) < 0$. Since the function p_2 is nonconvex and it is too ambitious to compute in one step its zero point λ_*, we replace this function by the easier convex function $p_{1,x}(.)$ with x belonging to $S_{p_2}(\lambda)$. We know by the definition of $p_{1,x}$ and $S_{p_2}(\lambda)$ that $p_2(\lambda) = p_{1,x}(\lambda)$ and $p_{1,x}(.) \geq p_2(.)$. For the function $p_{1,x}(.)$ it is easy to compute its zero point. By Lemma 8.2 this is given by $\lambda_*(x)$. We now replace the original point λ in the parametric problem (P_λ) by the smaller value $\lambda_*(x) \geq \lambda_*$ and repeat the procedure.

Primal Dinkelbach-type algorithm.

1 Select $x_0 \in B$ and $k := 1$ and compute

$$\lambda_k := \lambda_*(x_0).$$

2 Determine $x_k \in S_{p_2}(\lambda_k)$.If $p_1(\lambda_k, x_k) \geq 0$stop and return $\lambda_k$and x_k. Otherwise compute

$$\lambda_{k+1} := \lambda_*(x_k),$$

let $k := k + 1$ and go to step 1.

To determine $\lambda_*(x)$ in step 1 and 2 one has to solve a single-ratio fractional program. If A is a finite set, then this is easy. Also in order to select $x_k \in S_{p_2}(\lambda_k)$, one has to solve for A finite a finite min-max problem. Algorithms for such a problem can be found in part 2 of [55]. In case A is not finite, one needs to solve a much more difficult problem, a semi-infinite min-max problem (cf. [27, 55]). Therefore to apply the above generic primal Dinkelbach-type algorithm in practice one needs to have an efficient algorithm to determine an element of the set $S_{p_2}(\lambda_k)$, and this is in most cases the bottleneck. In general one cannot expect that an efficient and fast algorithm exists. But for special cases this might be the case. Including the construction of approximate solutions of the problem (P_{λ_k}) by using smooth approximations of the max operator, thus speeding up the computations and at the same time bounding the errors (cf. [16]) seems to be an important topic for future research.

By Lemma 8.6 it is sufficient to find in step 2 of the primal Dinkelbach-type algorithm the solution of the equation $p_1(\lambda, x_k) = 0$. As already observed, we can give an easy geometrical interpretation of the above algorithm (cf. [5, 16]). The next result shows that the sequence λ_k generated by the primal Dinkelbach-type algorithm is strictly decreasing.

Lemma 8.7 *If Condition 8.5 holds, then the sequence λ_k generated by the primal Dinkelbach-type algorithm is strictly decreasing and satisfies $\lambda_k \geq \lambda_* \geq -\infty$ for every $k \in \mathbb{N}$.*

Proof. If the algorithm stops at $k = 1$, then by the stopping rule we know that $p_2(\lambda_1) \geq 0$. This implies by Theorem 8.1 for $\lambda_1 = \lambda_*(x_0)$ that $\lambda_*(x_0) \leq \lambda_*$ which shows that $\lambda_*(x_0) = \lambda_*$. If the algorithm does not stop at the first step, then $p_2(\lambda_1) < 0$. Since $S_{p_2}(\lambda_1)$ is nonempty, the algorithm finds some $x_1 \in S_{p_2}(\lambda_1)$. Hence

$$0 > p_2(\lambda_1) = p_1(\lambda_1, x_1) = \sup_{y \in A} p(\lambda_1, y, x_1). \qquad (8.29)$$

Thus for every $y \in A$ we obtain $f(y, x_1) - \lambda_1 g(y, x_1) < 0$, and so

$$\frac{f(y, x_1)}{g(y, x_1)} < \lambda_1$$

for every $y \in A$. This shows $\lambda_2 \leq \lambda_1$. To verify that $\lambda_*(x_1) = \lambda_2 < \lambda_1$ we assume by contradiction that $\lambda_*(x_1) = \lambda_1$. Since $x_1 \in S_{p_2}(\lambda_1)$, this yields by relation (8.29) and Lemma 8.6 that

$$0 > p_2(\lambda_1) = p_1(\lambda_1, x_1) = p_1(\lambda_*(x_1), x_1) = 0,$$

and we obtain a contradiction. Therefore $\lambda_2 < \lambda_1$, and by the definition of λ_2 it is obvious that $\lambda_2 \geq \lambda_*$. Applying now the same argument iteratively shows the desired result. □

By Lemma 8.7 it follows that the sequence λ_k generated by the primal Dinkelbach-type algorithm converges to some limit $w \geq -\infty$. In case the generated sequence is finite, it is easy to show the following result.

Lemma 8.8 *If Condition 8.5 holds and the primal Dinkelbach-type algorithm stops at λ_i, then $\lambda_* = \lambda_i = \lambda_{i+1}$ and $p_2(\lambda_i) = 0$.*

Proof. Since Condition 8.5 holds, we obtain $\lambda_* < \infty$. Also by the stopping rule of the Dinkelbach-type algorithm it follows that $p_2(\lambda_i) \geq 0$. This implies by Theorem 8.1 that $\lambda_i \leq \lambda_*$. Since always $\lambda_i \geq \lambda_*$, we obtain $\lambda_i = \lambda_*$. To show that $\lambda_{i+1} = \lambda_i$ with $\lambda_i := \lambda_*(x_{i-1})$ and $p_2(\lambda_i) = 0$, we observe by Lemma 8.6 and by using $x_i \in S_{p_2}(\lambda_i)$ that

$$0 \leq p_2(\lambda_i) = p_1(\lambda_i, x_i) \leq p_1(\lambda_i, x_{i-1}) = 0. \tag{8.30}$$

Hence it follows that $p_2(\lambda_i) = p_1(\lambda_i, x_i) = 0$. Applying again Lemma 8.6 we obtain $\lambda_{i+1} := \lambda_*(x_i) = \lambda_i$ which completes the proof. □

In the remainder of this subsection we only consider the case that the primal Dinkelbach-type algorithm generates an infinite sequence $\lambda_k, k \in \mathbb{N}$. By Lemma 8.7 it follows that $\lim_{k\uparrow\infty} \lambda_k = w \geq -\infty$ exists. Imposing some additional condition it will be shown in Lemma 8.9 that this limit equals λ_*. To simplify the notation in the following lemmas we introduce for the sequence $\{(\lambda_k, x_k) \in \mathbb{R} \times B : x_k \in S_{p_2}(\lambda_k)\}$ generated by the primal Dinkelbach-type algorithm the sequence $\{a_k : k \in \mathbb{N}\}$ with

$$a_k \in \partial p_{1,x_k}(\lambda_{k+1}) \tag{8.31}$$

and for λ_* finite the sequence $\{b_k : k \in \mathbb{N}\}$ with

$$b_k \in \partial p_{1,x_k}(\lambda_*). \tag{8.32}$$

By the observation after Lemma 8.4 these subgradient sets are nonempty. It is now possible to derive the next result.

Lemma 8.9 *If Condition 8.5 holds and there exists a subsequence* $\{a_{n_k} : k \in \mathbb{N}\}$ *satisfying* $\sum_{k=1}^{\infty} a_{n_k}^{-1} = -\infty$, *then* $\lim_{k\uparrow\infty} \lambda_k = \lambda_*$. *Moreover, for* λ_* *finite it follows that* $\lim_{k\uparrow\infty} p_2(\lambda_k) = 0 \leq p_2(\lambda_*)$.

Proof. By Lemma 8.7 the sequence $\{\lambda_k : k \in \mathbb{N}\}$ is strictly decreasing, and so $\lim_{k\uparrow\infty} \lambda_k := w \geq -\infty$ exists. If $w = -\infty$, we obtain using $\lambda_k \geq \lambda_*$ for every $k \in \mathbb{N}$ that $-\infty = w \geq \lambda_*$, and so for $w = -\infty$ the result is proved. Therefore assume that w is finite. Since $p_2(\lambda_k) = p_1(\lambda_k, x_k) < 0$ and the function p_2 and the sequence $\{\lambda_k : k \in \mathbb{N}\}$ are decreasing, it follows that the sequence $\{p_1(\lambda_k, x_k) : k \in \mathbb{N}\}$ is increasing and $-\infty < \alpha := \lim_{k\uparrow\infty} p_1(\lambda_k, x_k) \leq 0$ exists. If we assume that $\alpha < 0$, then one can find some $\epsilon > 0$ satisfying $p_1(\lambda_k, x_k) \leq -\epsilon$ for every $k \in \mathbb{N}$. By Lemma 8.2 we also know that $p_1(\lambda_{k+1}, x_k) = 0$. Applying the subgradient inequality to the convex function p_{1,x_k} we obtain for every $k \in \mathbb{N}$ that

$$a_k(\lambda_k - \lambda_{k+1}) \leq p_1(\lambda_k, x_k) - p_1(\lambda_{k+1}, x_k) = p_1(\lambda_k, x_k) \leq -\epsilon$$

with $a_k \in \partial p_{1,x_k}(\lambda_{k+1})$. Since by relation (8.25) it follows that $-\infty < a_k < 0$, the above inequality shows $\lambda_k - \lambda_{k+1} \geq -\epsilon a_k^{-1}$. This yields by our assumption and w finite that

$$\lambda_1 - w = \sum_{k=1}^{\infty} (\lambda_k - \lambda_{k+1}) \geq -\epsilon \sum_{k=1}^{\infty} a_k^{-1} \geq -\epsilon \sum_{k=1}^{\infty} a_{n_k}^{-1} = \infty,$$

and so $w = -\infty$. This contradicts that w is finite, and we have shown that $\lim_{k\uparrow\infty} p_2(\lambda_k) = 0$. Applying now Theorem 8.2 and Lemma 1.30 of [29] yields $p_2(w) \geq \limsup_{k\uparrow\infty} p_2(\lambda_k) = 0$. Then by Theorem 8.1 it follows that $w \leq \lambda_*$. Since by Lemma 8.7 it is obvious that $w = \lim_{k\uparrow\infty} \lambda_k \geq \lambda_*$, we obtain $w = \lambda_*$ completing the proof. □

By relation (8.25) it follows that

$$0 > a_k \geq -g_{\sup}(x_k)$$

for every $a_k \in \partial p_{1,x_k}(\lambda_{k+1})$, and so one can apply Lemma 8.9 in case $\sum_{k=1}^{\infty} g_{\sup}(x_{n_k})^{-1} = \infty$. To achieve a rate of convergence result for the sequence λ_k generated by the primal Dinkelbach-type algorithm, we need to assume in the proof that $p_2(\lambda_*) = 0$. To apply our procedure we always impose that $S_{p_2}(\lambda_*)$ is nonempty for λ_* finite. Then it follows by Theorem 8.3 that $p_2(\lambda_*) = 0$ if and only if the min-max fractional program (P) has an optimal solution in B or equivalently there exists some

$x_0 \in B$ satisfying $\lambda_* = \lambda_*(x_0)$. However, if the condition of Lemma 8.9 holds, we conjecture for λ_* finite that the min-max fractional program *(P)* might not have an optimal solution in *B*, and so $p_2(\lambda_*)$ is not equal to zero. Using a stronger condition than in Lemma 8.9, we show in the next lemma for finite λ_* that the sequence $\{p_2(\lambda_k) : k \in \mathbb{N}\}$ generated by the primal Dinkelbach-type algorithm satisfies $\lim_{k\uparrow\infty} p_2(\lambda_k) = p_2(\lambda_*) = 0$. This sufficient condition implies the existence of an optimal solution of the (primal) min-max fractional program (P) in *B*.

Lemma 8.10 *If Condition 8.5 holds, λ_* is finite and there exists a subsequence $\{b_{n_k} : k \in \mathbb{N}\}$ satisfying $\inf_{k\in\mathbb{N}} b_{n_k} > -\infty$, then $\lim_{k\uparrow\infty} \lambda_k = \lambda_*$ and $\lim_{k\uparrow\infty} p_2(\lambda_k) = 0 = p_2(\lambda_*)$.*

Proof. By the convexity of the function p_{1,x_k} and the subgradient inequality we obtain for every $k \in \mathbb{N}$ that

$$0 \geq p_2(\lambda_k) \geq p_1(\lambda_*, x_k) + b_k(\lambda_k - \lambda_*) \geq p_2(\lambda_*) + b_k(\lambda_k - \lambda_*) \quad (8.33)$$

with $b_k \in \partial p_{1,x_k}(\lambda_*)$. Since $\lambda_{k+1} > \lambda_*$, it follows by our assumption and the monotonicity of the subgradient sets as shown in Lemma 8.5 that one can find some finite *M* satisfying $M \leq b_{n_k} \leq a_{n_k} < 0$ for every $k \in \mathbb{N}$ and every sequence $\{a_{n_k} : k \in \mathbb{N} \text{ and } a_{n_k} \in \partial p_{1,x_k}(\lambda_{k+1})\}$. This shows

$$0 > M^{-1} \geq b_{n_k}^{-1} \geq a_{n_k}^{-1} \quad (8.34)$$

for every $k \in \mathbb{N}$, and so $\sum_{k=1}^{\infty} a_{n_k}^{-1} = -\infty$. Hence by Lemma 8.9 we obtain $\lim_{k\uparrow\infty} \lambda_k = \lambda_*$. Using relations (8.34) and (8.33) yields $0 \geq \lim_{k\uparrow\infty} p_2(\lambda_{n_k}) \geq p_2(\lambda_*)$. Since by Theorem 8.1 we know that $p_2(\lambda_*) \geq 0$, the proof is completed. □

By relation (8.25) it follows in case $\sup_{k\in\mathbb{N}} g_{\sup}(x_k) < \infty$ that the condition of Lemma 8.10 is satisfied. A similar condition is also given in [22] for a generalized fractional program. In the next lemma we consider the generated sequence $\{x_k : x_k \in S_{p_2}(\lambda_k)\}_{k\in\mathbb{N}}$ and show for *B* compact and some additional topological properties on the functions f and g that this sequence contains a converging subsequence.

Lemma 8.11 *If Condition 8.5 holds, the functions f and g are finite-valued and continuous on some open set $W \subseteq \mathbb{R}^{m+n}$ containing $A \times B$, the set B is compact and there exists a subsequence $\{a_{n_k} : k \in \mathbb{N}\}$ satisfying $\sum_{k=1}^{\infty} a_{n_k}^{-1} = -\infty$, then the sequence $\{x_k : x_k \in S_{p_2}(\lambda_k)\}_{k\in\mathbb{N}}$ has a converging subsequence and every limit point x_∞ of the sequence $\{x_k : k \in \mathbb{N}\}$ satisfies $\lambda_* = \lambda_*(x_\infty)$ with λ_* finite. Additionally, if there exist a unique $x_* \in B$ satisfying $\lambda_* = \lambda_*(x_*)$, then $\lim_{k\uparrow\infty} x_k =$*

x_*. *Moreover, for $A \times B$ compact, the generated sequence $\{(y_k, x_k) : (y_k, x_k) \in S_p(\lambda_k)\}_{k \in \mathbb{N}}$ has a converging subsequence and every limit point of the sequence $\{(y_k, x_k) : k \in \mathbb{N}\}$ is an optimal solution of problem (P). If the optimization problem (P) has a unique optimal solution (y_*, x_*), then $\lim_{k \uparrow \infty} x_k = x_*$ and $\lim_{k \uparrow \infty} y_k = y_*$.*

Proof. To verify that λ_* is finite we obtain by Condition 8.5 and f, g continuous that the finite-valued function $x \to \lambda_*(x)$ is lower semicontinuous. By the compactness of B this implies, using Corollary 1.2 of [3], that there exists some $x \in B$ satisfying $\lambda_* = \lambda_*(x)$, and so λ_* is finite. Again by the compactness of B it is also obvious that the sequence $\{x_k : k \in \mathbb{N}\}$ contains a convergent subsequence. To show that every limit point x_∞ of the sequence $x_k, k \in \mathbb{N}$ satisfies $\lambda_* = \lambda_*(x_\infty)$ we observe by Lemma 8.9 that $\lim_{k \uparrow \infty} \lambda_k = \lambda_*$. This implies by Lemma 8.3 that $x_\infty \in S_{p_2}(\lambda_*)$. Using now Theorem 8.3 we obtain $\lambda_* = \lambda_*(x_\infty)$. If there exists a unique $x_* \in B$ satisfying $\lambda_* = \lambda_*(x_*)$, then again by Theorem 8.3 we obtain $S_{p_2}(\lambda_*) = \{x_*\}$. Since every converging subsequence of the sequence $x_k, k \in \mathbb{N}$ converges to an element of $S_{p_2}(\lambda_*)$, it follows that every convergent subsequence converges to the element x_*. By contradiction and B compact we obtain $\lim_{k \uparrow \infty} x_k = x_*$, and the proof of the first part is completed. If $A \times B$ is compact, then by the continuity of the function g we obtain

$$\sup_{(x,y) \in A \times B} g(y, x) < \infty.$$

Again by the observation after Lemma 8.10 we obtain $\lambda_k \downarrow \lambda_*$. By Lemma 8.3 the set-valued mapping S_p is closed and using a similar proof as for the first part one can show the last part. □

If we consider a generalized fractional program, then clearly A is compact, and if additionally the conditions of Lemma 8.11 hold, then the second part of this lemma applies. Unfortunately it is not clear to the authors whether in the first part of this lemma the condition $\sum_{k=1}^{\infty} a_{n_k}^{-1} = -\infty$ can be omitted.

We now want to investigate how fast the sequence λ_k converges to λ_*. Before discussing this in detail, we list for λ_* finite the following inequality for the sequence $\{\lambda_k : k \in \mathbb{N}\}$ generated by the primal Dinkelbach-type algorithm. A similar inequality can also be derived for the dual Dinkelbach-type algorithm to be discussed in subsection 6.4.

Theorem 8.5 *If Condition 8.5 holds and there exists some $x \in B$ satisfying $\lambda_* = \lambda_*(x)$, then it follows for every $c_k \in \partial p_{1,x}(\lambda_k)$ and*

$a_k \in \partial p_{1,x_k}(\lambda_{k+1})$ *that*

$$0 \leq \frac{\lambda_{k+1} - \lambda_*}{\lambda_k - \lambda_*} \leq (1 - c_k a_k^{-1}).$$

Proof. Since $\lambda_* = \lambda_*(x)$ for some $x \in B$, we obtain by Lemma 8.6 that $p_1(\lambda_*, x) = p_1(\lambda_*(x), x) = 0$. Applying now the subgradient inequality to the function $p_{1,x}$ at the point λ_* it follows for $c_k \in \partial p_{1,x}(\lambda_k)$ that

$$-p_1(\lambda_k, x) = p_1(\lambda_*, x) - p_1(\lambda_k, x) \geq c_k(\lambda_* - \lambda_k).$$

Hence

$$p_2(\lambda_k) \leq p_1(\lambda_k, x) \leq c_k(\lambda_k - \lambda_*). \tag{8.35}$$

Moreover, for every $x_k \in S_{p_2}(\lambda_k)$ and $\lambda_{k+1} = \lambda_*(x_k)$ we obtain again by Lemma 8.6 that $p_1(\lambda_{k+1}, x_k) = 0$. Applying now the subgradient inequality to the function p_{1,x_k} at the point λ_{k+1} yields for $a_k \in \partial p_{1,x_k}(\lambda_{k+1})$ that

$$p_2(\lambda_k) = p_1(\lambda_k, x_k) - p_1(\lambda_{k+1}, x_k) \geq a_k(\lambda_k - \lambda_{k+1}). \tag{8.36}$$

Hence by relations (8.35) and (8.36) we obtain $-a_k(\lambda_{k+1} - \lambda_k) \leq p_2(\lambda_k) \leq c_k(\lambda_k - \lambda_*)$. Since $a_k < 0$, this implies

$$\lambda_{k+1} - \lambda_k \leq -c_k a_k^{-1}(\lambda_k - \lambda_*). \tag{8.37}$$

Using relation (8.37) it follows that

$$\lambda_{k+1} - \lambda_* = \lambda_k - \lambda_* + \lambda_{k+1} - \lambda_k \leq (1 - c_k a_k^{-1})(\lambda_k - \lambda_*),$$

and this completes the proof. □

In case of a single-ratio fractional program the function $\lambda \to p_{1,x}(\lambda)$ reduces to $p_{1,x}(\lambda) = f(x) - \lambda g(x)$, and so for every $\lambda \in \mathbb{R}$ it follows that $\partial p_{1,x}(\lambda) = \{-g(x)\}$. Hence we obtain that the inequality in Theorem 8.5 reduces to

$$0 \leq \frac{\lambda_{k+1} - \lambda_*}{\lambda_k - \lambda_*} \leq (1 - \frac{g(x_0)}{g(x_k)}) \tag{8.38}$$

for any optimal solution x_0 of the optimization problem

$$\inf_{x \in B} f(x)(g(x))^{-1}$$

(cf. [67]).

Before introducing convergence results for the primal Dinkelbach-type algorithm, we need the following definition (cf. [54]).

Definition 8.2 *A sequence* $\{s_k : k \in \mathbb{N}\} \subseteq \mathbb{R}^n$ *with limit* s_∞ *converges Q-linearly if there exists some* $0 < r < 1$ *such that*

$$\limsup_{k\uparrow\infty} \frac{\|s_{k+1} - s_\infty\|}{\|s_k - s_\infty\|} \leq r.$$

The sequence $\{s_k : k \in \mathbb{N}\}$ *converges Q-super linearly if*

$$\lim_{k\uparrow\infty} \frac{\|s_{k+1} - s_\infty\|}{\|s_k - s_\infty\|} = 0.$$

If a slightly stronger condition as used in Lemma 8.10 holds, then one can show that the sequence $\{\lambda_k : k \in \mathbb{N}\}$ generated by the primal Dinkelbach-type algorithm converges Q-linearly. The same result was shown for a generalized fractional program in [22].

Theorem 8.6 *If Condition 8.5 holds,* λ_* *is finite and the sequence* $\{b_k : k \in \mathbb{N}\}$ *satisfies* $\inf_{k\in\mathbb{N}} b_k > -\infty$, *then* $\lim_{k\uparrow\infty} \lambda_k = \lambda_*$ *and* $\{\lambda_k : k \in \mathbb{N}\}$ *converges Q-linearly.*

Proof. By Lemma 8.10 we obtain $p_2(\lambda_*) = 0$. Since Condition 8.5 holds, one can find some $x \in B$ satisfying $0 = p_2(\lambda_*) = p_1(\lambda_*, x)$, and this shows by Lemma 8.6 that $\lambda_* = \lambda_*(x)$. Hence the set $\{x \in B : \lambda_* = \lambda_*(x)\}$ is nonempty, and for every x belonging to this set it follows by Theorem 8.5 that

$$0 \leq \frac{\lambda_{k+1} - \lambda_*}{\lambda_k - \lambda_*} \leq (1 - c_k a_k^{-1}) \tag{8.39}$$

with $c_k \in \partial p_{1,x}(\lambda_k)$ and $a_k \in \partial p_{1,x_k}(\lambda_{k+1})$. Since $\{\lambda_k : k \in \mathbb{N}\}$ is strictly decreasing and $\lambda_k > \lambda_*$, it follows by Lemma 8.5 that the sequence $\{c_k : k \in \mathbb{N}\}$ is decreasing and satisfies $0 > c_k \geq \sigma$ with $\sigma := \max\{t : t \in \partial p_{1,x}(\lambda_*)\}$ This shows that $\lim_{k\uparrow\infty} c_k = c_\infty$ exists. To identify c_∞ we observe in view of $c_k \in \partial p_{1,x}(\lambda_k)$ that

$$p_1(\lambda, x) \geq p_1(\lambda_k, x) + c_k(\lambda - \lambda_k)$$

for every $\lambda \in \mathbb{R}$. Since the function $p_{1,x}$ is continuous, this yields using $\lambda_k \downarrow \lambda_*$ and $\lim_{k\uparrow\infty} c_k = c_\infty$ that

$$p_1(\lambda, x) \geq p_1(\lambda_*, x) + c_\infty(\lambda - \lambda_*)$$

for every $\lambda \in \mathbb{R}$, and so $c_\infty \in \partial p_{1,x}(\lambda_*)$. Therefore $c_\infty = \sigma$, and we have identified this limit. Also by our assumption we obtain that there exists some $-\infty < M \leq b_k \leq a_k$, and this shows

$$\limsup_{k\uparrow\infty}(1 - c_k a_k^{-1}) \leq 1 - \frac{\sigma}{M} < 1.$$

Applying now relation (8.39) yields the desired result. □

If the conditions of Theorem 8.5 hold and additionally we assume that $\inf_{k\in\mathbb{N}} a_k > -\infty$, then it can be shown in view of the proof of Theorem 8.6 that the sequence λ_k converges Q-linearly to λ_*. This condition is slightly weaker than the one used in Theorem 8.6. Observe the condition $\inf_{k\in\mathbb{N}} b_k > -\infty$ was used in the proof of Lemma 8.10 to show that $p_2(\lambda_*) = 0$, and this implies as shown in the first part of the proof of Theorem 8.6 that $\lambda_* = \lambda_*(x)$ for some $x \in B$. Therefore, if there exists some $x \in B$ satisfying $\lambda_*(x) = \lambda_*$ and $\inf_{k\in\mathbb{N}} a_k > -\infty$, then assuming Condition 8.5 holds the sequence λ_k converges Q-linearly to λ_*. A disadvantage of the first part of the previous assumption is that in general we do not know looking at a min-max problem whether there exists some $x \in B$ satisfying $\lambda_* = \lambda_*(x)$. Hence we imposed some stronger algorithmic condition on the sequence b_k, $k \in \mathbb{N}$ implying this result. In case the (primal) min-max fractional program *(P)* has a unique optimal solution and some additional topological properties are satisfied, then one can show that the sequence $\{\lambda_k : k \in \mathbb{N}\}$ converges superlinearly.

Theorem 8.7 *If Condition 8.5 holds, the functions f and g are continuous on some open set $W \subseteq \mathbb{R}^{m+n}$ containing the compact set $A \times B$ and the min-max fractional program (P) has a unique optimal solution (y_*, x_*), then $\lim_{k\uparrow\infty} x_k = x_*$, $\lim_{k\uparrow\infty} y_k = y_*$ and $\lim_{k\uparrow\infty} \lambda_k = \lambda_*$, and the sequence λ_k converges Q-superlinearly.*

Proof. Using Lemmas 8.9 and 8.11 the first part follows, and so we only have to show that λ_k converges superlinearly. Considering the proof of Theorem 8.6 it follows that

$$\limsup_{k\uparrow\infty}(1 - c_k a_k^{-1}) = 1 - \sigma(\limsup_{k\uparrow\infty} a_k)^{-1}$$

with $\sigma := \max\{t : t \in \partial p_{1,x_*}(\lambda_*)\}$ and $a_k \in \partial p_{1,x_k}(\lambda_{k+1}), k \in \mathbb{N}$. Since a_k is uniformly bounded by the compactness of $A \times B$ and the function g is continuous, there exists a converging subsequence a_{n_k} satisfying $a_\infty = \lim_{k\uparrow\infty} a_{n_k} = \limsup_{k\uparrow\infty} a_k$. To identify a_∞ we observe for every $k \in \mathbb{N}$ that

$$p_1(\lambda, x_k) \geq p_1(\lambda, x_k) - p_1(\lambda_{k+1}, x_k) \geq a_k(\lambda - \lambda_{k+1}) \tag{8.40}$$

with $a_k \in \partial p_{1,x_k}(\lambda_{k+1})$. Since B is compact and p continuous, it follows by Proposition 1.7 of [3] that $x \to p_1(\lambda, x)$ is upper semicontinuous, and this implies by relation (8.40) that

$$p_1(\lambda, x_*) \geq \limsup_{k\uparrow\infty} p_1(\lambda, x_k) \geq a_\infty(\lambda - \lambda_*) \tag{8.41}$$

Since $x_* \in S_{p_2}(\lambda_*)$, we obtain $p_1(\lambda_*, x_*) = p_2(\lambda_*) = 0$, and this shows by relation (8.41) that $a_\infty \in \partial p_{1,x_*}(\lambda_*)$. By the uniqueness of the optimal solution and Lemma 8.5 we obtain $a_\infty = \sigma$. This shows the desired result. □

In case we consider a single-ratio fractional program with B compact and the functions f, g continuous it follows by Lemma 8.11 that

$$\limsup_{k\uparrow\infty} g(x_k) = \lim_{k\uparrow\infty} g(x_{n_k}) = g(x_*)$$

with x_* an optimal solution of this fractional programming problem. Replacing now in relation (8.38) x_0 by x_* we obtain for a single-ratio fractional program with B compact and f, g continuous that the sequence $\{\lambda_k : k \in \mathbb{N}\}$ always converges Q-superlinearly. Clearly in practice the (primal) Dinkelbach-type algorithm stops in a finite number of steps, and so we need to derive a practical stopping rule. Such a rule is constructed in the next lemma. For other practical stopping rules yielding so-called ϵ-optimal solutions the reader may consult [16].

Lemma 8.12 *If Condition 8.5 holds and there exists some subsequence* $\{a_{n_k} : k \in \mathbb{N}\}$ *satisfying* $\sum_{k=1}^{\infty} a_{n_k}^{-1} = -\infty$ *and some* $x \in B$ *satisfying* $\lambda_* = \lambda_*(x)$, *then the sequence* $\{c_k^{-1} p_2(\lambda_k) : c_k \in \partial p_{1,x}(\lambda_k)\}_{k\in\mathbb{N}}$ *is decreasing and its limit equals* 0. *Moreover, it follows for every* $k \in \mathbb{N}$ *that*

$$\lambda_* \le \lambda_k \le \lambda_* + c_k^{-1} p_2(\lambda_k).$$

Proof. By Lemma 8.7 the sequence λ_k is strictly decreasing, and this implies by Lemma 8.5 that the negative sequence c_k is decreasing. Also, since p_2 is decreasing, we obtain that the negative sequence $p_2(\lambda_k)$ is increasing and so the positive sequence $c_k^{-1} p_2(\lambda_k)$ is decreasing. Applying now Lemma 8.9 it follows that $\lim_{k\uparrow\infty} c_k^{-1} p_2(\lambda_k) = 0$, while the listed inequality is an immediate consequence of Lemma 8.9 and relation (8.35). □

Using Lemma 8.12 a stopping rule for the (primal) Dinkelbach-type algorithm is given by $c_k^{-1} p_2(\lambda_k) \le \epsilon$ for some predetermined $\epsilon > 0$. Finally we observe that the (primal) Dinkelbach-type algorithm applied to a generalized fractional program can be regarded as a cutting plane algorithm (cf. [10]). This generalizes a similar observation by Sniedovich (cf. [70]) who showed the result for the (primal) Dinkelbach-type algorithm applied to a single-ratio fractional program.

In the next section we investigate the dual max-min fractional program *(D)* and its relation to the primal min-max fractional program *(P)*.

6.3 Duality Results for Primal Min-Max Fractional Programs.

In this subsection we first investigate under which conditions the optimal objective function values of the primal min-max fractional program *(P)* and the dual max-min fractional program (*D*) coincide. To start with this analysis, we introduce the following class of bifunctions.

Definition 8.3 *The function* $h : \mathbb{R}^m \times \mathbb{R}^n \to [-\infty, \infty]$ *is called a concave/convex bifunction on the convex set* $C_1 \times C_2$ *with* $C_1 \subseteq \mathbb{R}^m$ *and* $C_2 \subseteq \mathbb{R}^n$ *if for every* $x \in C_2$ *the function* $y \to h(y, x)$ *is concave on* C_1 *and for every* $y \in C_1$ *the function* $x \to h(y, x)$ *is convex on* C_2. *Moreover, a function* $h : \mathbb{R}^m \times \mathbb{R}^n \to [-\infty, \infty]$ *is called a convex/concave bifunction on* $C_1 \times C_2$ *if* $-h$ *is a concave/convex bifunction on the same set. It is called an affine/affine bifunction if it is both a concave/convex and a convex/concave bifunction.*

To guarantee that μ_* equals λ_*, we introduce the following sufficient condition.

Condition 8.6 *The set* $B \subseteq R^n$ *is a closed convex set and* $A \subseteq R^m$ *is a compact convex set. Moreover, there exists some open convex set* $A_1 \times B_1$ *containing* A × B *such that* g *is a positive finite-valued convex/concave bifunction and* f *a positive finite-valued concave/convex bifunction on* $A_1 \times B_1$. *If the function* g *is a positive affine/affine bifunction, then* f *is a finite-valued concave/convex bifunction.*

If the set *B* is given by relation (8.1), one can also introduce another dual max-min fractional program. To guarantee that for this problem strong duality holds, we need the following slightly stronger condition.

Condition 8.7 *The set* $B \subseteq R^n$ *is a closed convex set and* $A \subseteq R^m$ *is a compact convex set. Moreover, there exists some open convex set* $A_1 \times C_1$ *containing* A×C *such that* g *is a positive finite-valued convex/concave bifunction and* f *a positive finite-valued concave/convex bifunction on* $A_1 \times C_1$. *If the function* g *is a positive affine/affine bifunction, then* f *is a finite-valued concave/convex bifunction*

If Condition 8.6 holds, then by Theorem 1.15 of [29] we obtain that the function $y \to f(y, x)$ is continuous on A_1 for every $x \in B$ and $x \to f(x, y)$ is continuous on B_1 for every $y \in A$. The same property also holds for the function g. By the compactness of *A* this implies

$$0 < g_{\inf}(x) \leq g_{\sup}(x) < \infty$$

for every $x \in B$, and so Condition 8.6 implies Condition 8.1. Also, since for every $x \in B$ the function $y \to f(y, x)(g(y, x))^{-1}$ is continuous on A and the set A is compact, we obtain that $\lambda_*(x)$ is finite for every $x \in B$ implying $\lambda_* < \infty$. For $\lambda_* < \infty$ we derive in Theorem 8.8 that the optimal objective function value of the (primal) min-max fractional program (P) equals the optimal objective function value of the (dual) max-min fractional program *(D).* Contrary to the proof of the same result in [5] for generalized fractional programs based on Sion's minimax result (cf. [31, 69]) the present proof is an easy consequence of the easier-to-prove minimax result by Ky Fan (cf. [26, 27, 32]) and Theorem 8.1. Note that we do not assume that there exists some $x \in B$ satisfying $\lambda_* = \lambda_*(x)$.

Theorem 8.8 *If Condition 8.6 holds, then there exists some* $y_0 \in A$ *satisfying*

$$\lambda_* = \mu_* = \mu_*(y_0).$$

Proof. Since we know that $\mu_* \leq \lambda_* < \infty$, it follows for $\lambda_* = -\infty$ that $-\infty = \lambda_* = \mu_* \geq \mu_*(y)$ for every $y \in A$. This shows the desired result for $\lambda_* = -\infty$. If λ_* is finite, then we need to verify that $\lambda_* \leq \mu_*$. Since λ_* is finite, we obtain by Condition 8.6 that the function $(y, x) \to p(\lambda_*, y, x)$ is a concave/convex bifunction on $A \times B$ and for every $x \in B$ the function $y \to p(\lambda_*, y, x)$ is continuous on A_1. Applying now Theorem 3.2 of [32] (see also [27]) we obtain

$$p_2(\lambda_*) = \inf_{x \in B} \sup_{y \in A} p(\lambda_*, y, x) = \max_{y \in A} \inf_{x \in B} p(\lambda_*, y, x).$$

This shows by Theorem 8.1 and the remark after Condition 8.6 that

$$0 \leq p_2(\lambda_*) = \max_{y \in A} \inf_{x \in B} p(\lambda_*, y, x) = \inf_{x \in B} p(\lambda_*, y_0, x) \qquad (8.42)$$

for some $y_0 \in A$. Since $g(y_0, x) > 0$ for every $x \in B$, we obtain

$$\frac{f(y_0, x)}{g(y_0, x)} \geq \lambda_*$$

for every $x \in B$. Hence

$$\mu_* = \sup_{y \in A} \inf_{x \in B} \frac{f(y, x)}{g(y, x)} \geq \inf_{x \in B} \frac{f(y_0, x)}{g(y_0, x)} \geq \lambda_*. \qquad (8.43)$$

Using now relation (8.43) the desired result follows. □

Since there are rather general necessary and sufficient conditions on the bifunctions such that for those functions min-max equals max-min (cf. [28, 30]), the above result holds for a much larger class than the class of concave/convex bifunctions. However, since the class of concave/convex bifunctions is most known, we have restricted ourselves to this well-known class. An easy consequence of Theorem 8.8 is given by the next result.

Lemma 8.13 *If Condition 8.6 holds and there exists some $x_0 \in B$ satisfying $\lambda_* = \lambda_*(x_0)$ and some $y_0 \in A$ satisfying $\mu_* = \mu_*(y_0)$, then the vector (y_0, x_0) is an optimal solution of the (primal) min-max fractional program (P) and an optimal solution of the (dual) max-min fractional program (D).*

Proof. By the definition of $\mu_*(y)$ and $\lambda_*(x)$ it is clear that for every vector $(y, x) \in A \times B$ that

$$\mu_*(y) \leq \frac{f(y,x)}{g(y,x)} \leq \lambda_*(x).$$

This implies by Theorem 8.8 for the given vector $(y_0, x_0) \in A \times B$ that

$$\mu_* = \mu_*(y_0) = \frac{f(y_0,x_0)}{g(y_0,x_0)} = \lambda_*(x_0) = \lambda_*.$$

Hence (y_0, x_0) is an optimal solution of the (primal) min-max fractional program (P) and an optimal solution of the (dual) max-min fractional program *(D)*. □

If the (dual) max-min fractional program *(D)* has a unique optimal solution and the optimal solution set of the (primal) min-max fractional program *(P)* is nonempty, then by Lemma 8.13 the unique optimal solution of *(D)* is an optimal solution of *(P)*. If Condition 8.6 holds and we use the so-called dual Dinkelbach-type algorithm to *be* discussed in subsection 6.4 for identifying λ_*, this observation will be useful. To analyze the properties of the optimization problem *(D)* and at the same time construct some generic algorithm to solve problem *(D),* we introduce similar parametric optimization problems as done for problem *(P)* at the beginning of subsection 6.1. For every $(\lambda, y) \in \mathbb{R} \times A$ consider the parametric optimization problem

$$d_1(\lambda, y) := \inf_{x \in B} p(\lambda, y, x). \qquad (D_\lambda^y)$$

For every $y \in A$ the function $d_{1,y} : \mathbb{R} \to (-\infty, \infty]$ is now given by

$$d_{1,y}(\lambda) := d_1(\lambda, y).$$

Since $g > 0$ on $A \times B$ and $d_{1,y}$ is the infimum of affine functions, it is obvious that $d_{1,y}$ is a decreasing upper semicontinuous concave function. The so-called effective domain $dom(d_{1,y})$ is defined by

$$dom(d_{1,y}) := \{\lambda \in \mathbb{R} : d_{1,y}(\lambda) > -\infty\} \subseteq \mathbb{R}.$$

By the finiteness of p on $\mathbb{R} \times A \times B$ it is obvious for every $y \in A$ that actually $dom(d_{1,y}) = \{\lambda \in \mathbb{R} : d_{1,y}(\lambda) \text{ finite}\}$. A more difficult optimization problem than problem (D_λ^y) is now given by the parametric optimization problem

$$d_2(\lambda) = \sup_{y \in A} d_1(\lambda, y). \qquad (D_\lambda)$$

As for the concave function $d_{1,y}$ we also introduce the effective domain $dom(d_2)$ of the function d_2 given by

$$dom(d_2) := \{\lambda \in \mathbb{R} : d_2(\lambda) > -\infty\}.$$

It should be clear to the reader that we actually apply the Dinkelbach-type approach to the (dual) max-min fractional program *(D)* while at the beginning of subsection 6.1 we applied the same approach to the (primal) min-max fractional program *(P)*. It is easy to show that

$$\sup_{y \in A} \inf_{x \in B} p(\lambda, y, x) \le \inf_{x \in B} \sup_{y \in A} p(\lambda, y, x), \qquad (8.44)$$

and so we obtain $d_2(\lambda) \le p_2(\lambda)$ for every $\lambda \in \mathbb{R}$. If the optimization problem *(P)* is a single-ratio fractional program, then the set A consists of one element, and as already observed there is no difference in the representation of the (primal) min-max fractional program *(P)* and the (dual) max-min fractional program *(D)*. Hence for A consisting of one element it is not surprising that also the functional representation of the functions d_2 and p_2 are the same. If the set A consists of more than one element, then we want to know, despite the different functional representations of the functions d_2 and p_2, under which conditions $d_2(\lambda) = p_2(\lambda)$ for some λ. It should come as no surprise that this equality holds under the same conditions as used in Theorem 8.8. Note that in the next result we do not assume that the set $S_{p_2}(\lambda)$ is nonempty.

Theorem 8.9 *Assume Condition 8.6 holds where g is a convex/concave bifunction on $A \times B$. Then it follows for every $\lambda \ge 0$ that there exists some $y_\lambda \in A$ satisfying*

$$p_2(\lambda) = d_2(\lambda) = d_1(\lambda, y_\lambda).$$

Moreover, if g is an affine/affine bifunction, the same result holds for every $\lambda \in \mathbb{R}$.

Proof. Since $\lambda_* < \infty$, we obtain by Lemma 8.1 that $p_2(\lambda) < \infty$ for every $\lambda \in \mathbb{R}$. Also, for a convex/concave bifunction g it follows by Condition 8.6 and $\lambda \geq 0$ that the function $(y, x) \to p(\lambda, y, x)$ is a concave/convex bifunction on $A \times B$ and $y \to p(\lambda, y, x)$ is continuous on A_1 for every $(\lambda, x) \in \mathbb{R}_+ \times B$. A similar observation holds for $\lambda \in \mathbb{R}$, if g is an affine/affine bifunction. Since A is compact, we can now apply Theorem 3.2 of [32]. This shows

$$\begin{aligned} p_2(\lambda) &= \inf_{x \in B} \sup_{y \in A} p(\lambda, y, x) \\ &= \max_{y \in A} \inf_{x \in B} p(\lambda, y, x) = d_2(\lambda). \end{aligned} \tag{8.45}$$

Hence by relation (8.45) there exists for $\lambda \geq 0$ and a convex/concave bifunction g or $\lambda \in \mathbb{R}$ and an affine/affine bifunction g some $y_\lambda \in A$ satisfying $d_1(\lambda, y_\lambda) = d_2(\lambda)$. This completes the proof. □

Applying similar proofs as in Lemma 8.1 and Theorem 8.1 one can verify the following results.

Lemma 8.14 *Assume Condition 8.2 holds. Then* $\mu_* > -\infty$ *if and only if* $dom(d_2) = \mathbb{R}$, *and* $\mu_*(y) > -\infty$ *if and only if* $dom(d_{1,y}) = \mathbb{R}$.

Clearly Lemma 8.14 can be compared with Lemma 8.1 while the next result is the counterpart of Theorem 8.1.

Theorem 8.10 *Assume Condition 8.2 holds and* $\mu_* > -\infty$. *Then* $\lambda < \mu_*$ *if and only if* $d_2(\lambda) > 0$. *Moreover, if* $\mu_*(y) > -\infty$, *then* $\lambda < \mu_*(y)$ *if and only if* $d_1(\lambda, y) > 0$.

A direct consequence of the above results is given by the following.

Theorem 8.11 *Assume Condition 8.6 holds where* g *is a positive convex/concave bifunction on* $A \times B$. *Then it follows that* $0 \leq \lambda_* = \mu_* < \infty$, $p_2(\lambda) = d_2(\lambda)$ *for every* $\lambda \geq 0$, *and these functions are finite-valued on* $(-\infty, \lambda_*]$. *Moreover, if* g *is a positive affine/affine bifunction on* $A \times B$ *and* λ_* *is finite, then* $\mu_* = \lambda_*$, $p_2(\lambda) = d_2(\lambda)$ *for every* $\lambda \in \mathbb{R}$, *and these functions are finite-valued on* $(-\infty, \lambda_*]$.

Proof. If g is a positive convex/concave bifunction on $A \times B$, then by Condition 8.6 the function f must be a positive concave/convex bifunction on $A \times B$. Then automatically $0 \leq \lambda_* < \infty$. Also, by Theorem 8.8 and 8.9 we obtain $\mu_* = \lambda_*$ and $p_2(\lambda) = d_2(\lambda)$ for every $\lambda \geq 0$. Since Condition 8.6 implies Condition 8.1, it follows by the remark after Theorem 8.1 that $p_2(\lambda)$ is finite for every $\lambda \leq \lambda_*$. This yields $d_2(\lambda) = p_2(\lambda)$ is finite-valued on $[0, \lambda_*]$. Using the monotonicity of d_2, we see

$$\infty > p_2(\lambda) \geq d_2(\lambda) \geq d_2(0) = p_2(0) \geq 0$$

for every $\lambda \leq 0$. Hence the first part follows. The second part can be proved similarly, and its proof is therefore omitted. $\square$

If Condition 8.6 holds and hence also Condition 8.1 and λ_* is finite, then it might happen (as shown in Example 8.1) that the value $p_2(\lambda_*)$ is not equal to zero. If additionally there exists some $x_0 \in B$ satisfying $\lambda_* = \lambda_*(x_0)$, then by Theorems 8.3 and 8.11 we know that $d_2(\mu_*) = d_2(\lambda_*) = p_2(\lambda_*) = 0$, and we need this assumption in combination with Condition 8.6 to identify λ_* by the so-called dual Dinkelbach-type algorithm to be discussed in the next subsection. The next result is the counterpart of Theorem 8.2. It can be proved by similar techniques.

Theorem 8.12 *Assume Condition 8.2 holds. Then the decreasing function* $d_2 : \mathbb{R} \to [-\infty, \infty]$ *is lower semicontinuous.*

Similarly as in Section 6.1 it follows by Theorem 8.12 that $\lim_{s \uparrow \lambda} d_2(s) = d_2(\lambda)$, and the function d_2 is right-continuous with left-hand limits.

As in Section 6.1 we now introduce the following set-valued mappings $S_{d_1} : \mathbb{R} \times A \to 2^B$ and $S_{d_2} : \mathbb{R} \to 2^A$ given by

$$S_{d_1}(\lambda, y) := \{x \in B : d_1(\lambda, y) = p(\lambda, y, x)\} \tag{8.46}$$

and

$$S_{d_2}(\lambda) := \{y \in A : d_2(\lambda) = d_1(\lambda, y)\}. \tag{8.47}$$

The set $S_{d_1}(\lambda, y)$ represents the set of optimal solutions of optimization problem (D_λ^y), while the set $S_{d_2}(\lambda)$ denotes the set of optimal solutions in A of optimization problem (D_λ). Also we consider the set-valued mapping $S_d : \mathbb{R} \to 2^{A \times B}$ given by

$$S_d := \{(y, x) \in A \times B : d_2(\lambda) = d_1(\lambda, y) = p(\lambda, y, x)\}. \tag{8.48}$$

This set represents the set of optimal solutions in $A \times B$ of optimization problem (D_λ). In the next result it is assumed that the sets $S_{d_1}(\lambda, y), S_{d_2}(\lambda)$ and $S_d(\lambda)$ are nonempty on their domain. Applying Theorem 8.12 and using a similar proof as in Lemma 8.3 we obtain the following counterpart of Lemma 8.3.

Lemma 8.15 *Assume Condition 8.2 holds and the functions f and g are finite-valued and continuous on some open set $W \subseteq \mathbb{R}^{m+n}$ containing $A \times B$. Then the set-valued mappings S_{d_1}, S_{d_2} and S_d are closed.*

Considering now the function $d_{1,y} : \mathbb{R} \to [-\infty, \infty)$ given by

$$d_{1,y}(\lambda) := d_1(\lambda, y)$$

one can show as in Lemma 8.4 the following result.

Lemma 8.16 *Assume Condition 8.2 holds and* $\mu_*(y)$ *is finite for* $y \in A$. *Then the function* $d_{1,y} : \mathbb{R} \to (-\infty, \infty)$ *is strictly decreasing and Lipschitz continuous with Lipschitz constant* $\overline{g}_{\text{sup}}(y)$ *and the function satisfies* $\lim_{\lambda \uparrow \infty} d_{1,y}(\lambda) = -\infty$ *and* $\lim_{\lambda \downarrow -\infty} d_{1,y}(\lambda) = \infty$.

As in Section 6.1 with respect to the function $p_{1,x}$ it follows in case Condition 8.2 holds that the subgradient set of the convex strictly increasing function $-d_{1,y}$ is nonempty for every $\lambda \in \mathbb{R}$ and this set satisfies

$$\partial(-d_{1,y})(\lambda) \subseteq [\underline{g}_{\text{inf}}(y), \overline{g}_{\text{sup}}(y)]. \tag{8.49}$$

Moreover, the subgradient inequality is given by

$$-d_{1,y}(\mu) \geq -d_{1,y}(\lambda) + a(\mu - \lambda) \tag{8.50}$$

for every $a \in \partial(-d_{1,y})(\lambda)$. Also one can show the following counterpart of Lemma 8.5.

Lemma 8.17 *Assume Condition 8.4 holds. Then it follows for every* $y \in A$ *that* $\mu_*(y)$ *is finite,* $S_{d_1}(\lambda, y)$ *is a nonempty compact set for every* $(\lambda, y) \in \mathbb{R} \times A$ *and*

$$\partial(-d_{1,y})(\lambda) = [\min_{x \in S_{d_1}(\lambda,y)} g(y, x), \max_{x \in S_{d_1}(\lambda,y)} g(y, x)].$$

Also for every $a_\lambda \in \partial(-d_{1,y})(\lambda)$ *and* $a_\mu \in \partial(-d_{1,y})(\mu)$ *and* $\lambda > \mu$ *it holds that* $a_\lambda \geq a_\mu > 0$.

The next result can be compared with Lemma 8.6.

Lemma 8.18 *Assume Condition 8.2 holds. Then the set* $\{\lambda \in \mathbb{R} : d_1(\lambda, y) = 0\}$ *is nonempty if and only if* $\mu_*(y) > -\infty$. *Moreover, if this set is nonempty, then it only contains the finite value* $\mu_*(y)$.

Up to now we did not assume that there exists some $y \in A$ satisfying $\mu_* = \mu_*(y) > -\infty$ or equivalently the dual max-min fractional program *(D)* has an optimal solution in *B*. In the next lemma the implications of this assumption are discussed. To do so, consider the (possibly empty) set $D_2 \subseteq \mathbb{R}$ given by

$$D_2 := \{\lambda \in \mathbb{R} : d_2(\lambda) = 0 \text{ and } S_{d_2}(\lambda) \text{ is nonempty}\}.$$

The counterpart of Theorem 8.3 is given by the following result.

Theorem 8.13 *Assume Condition 8.2 holds. Then* $\mu_* = \mu_*(y_0) > -\infty$ *for some* $y_0 \in A$ *if and only if* $D_2 = \{\mu_*\}$. *Moreover, if* $\mu_* = \mu_*(y_0) > -\infty$ *for some* $y_0 \in A$, *then the set* $S_{d_2}(\lambda_*)$ *is nonempty and*

$$S_{d_2}(\lambda_*) = \{y \in A : \mu_* = \mu_*(y)\}.$$

If we introduce the (possibly) empty set $D_3 \subseteq \mathbb{R}$ given by

$$D_3 := \{\lambda \in \mathbb{R} : d_2(\lambda) = 0 \text{ and } (D) \text{ has an optimal solution}\},$$

then without Condition 8.2 one can show the following counterpart of Theorem 8.4. Remember a vector (y, x) is an optimal solution of *(D)* if and only if $(y, x) \in A \times B$ and $\mu_* = \mu_*(y) = f(y, x)(g(y, x))^{-1}$.

Theorem 8.14 *The (dual) max-min fractional program (D) has an optimal solution if and only if* $D_3 = \{\mu_*\}$. *Moreover, if (D) has an optimal solution, then the set* $S_d(\mu_*)$ *is nonempty and*

$$S_d(\mu_*) = \{(y, x) \in A \times B : \mu_* = \mu_*(y) = \frac{f(y, x)}{g(y, x)}\}.$$

Finally we will consider in this section another dual max-min fractional program if the nonempty set *B* is given by (see also relation (8.1))

$$B = \{x \in C : h_k(x) \leq 0,\ k = 1, ..., l\}. \tag{8.51}$$

In case the set *B* is specified as in relation (8.51) we always assume for the corresponding primal min-max fractional program *(P)* that the function g is positive on $A \times C$. Introducing now the vector-valued function $h : \mathbb{R}^n \rightarrow \mathbb{R}^l$ given by $h(x)^\top = (h_1(x), ..., h_l(x))$, we consider for every $(y, z) \in A \times \mathbb{R}^l_+$ the single-ratio fractional program

$$\mu_*^p(y, z) := \inf_{x \in C} \frac{f(y, x) + z^\top h(x)}{g(y, x)}. \tag{$D_p^{(y,z)}$}$$

A more complicated optimization problem is now introduced by the so-called partial dual of the (primal) min-max fractional program given by

$$\mu_*^p := \sup_{y \in A, z \geq 0} \inf_{x \in C} \frac{f(y, x) + z^\top h(x)}{g(y, x)}. \tag{D_p}$$

Again this is a max-min fractional program, and using only $g > 0$ on $A \times C$ it is easy to show the following result.

Lemma 8.19 *If* g *is positive on* $A \times C$, *then it follows that* $\mu_*^p \leq \mu_* \leq \lambda_*$.

Proof. Since $B \subseteq C$ and $z^\top h(x) \leq 0$ for every $x \in B$ and $z \geq 0$, we obtain by the positivity of g on $A \times C$ that

$$\mu_*^p(y,z) \leq \inf_{x\in B} \frac{f(y,x) + z^\top h(x)}{g(y,x)} \leq \inf_{x\in B} \frac{f(y,x)}{g(y,x)}$$

for every $z \geq 0$ and $y \in A$. This shows

$$\mu_*^p = \sup_{y\in A,\ z\geq 0} \mu_*^p(y,z) \leq \sup_{y\in A} \inf_{x\in B} \frac{f(y,x)}{g(y,x)} = \mu_*,$$

and so the first inequality is verified. We already showed that $\mu_* \leq \lambda_*$. Hence the proof is complete. □

To verify that $\mu_*^p = \lambda_*$, it is clear from Lemma 8.19 that

$$\lambda_* = -\infty \Rightarrow \lambda_* = \mu_* = \mu_*^p = \mu_*^p(y,z) = -\infty$$

for every $(y,z) \in A \times \mathbb{R}_+^l$. If λ_* is finite and we want to ensure that $\mu_*^p = \lambda_*$, then the following so-called Slater-type condition on the non-empty set B should be considered. Before introducing this condition, we assume throughout the remainder of this section that the (possibly empty) set $I \subseteq \{1,...,l\}$ denotes the set of indices for which $h_k : \mathbb{R}^n \to \mathbb{R}$ is affine. Note that $ri(C)$ denotes the relative interior of the set C (cf. [29,58]).

Condition 8.8 *There exists some $x \in ri(C)$ where C is a closed convex set satisfying $h_k(x) < 0$ for every $k \notin I$ and $h_k(x) \leq 0$ for every $k \in I$. Moreover, for every $k \notin I$ the functions $h_k : \mathbb{R}^n \to \mathbb{R}$ are convex.*

To show under which conditions the equality $\mu_*^p = \lambda_*$ and the finiteness of λ_* holds, we first need to prove the following Lagrangean duality result.

Lemma 8.20 *Assume Condition 8.8 holds and for a given $y \in A$ the function $x \to f(y,x)$ is convex on C and $x \to g(y,x)$ is concave on C. Then it follows for every $\lambda \geq 0$ that there exists some $z_{\lambda,y} \geq 0$ satisfying*

$$\inf_{x\in B} p(\lambda,y,x) = \inf_{x\in C}\{f(y,x) - \lambda g(y,x) + z_{\lambda,y}^\top h(x)\}$$

with B defined in relation (8.51). Moreover, the same result holds for every $\lambda \in \mathbb{R}$ if $x \to f(y,x)$ is convex and $x \to g(y,x)$ is affine.

Proof. Using the definition of the set B and $z \geq 0$, it is easy to see that

$$\inf_{x\in B} p(\lambda,y,x) \geq \inf_{x\in C}\{f(y,x) - \lambda g(y,x) + z^\top h(x)\}.$$

Moreover, for either $\lambda \geq 0$ and $x \to g(y,x)$ concave or $\lambda \in \mathbb{R}$ and $y \to g(y,x)$ affine we see that the function $x \to p(\lambda, y, x)$ is convex on *C*. Applying now Theorem 28.2 of [58] or Theorem 1.25 of [29] we obtain that there exists some dual solution $z_{\lambda,y} \geq 0$ such that the above inequality is actually an equality. □

Using Lemma 8.20 it is now possible to show that the optimal objective function value of the partial dual equals λ_*.

Theorem 8.15 *Assume Conditions 8.7 and 8.8 hold. Then there exists some* $(y_0, z_0) \in A \times \mathbb{R}^l_+$ *satisfying*

$$\lambda_* = \mu_*^p = \mu_*^p(y_0, z_0).$$

Proof. For $\lambda_* = -\infty$ we know by the remark after Lemma 8.19 that the result holds. Hence we only need to verify the result for λ_* finite. To start we observe by relation (8.42) that

$$0 \leq p_2(\lambda_*) = \inf_{x \in B} p(\lambda_*, y_0, x)$$

for some $y_0 \in A$. Applying now Lemma 8.20 one can find some $z_0 \geq 0$ satisfying

$$\inf_{x \in B} p(\lambda_*, y_0, x) = \inf_{x \in C}\{f(y_0, x) - \lambda_* g(y_0, x) + z_0^\top h(x)\}.$$

This shows

$$0 \leq p_2(\lambda_*) = \inf_{x \in C}\{f(y_0, x) - \lambda_* g(y_0, x) + z_0^\top h(x)\}. \tag{8.52}$$

By relation (8.52) and $g(y_0, x) > 0$ for every $x \in C$ we obtain $\mu_*^p(y_0, z_0) \geq \lambda_*$ which completes the proof. □

In case we use the partial dual (D_p) it follows that the partial dual of the single-ratio fractional program

$$\inf_{x \in B} \frac{f(x)}{g(x)}$$

with *B* given by relation (8.51) is given by

$$\sup_{z \geq 0} \inf_{x \in C} \frac{f(x) + z^\top h(x)}{g(x)}.$$

Thus for this (Lagrangean) dual (cf. [66, 68]) the single-ratio fractional program and its dual have a different representation. If Theorem 8.15 holds, one can always apply a Dinkelbach-type algorithm to the partial dual (D_p) to find λ_*. This is discussed in detail in [6] and [9]. In the next subsection we will introduce a similar Dinkelbach-type algorithm applied to the (dual) max-min problem *(D)*.

6.4 The Dual Dinkelbach-Type Algorithm.

In this section we apply the Dinkelbach-type approach to the (dual) max-min fractional program *(D)*. Parallel to subsection 6.2 we assume that the next condition holds. Note that this condition is the counterpart of Condition 8.5 used in the primal Dinkelbach-type algorithm which was applied to the (primal) min-max fractional program *(P)*.

Condition 8.9

- Condition 8.2 holds and $\mu_*(y)$ is finite for every $y \in A$.
- If μ_* is finite, then for every $\lambda \leq \mu_*$ the set $S_{d_2}(\lambda)$ is nonempty while for $\mu_* = -\infty$ the set $S_{d_2}(\lambda)$ is nonempty for every $\lambda \in \mathbb{R}$.

If condition 8.9 holds, then one can execute the following so-called dual Dinkelbach-type algorithm. As for the (primal) Dinkelbach-type algorithm introduced in Section 6.2 one can give a similar geometrical interpretation of the next algorithm.

Dual Dinkelbach-type algorithm.

1 Select $y_0 \in A$ and $k := 1$ and compute

$$\mu_k := \mu_*(y_0).$$

2 Determine $y_k \in S_{d_2}(\lambda_k)$. If $d_1(\mu_k, y_k) \leq 0$ stop and return μ_k and y_k. Otherwise compute

$$\mu_{k+1} := \mu_*(y_k),$$

let $k := k + 1$ and go to 1.

Observe in Step 1 and 2 one has to solve a single-ratio fractional program. If B is a finite set, then solving such a problem is easy. Moreover, by Lemma 8.18 it is sufficient to find in step 2 of the primal Dinkelbach-type algorithm the solution of the equation $d_1(\lambda, y_k) = 0$. As already observed, this yields an easy geometrical interpretation of the above algorithm (see also [5]). The next result shows that the sequence μ_k generated by the dual Dinkelbach-type algorithm is strictly increasing. The proof of this result is similar to the proof of the corresponding result for the primal Dinkelbach-type algorithm in Lemma 8.7. This also shows that the primal Dinkelbach-type algorithm approaches the optimal objective function value from above while the dual Dinkelbach-type algorithm approaches it from below.

Lemma 8.21 *If Condition 8.9 holds, then the sequence μ_k generated by the dual Dinkelbach-type algorithm is strictly increasing and satisfies $\mu_k \leq \mu_* \leq \infty$ for every $k \in \mathbb{N}$.*

By Lemma 8.21 we obtain that the sequence μ_k generated by the dual Dinkelbach-type algorithm converges to some limit $v \leq \infty$. Using a similar proof as in Lemma 8.8 one can show the following result in case the generated sequence is finite. If strong duality holds and so $\mu_* = \lambda_*$, one can also use this algorithm to approximate λ_*.

Lemma 8.22 *If Condition 8.9 holds and the dual Dinkelbach-type algorithm stops at μ_k, then $\mu_* = \mu_k = \mu_{k+1}$ and $d_2(\mu_k) = 0$.*

In the remainder of this subsection we only consider the case where the dual Dinkelbach-type algorithm generates an infinite sequence $\mu_k, k \in \mathbb{N}$. By Lemma 8.21 it follows that $\lim_{k\uparrow\infty} \mu_k = v \leq \infty$ exists. Imposing some additional condition it will be shown in Lemma 8.23 that this limit equals μ_*. To simplify the notation in the following lemmas, we introduce for the sequence $\{(\mu_k, y_k) \in \mathbb{R} \times A : y_k \in S_{d_2}(\mu_k)\}$ generated by the primal Dinkelbach-type algorithm the sequence $\{\underline{a}_k : k \in \mathbb{N}\}$ given by

$$-\underline{a}_k \in \partial(-d_{1,x_k})(\mu_{k+1}) \tag{8.53}$$

and for μ_* finite the sequence $\{\underline{b}_k : k \in \mathbb{N}\}$ given by

$$-\underline{b}_k \in \partial(-d_{1,x_k})(\mu_*). \tag{8.54}$$

By the observation after Lemma 8.16 these subgradient sets are nonempty. Using a similar proof as in Lemma 8.9 it is possible to verify the next result.

Lemma 8.23 *If Condition 8.9 holds and there exists a subsequence $\{\underline{a}_{n_k} : k \in \mathbb{N}\}$ satisfying $\sum_{k=1}^{\infty} \underline{a}_{n_k}^{-1} = -\infty$, then $\lim_{k\uparrow\infty} \mu_k = \mu_*$. Moreover for μ_* finite it follows that $\lim_{k\uparrow\infty} d_2(\mu_k) = 0 \geq d_2(\mu_*)$.*

By relation (8.49) it follows that

$$0 > \underline{a}_k \geq -\overline{g}_{\sup}(y_k) \tag{8.55}$$

for every $-\underline{a}_k \in \partial(-d_{1,y_k})(\mu_{k+1})$. Hence one can apply Lemma 8.23 in case $\sum_{k=1}^{\infty} \overline{g}_{\sup}(y_{n_k})^{-1} = \infty$. To show that $d_2(\mu_*) = 0$, we can follow the proof of Lemma 8.10 and obtain the following result.

Lemma 8.24 *If Condition 8.9 holds, μ_* is finite and there exists a subsequence $\{\underline{b}_{n_k} : k \in \mathbb{N}\}$ satisfying $\inf_{k\in\mathbb{N}} \underline{b}_{n_k} > -\infty$, then $\lim_{k\uparrow\infty} \mu_k = \mu_*$ and $\lim_{k\uparrow\infty} d_2(\mu_k) = 0 = d_2(\mu_*)$.*

By relation (8.55) it follows in case $\sup_{k\in\mathbb{N}} \bar{g}_{\sup}(y_k) < \infty$ that the condition of Lemma 8.24 is satisfied. The next result should be contrasted with Lemma 8.11.

Lemma 8.25 *If Condition 8.9 holds, the functions f and g are finite-valued and continuous on some open set $W \subseteq \mathbb{R}^{m+n}$ containing $A \times B$, the set A is compact and there exists a subsequence $\{\underline{a}_{n_k} : k \in \mathbb{N}\}$ satisfying $\sum_{k=1}^{\infty} \underline{a}_{n_k}^{-1} = -\infty$, then the sequence $\{y_k : y_k \in S_{d_2}(\mu_k)\}_{k\in\mathbb{N}}$ has a converging subsequence and every limit point y_∞ of the sequence $\{y_k : k \in \mathbb{N}\}$ satisfies $\mu_* = \mu_*(y_\infty)$ with μ_*finite. Additionally, if there exist a unique $y_* \in A$ satisfying $\mu_* = \mu_*(y_*)$, then $\lim_{k\uparrow\infty} y_k = y_*$. Moreover, for $A\times B$ compact the generated sequence $\{(y_k, x_k) : (y_k, x_k) \in S_d(\mu_k)\}_{k\in\mathbb{N}}$ has a converging subsequence and every limit point of the sequence$\{(y_k, x_k) : k \in \mathbb{N}\}$ is an optimal solution ofproblem (D). If the optimization problem (D) has a unique optimal solution (y_*, x_*), then $\lim_{k\uparrow\infty} x_k = x_*$ and $\lim_{k\uparrow\infty} y_k = y_*$.*

We now want to investigate how fast the sequence μ_k converges to μ_*. Before discussing this in detail, we list for μ_* finite the following inequality for the sequence $\{\mu_k : k \in \mathbb{N}\}$ generated by the dual Dinkelbach-type algorithm. The proof is similar to the proof of the corresponding result listed in Theorem 8.5 for the primal Dinkelbach-type algorithm.

Theorem 8.16 *If Condition 8.9 holds and there exists some $y \in A$ satisfying $\mu_* = \mu_*(y)$, then it follows for every $-\underline{c}_k \in \partial(-d_{1,y})(\mu_k)$ and $-\underline{a}_k \in \partial(-d_{1,y_k})(\mu_{k+1})$ that*

$$0 \leq \frac{\mu_* - \mu_{k+1}}{\mu_* - \mu_k} \leq (1 - \underline{c}_k \underline{a}_k^{-1}).$$

If a slightly stronger condition as used in Lemma 8.24 holds, then one can show that the sequence $\{\mu_k : k \in \mathbb{N}\}$ generated by the primal Dinkelbach-type algorithm converges Q-linearly. The same result was shown for the dual generalized fractional program in [5] and [8]. The proof of the next result is similar as the proof of the corresponding result for the primal Dinkelbach-type algorithm given in Theorem 8.6

Theorem 8.17 *If Condition 8.9 holds, μ_* is finite and the sequence $\{\underline{b}_k : k \in \mathbb{N}\}$ satisfies $\inf_{k\in\mathbb{N}} \underline{b}_k > -\infty$, then $\lim_{k\uparrow\infty} \mu_k = \mu_*$ and the sequence μ_k converges Q-linearly.*

Finally we show in case the dual (max-min) fractional program *(D)* has a unique optimal solution and some other topological conditions hold that the sequence $\{\mu_k : k \in \mathbb{N}\}$ converges Q-superlinearly. In case also strong duality holds, then we know by the remark after Lemma 8.13 that this unique optimal solution of *(D)* is also an optimal solution of the primal min-max fractional program P assuming this set is nonempty. By the compactness of $A \times B$ in the next result the set of optimal solutions of (P) is nonempty.

Theorem 8.18 *If Condition 8.9 holds, the functions f and g are continuous on some open set W containing the compact set $A \times B$ and the max-min fractional program (D) has a unique optimal solution (y_*, x_*), then $\lim_{k\uparrow\infty} x_k = x_*$, $\lim_{k\uparrow\infty} y_k = y_*$ and $\lim_{k\uparrow\infty} \lambda_k = \lambda_*$ and the sequence μ_k converges Q-superlinearly.*

If strong duality holds, then it is clear that one can also use the dual Dinkelbach-type algorithm to determine the value λ_*. This is the main use of this algorithm in the literature (cf. [8, 9]). Also one could combine the dual and primal approach in case strong duality holds and use simultaneously both. An example of such an approach applied to a generalized fractional program with an easy geometrical interpretation is discussed by Gugat (cf. [39, 41]). In [39] it is shown under slightly stronger conditions that always a Q-superlinear convergence rate holds. This concludes our discussion of the parametric approach used in min-max fractional programming which was a major emphasis in this chapter on fractional programming.

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II

GENERALIZED MONOTONICITY