Chapter 9

THE ANALYTIC HIERARCHY AND ANALYTIC NETWORK PROCESSES FOR THE MEASUREMENT OF INTANGIBLE CRITERIA AND FOR DECISION-MAKING

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Abstract   The Analytic Hierarchy Process (AHP) and its generalization to dependence and feedback, the Analytic Network Process (ANP), are theories of relative measurement of intangible criteria. With this approach to relative measurement, a scale of priorities is derived from pairwise comparison measurements only after the elements to be measured are known. The ability to do pairwise comparisons is our biological heritage and we need it to cope with a world where everything is relative and constantly changing. In traditional measurement one has a scale that one applies to measure any element that comes along that has the property the scale is for, and elements are measured one by one, not by comparing them with each other. In the AHP paired comparisons are made with judgments using numerical values taken from the AHP absolute fundamental scale of 1-9. A scale of relative values is derived from all these paired comparisons and it also belongs to an absolute scale that is invariant under the identity transformation like the system of real numbers. The AHP/ANP is useful for making multicriteria decisions involving benefits, opportunities, costs and risks. The ideas are developed in stages and illustrated with examples of real life decisions. The subject is transparent and despite some mathematics, it is easy to understand why it is done the way it is along the lines discussed here.

1.   Introduction

The purpose of decision-making is to help people make decisions according to their own understanding. They would then feel that they really made the decision themselves justified completely according to their individual or group values, beliefs, and convictions even as one tries to make them understand these better. Because decision-making is the most frequent activity of all people all the time, the techniques used today to help people make better decisions should probably remain closer to the biology and psychology of people than to the techniques conceived and circulated at a certain time and that are likely to become obsolete, as all knowledge does, even though decisions go on and on forever. This suggests that methods offered to help make better decisions should be closer to being descriptive and considerably transparent. They should also be able to capture standards and describe decisions made normatively. Natural science, like decision-making, is mostly descriptive and predictive to help us cope intelligently with a complex world.

Not long ago, people believed that the human mind is an unreliable instrument for performing measurement and that the only meaningful measurement is obtained on a physical scale like the meter and the kilogram invented by clever people who care about precision and objective truth. They did not think how the measurements came to have meaning for people and that this meaning depends on people's purpose each time they obtain a reading on that scale. In the winter, ice may be a source of discomfort but an ice drink in the summer can be a refreshing source of comfort. A number has no meaning except that assigned to it by someone. We may all agree on the numerical value of a reading on a physical scale, but not on what exactly that number means to each of us in practical terms. We tend to parrot abstractions that define a number but often forget that numbers are meant to serve some need that is inevitably subjective, which is ultimately more important for our survival. Thus it is our subjective values that are essential for interpreting the readings obtained through measurement. This interpretation depends on what one has in mind at the time and different people may interpret the same reading differently for the same situation depending on their goal. The reading may be called objective, but the interpretation is predominantly subjective. In this sense subjectivity is important, because without it objectivity has no intrinsic meaning. If the mind of an expert can produce measurement close to what we obtain through measuring instruments, then it has greater power than instruments to deal with a complexity for which we have no way to measure. What we have to do is examine the possibility and validity of this assumption as critically as we can. It turns out that when we have knowledge and experience, our brains are very good measuring instruments. That does not mean that we should discard what we use in science that enhances our

understanding, but rather we should use it to support and strengthen what we do directly with our minds.

The subject of this chapter is the Analytic Hierarchy Process (AHP), the original theory of prioritization that derives relative scales of absolute numbers known as priorities from judgments expressed numerically on an absolute fundamental scale. It is also about a more general approach to decisions that is a generalization of hierarchies to networks with dependence and feedback, the Analytic Network Process (ANP). Both the AHP and ANP are descriptive approaches to decision-making. The AHP/ANP evolved out of my experience at the Arms Control and Disarmament Agency (ACDA) in the Department of State during the Kennedy and Johnson years. ACDA negotiated arms agreements with the Soviets in Geneva. I was invited to join ACDA, I think because of work I had done for the military using Operations Research mathematics. I published on it and wrote the first book on mathematical methods of operations research. At ACDA I supervised a team of foremost internationally known scientists, economists and game theorists (including three people who later won the Nobel Prize in economics: Debreu, Harsanyi and Selten) who advised ACDA on arms tradeoffs, but we had some insurmountable difficulties in making lucid and usable recommendations to our highly intelligent and experienced negotiators who were guided by strong intuition deriving from long practice.

The basic problem is that we need to quantify intangibles of which there is nearly an infinite number and we can only do it by making comparisons in relative terms. Even if everything were measurable, we would still need to compare the different types of measurements on the different scales and determine how important they are to us to make tradeoffs among them and reach a final answer. If we use tangibles and their measurements we would need to reduce them to a common relative frame of reference and then weight and combine them along with intangibles. Combining priorities of measurable quantities with those of non-measurable qualities needs ratio or even the stronger absolute scales, because we can then multiply and add the outcomes particularly when there is interdependence among all the elements involved in a decision.

The AHP is a theory of relative measurement on absolute scales of both tangible and intangible criteria based both on the judgment of knowledgeable and expert people and on existing measurements and statistics needed to make a decision. How to measure intangibles is the main concern of the mathematics of the AHP. The AHP has been mostly applied to multi-objective, multi-criteria and multiparty decisions because decision-making has this diversity. To make tradeoffs among the many intangible objectives and criteria, the judgments that are usually made in qualitative terms are expressed numerically. To do this, rather than simply assign a score out of a person's memory that is hard to justify, one must make reciprocal pairwise comparisons in a carefully designed

scientific way. In the end, we must fit our entire world experience into our system of priorities if we are going to understand it. The AHP is based on four axioms: (1) reciprocal judgments, (2) homogeneous elements, (3) hierarchic or feedback dependent structure, and (4) rank order expectations. The synthesis of the AHP combines multidimensional scales of measurement into a single "unidimensional" scale of priorities. Decisions are determined by a single number for the best outcome or by a vector of priorities that gives a proportionate ordering of the different possible outcomes to which one can then allocate resources in an optimal way subject to both tangible and intangible constraints. We can also combine the judgments obtained from a group when several people are involved in a decision. It is known that with the reciprocal condition, the geometric mean is a necessary condition for combining individual judgments and that, contrary to the impossibility of combining individual judgments into a social welfare function when ordinals are used subject to certain conditions, with absolute judgments it is possible to construct with the AHP such a social welfare function that satisfies these conditions [8].

It is not idiosyncratic for one to believe that making a decision is more complex than just listing all the factors, good and bad, that one can think of and then plunge into numerical manipulations that surface a best outcome according to some plausible way of analysis. Nor is it less idiosyncratic to confine the analysis of decisions to risk and use risk aversion as a way to justify how to make a good choice. For every decision there are positive and negative factors to consider, usually interpreted psychologically in the form of benefits (gains) and opportunities (potential gains), and costs (losses) and risks (potential losses). How to evaluate a decision according to these merits (demerits) and how to combine them into a single overall answer is not easy to do and is something that leaders in business and government do qualitatively with the help of advisors to satisfy the broad goals that they serve. Multicriteria decision-making needs to provide meaningful quantitative assistance on this important, complex, and inevitable concern with its many intangibles.

2.   Pairwise Comparisons; Inconsistency and the Principal Eigenvector

The psychologist Arthur Blumenthal writes in his book *The Process of Cognition,* Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1977, that there are two types of judgment: "Comparative judgment which is the identification of some relation between two stimuli both present to the observer, and absolute judgment which involves the relation between a single stimulus and some information held in short term memory about some former comparison stimuli or about some previously experienced measurement scale using which the observer rates the single stimulus."

Comparative or relative judgment is made on pairs of elements to ensure accuracy. In paired comparisons, the smaller or lesser element is used as the unit, and the larger or greater element is estimated as a multiple of that unit with respect to the common property or criterion for which the comparisons are made. In this sense, measurement with many pairwise comparisons is made more scientifically than by assigning numbers more or less arbitrarily through guessing. What is really the scale to which such numbers belong so they can be operated on arithmetically in a legitimate way? For example, one cannot simply add numbers that belong to an ordinal or an interval scale. Because our brains are limited in size and the firings of their neurons are limited in intensity, it is clear that there is a limit to their ability to compare the very small with the very large. It is precisely for this reason that pairwise comparisons are made on elements or alternatives that are close or homogeneous and the more separated they are, the more need there is to put them in different groups and link these groups with a common element from one group to an adjacent group of slightly greater or slightly smaller elements. One can then compare the elements in each homogeneous group and then combine them through appropriate use of the measurement of the elements (pivots) that are common to consecutive groups.

We learn from making paired comparisons in the AHP that if A is 5 times larger in size than B and B is 3 times larger in size than C, then A is 15 times larger in size than C and thus we say that A dominates C 15 times. That is different from A having 5 dollars more than B and B having 3 dollars more than C implies that A has 8 dollars more than C. Defining intensity along the arcs of a graph and raising the resulting matrix of comparisons to powers measures the first kind of dominance precisely and never the second. It has definite meaning and as we shall see, because of the inconsistency inherent in making judgments , in the limit it is measured uniquely by the principal eigenvector. There is a useful connection between what we do with dominance priorities in the AHP and what is done with transition probabilities both of which use matrix algebra to find their answers. Transitions between states are multiplied and added. To compose the priorities of the alternatives of a decision with respect to different criteria, it is also necessary that the priorities of the alternatives with respect to each criterion be multiplied by the priority of that criterion and then added over all the criteria.

Paired comparisons deal with comparative judgment. However, in conformity with Blumenthal's observation above, the AHP also provides a way to rate alternatives one at a time to deal with absolute judgment. In absolute judgment the criteria are first prioritized through comparisons and then for each criterion one creates a scale of relative intensities possibly of widely ranging orders of magnitude. The priorities of these intensities are again appropriately derived through paired comparisons with respect to their criterion, and in the end the

alternatives are rated one at a time by assigning each one an intensity level for each criterion, then weighting by the priorities of the criteria and adding to obtain their overall rating priority [2]. Thus rating only applies to alternatives taken one at a time and relies on standards (good or poor) in the memory of the decision maker to rate the alternatives. It is useful when the number of alternatives is large and we want to standardize our treatment of them. When alternatives are fundamentally new, different and not fully understood, paired comparisons are essential because there are no familiar and widely accepted standards on which they can be rated.

To derive priorities for criteria or attributes we either think of a need to be satisfied, or of a property of alternatives that we already have. In either case when there are several criteria we need to establish their priorities to select the best alternative that meets all the requirements.

Assume that one is given n stones, $A_1, \cdots, A_n$, with known weights $w_1, \cdots, w_n$, respectively, and suppose that a matrix of pairwise ratios is formed whose rows give the ratios of the weights of each stone with respect to all others. We have:

$$
\begin{array}{c}
\begin{array}{cc} A_1 & \cdots & A_n \end{array} \\
\begin{array}{c} A_1 \\ \vdots \\ A_n \end{array}
\left[
\begin{array}{ccc}
w_1/w_1 & \cdots & w_1/w_n \\
\vdots & & \vdots \\
w_n/w_1 & \cdots & w_n/w_n
\end{array}
\right].
\end{array}
$$

To recover the vector $w = (w_1, ..., w_n)$ we introduce the system of equations:

$$
Aw =
\left[
\begin{array}{ccc}
w_1/w_1 & \cdots & w_1/w_n \\
\vdots & & \vdots \\
w_n/w_1 & \cdots & w_n/w_n
\end{array}
\right]
\left[
\begin{array}{c}
w_1 \\ \vdots \\ w_n
\end{array}
\right]
= n
\left[
\begin{array}{c}
w_1 \\ \vdots \\ w_n
\end{array}
\right]
= nw,
$$

where A has been multiplied on the right by the vector of weights w. The result of this multiplication is nw. To recover the scale from the matrix of ratios, one must solve the problem $Aw = nw$ or $(A - nI)w = 0$. This is a system of homogeneous linear equations. It has a nontrivial solution if and only if the determinant of $A - nI$ vanishes, that is, n is an eigenvalue of A. Now A has unit rank since every row is a constant multiple of the first row. As a result, all its eigenvalues except one are zero. The sum of the eigenvalues of a matrix is equal to its trace, the sum of its diagonal elements, and in this case the trace of A is equal to n. Thus n is an eigenvalue of A, and one has a nontrivial solution. The solution consists of positive entries and is unique to within a multiplicative constant.

To make w unique, we can normalize its entries by dividing by their sum. Thus, given the comparison matrix, we can recover the scale. In this case,

the solution is any column of A normalized. Notice that in A the reciprocal property $a_{ji} = 1/a_{ij}$ holds; thus, also $a_{ii} = 1$. Another property of A is that it is consistent: its entries satisfy the condition $a_{jk} = a_{ik}/a_{ij}$. The entire matrix can be constructed from a set of n elements that form a chain across the rows and columns of A.

In the general case, the precise value of w_i/w_j cannot be given, but instead only an estimate of it as a judgment. For the moment, consider an estimate of these values by an expert whose judgments are small perturbations of the coefficients w_i/w_j. This implies small perturbations of the eigenvalues.

Let us for generality call $A_1, \ldots, A_n$ stimuli instead of stones. The quantified judgments on pairs of stimuli A_i, A_j, are represented by an n-by-n matrix $A' = (a_{ij})$, $ij = 1, 2, \ldots, n$. The entries a_{ij} are defined by the following entry rules.

Rule 1. If $a_{ij} = a,$ then $a_{ji} = 1/a, a > 0.$

Rule 2. If A_i is judged to be of equal relative intensity to A_j then $a_{ij} = 1$, $a_{ji} = 1$; in particular, $a_{ii} = 1$ for all i.

Thus the matrix A' has the form:

$$A' = \begin{pmatrix} 1 & a_{12} & \cdots & a_{1n} \\ 1/a_{12} & 1 & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 1/a_{1n} & 1/a_{2n} & \cdots & 1 \end{pmatrix}.$$

Having recorded the quantified judgments on pairs of stimuli (A_i, A_j) as numerical entries a_{ij} in the matrix , the problem now is to assign to the n stimuli $A_1, A_2, \ldots, A_n$ a set of numerical weights that would "reflect the recorded judgments." In order to do that, the vaguely formulated problem must first be transformed into a precise mathematical one. This essential, and apparently harmless, step is the most crucial one in any problem that requires the representation of a real life situation in terms of an abstract mathematical structure. It is particularly crucial in the present problem where the representation involves a number of transitions that are not immediately discernible. It appears, therefore, desirable in the present problem to identify the major steps in the process of representation and to make each step as explicit as possible to enable the potential user to form his own judgment as to the *meaning and value* of the method in relation to *his* problem and *his* goal.

Why we must solve the principal eigenvalue problem in general has a simple justification based on the idea of dominance among the elements represented by the coefficients of the matrix. Dominance between two elements is obtained as the normalized sum of path intensities defined by the numerical judgments assigned to the arcs along a path. The overall dominance of an element is the

sum of the entries in its row given by Ae, $e = (1, \ldots, 1)$ when A is consistent because then $A^k = n^{k-1}A$. When A is inconsistent, we must consider paths of dominance of all lengths between the two points. All the paths of a given length k are obtained by raising the matrix to the power k. According to Cesaro summability, the limit of the average or Cesaro sum $\lim_{N \to \infty} 1/N \sum_{k=0}^{N} A^k e$ that represents the average of all order dominance vectors up to N, is the same as the limit of the sequence of the powers of the matrix i.e. $(\lim_{k \to \infty} A^k)e$. Now we know from Perron theory that the sequence A^k converges to a matrix all whose columns are identical and are proportional to the principal right eigenvector of A. Thus $(\lim_{k \to \infty} A^k)e$ is also proportional to the principal right eigenvector of A.

Without the theory of Perron, the proof (not given here but known in eigenvalue theory) of how to go from $Aw = nw$ to $Aw = \lambda_{\max}w$, is related to small perturbation theory and the amount of inconsistency one allows. A modicum of inconsistency is necessary to change our mind about old relations when we learn new things.

Another way to prove the necessity of the principal eigenvector is based on the need for the invariance of priorities. No matter what method we use to derive the weights , by using them to weight and add the entries in each row to determine the dominance of the element represented in that row, we must get these priorities back as proportional to the expression $\sum_{j=1}^{n} a_{ij}w_j$ $i = 1, 2 \cdots n$, that is, we must solve $\sum_{j=1}^{n} a_{ij}w_j = cw_i, i = 1, 2 \cdots n$, because in the end they can be normalized. Otherwise $\sum_{j=1}^{n} a_{ij}w_j, i = 1, \ldots, n$ would yield another set of different weights and they in turn can be used to form new expressions $\sum_{j=1}^{n} a_{ij}w_j, i = 1, 2 \ldots n$, and so on ad infinitum violating the need to have priorities that are invariant, unless in any case we solve the principal eigenvalue problem.

Our general problem takes the form:

$$A'w = \begin{bmatrix} 1 & a_{12} & \cdots & a_{1n} \\ 1/a_{12} & 1 & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 1/a_{1n} & 1/a_{2n} & \cdots & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = cw.$$

We now show that the perturbed eigenvalue from the consistent case is the principal eigenvalue of A'. Our argument involves both left and right eigenvectors of A'. Two vectors $x = (x_1, \cdots, x_n)$, $y = (y_1, \cdots, y_n)$ are orthogonal if their scalar product $x_1y_1 + \cdots x_ny_n$ is equal to zero. It is known that any left eigenvector of a matrix corresponding to an eigenvalue is orthogonal to any right eigenvector corresponding to a different eigenvalue. This property is known as bi-orthogonality using which we can prove:

THEOREM 1 *For a given positive matrix A, the only positive vector w and only positive constant c that satisfy $Aw = cw$, is a vector w that is a positive multiple of the principal eigenvector of A, and the only such c is the principal eigenvalue of A.*

Thus we see that both requirements of dominance and invariance lead us to the principal right eigenvector. The problem now is how good is the estimate of w. Notice that if w is obtained by solving this problem, the matrix whose entries are w_i/w_j is a consistent matrix. It is a consistent estimate of the matrix A'. The matrix itself need not be consistent. In fact, the entries of A' need not even be transitive; that is, A_1 may be preferred to A_2 and A_2 to A_3 but A_3 may be preferred to A_1. What we would like is a measure of the error due to inconsistency. It turns out that A' is consistent if and only if $\lambda_{\max} = n$ and that we always have $\lambda_{\max} \geq n$ when we solve the system of equations $Aw = \lambda_{\max}w$ for a non-negative reciprocal matrix A to obtain the priorities.

Thus the story is very different if the judgments are inconsistent, and as we said before, we need to allow inconsistent judgments for good reasons. In sports, team A beats team B, team B beats team C, but team C beats team A. How would we admit such an occurrence in our attempt to explain the real world if we do not allow inconsistency? So far we have legislated inconsistency, which is natural in making judgments, by assuming axiomatically that it should not exist particularly with regard to transitivity!

The priorities that we seek are concerned with the order to be captured from dominance judgments involving all order transitivity. Thus the problem of deriving unique priorities in decision-making by solving the principal eigenvalue problem of A' belongs to the field of mathematics known as *order topology*. In general priorities are not obtainable directly by the many methods of *metric topology* involving minimization of a metric such as the method of least squares (LSM) which determines a priority vector by minimizing the Frobenius norm of the difference between A and a positive rank one reciprocal matrix $[y_i/y_j]$:

$$\min_{y>0} \sum_{i,j=1}^{n} \left(a_{ij} - \frac{y_i}{y_j} \right)^2 \tag{9.1}$$

and the method of logarithmic least squares (LLSM) which determines a vector by minimizing the Frobenius norm of $[\log(a_{ij}x_j/x_i)]$:

$$\min_{x>0} \sum_{i,j=1}^{n} \left[\log a_{ij} - \log \left(\frac{x_i}{x_j} \right) \right]^2 . \tag{9.2}$$

Metric methods not only ignore transitivity, but also yield a variety of different answers thus violating the overall justification of the need for a single unique set of priorities. There is however a connection between order and optimization.

Solving the principal eigenvalue problem to obtain priorities is equivalent to the two problems of optimization that follow: Find w_i, $i = 1, \cdots, n$ which

1 maximize $\frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} w_j / w_i$, or, in the simpler linear optimization setting,

2 maximize $\sum_{j=1}^{n} w_j \sum_{i=1}^{n} a_{ij}$, obtained by multiplying the sum of each column j by its corresponding w_j and summing over j, subject to $\sum_{i=1}^{n} w_i = 1$.

3.   Stimulus Response and the Fundamental Scale

What numbers should we use when we only have qualitative judgments to express our understanding in making pairwise comparisons of elements that are close or homogeneous? We note that to be able to perceive and sense objects in the environment our brains miniaturize them within our system of neurons so that we have a proportional relationship between what we perceive and what is out there. Without proportionality we cannot coordinate our thinking with our actions with the accuracy needed to control the environment. Proportionality with respect to a single stimulus requires that our response to a proportionately amplified or attenuated stimulus we receive from a source should be proportional to what our response would be to the original value of that stimulus. If $w(s)$ is our response to a stimulus of magnitude s, then the foregoing gives rise to the functional equation $w(as) = b\,w(s)$. This equation can also be obtained as the necessary condition for solving the Fredholm equation of the second kind:

$$\int_a^b K(s,t)w(t)dt = \lambda_{max} w(s)$$

obtained as the continuous generalization of the discrete formulation $Aw = \lambda_{max} w(s)$. The solution of this functional equation in the real domain is given by

$$w(S) = C e^{\log b \frac{\log s}{\log a}} P\left(\frac{\log s}{\log a}\right),$$

where *P* is a periodic function of period 1 and $P(0) = 1$. One of the simplest such examples with $u = \log a / \log a$ is $P(u) = \cos(u/2\pi)$ for which $P(0) = 1$ and from which the logarithmic law of response to stimuli can be obtained as a first order approximation as:

$$v(u) = C_1 e^{-\beta u} P(u) \approx C_2 \log s + C_3$$

$\log ab = -\beta, \beta > 0$. The expression on the right is the well-known Weber-Fechner law of logarithmic response $M = a \log s + b$, $a \neq 0$ to a stimulus of magnitude s. It belongs to an interval scale. The larger the stimulus, the larger

a change in it is needed for that change to be detectable. The ratio of successive just noticeable differences (the well-known "jnd" in psychology) is equal to the ratio of their corresponding successive stimuli values. Proportionality is maintained. Thus, starting with a stimulus s_0 successive magnitudes of the new stimuli take the form:

$$
\begin{aligned}
s_1 &= s_0 + \Delta s_0 &= s_0 + \frac{\Delta s_0}{s_0} &= s_0(1 + r) \\
s_2 &= s_1 + \Delta s_1 &= s_0(1 + r)^2 &\equiv s_0\alpha^2 \\
&\vdots \\
s_n &= s_{n-1}\alpha &= s_0\alpha^n (n = 0, 1, 2, \ldots).
\end{aligned}
$$

We consider the responses to these stimuli to be measured on a ratio scale ($b = 0$). A typical response has the form $M_i = a \log \alpha^i$, $i = 1, \cdots, n$, or one after another they have the form:

$$M_1 = a \log \alpha, M_2 = 2a \log \alpha, \cdots M_n = na \log \alpha.$$

We take the ratios M_i/M_1, $i = 1, \cdots, n$ of these responses in which the first is the smallest and serves as the unit of comparison, thus obtaining the integer values $1, 2, \cdots, n$ of the fundamental scale of the AHP.

A person may not be schooled in the use of numbers but still have feelings, judgment and understanding that enable him or her to make accurate comparisons (equal, moderate, strong, very strong and extreme and compromises between these intensities). Such judgments can be applied successfully to compare stimuli that are not too disparate but homogeneous in magnitude. By homogeneous we mean that they fall within specified bounds. The foregoing may be summarized to represent the fundamental scale for paired comparisons shown in Table 9.1.

We know now that a judgment or comparison is the numerical representation of a relationship between two elements that share a common parent. We also know that the set of all such judgments can be represented in a square matrix in which the set of elements is compared with itself. Each judgment represents the dominance of an element in the column on the left over an element in the row on top. It reflects the answers to two questions: which of the two elements is more important with respect to a higher level criterion, and how strongly, using the 1-9 scale shown in Table 9.1 for the element on the left over the element at the top of the matrix. If the element on the left is less important than that on the top of the matrix, we enter the reciprocal value in the corresponding position in the matrix. It is important to note that the lesser element is always used as the unit and the greater one is estimated as a multiple of that unit. From all the paired comparisons we calculate the priorities and exhibit them on the right of the matrix. For a set of n elements in a matrix one needs $n(n - 1)/2$

Table 9.1. The fundamental scale of absolute numbers.

Intensity of Importance	Definition	Explanation
1	Equal importance	Two activities contribute equally to the objective
2	Weak	
3	Moderate importance	Experience and judgment slightly favor one activity over another
4	Moderate plus	
5	Strong importance	Experience and judgment strongly favor one activity over another
6	Strong plus	
7	Very strong or demonstrated importance	An activity is favored very strongly over another; its dominance demonstrated in practice
8	Very, very strong	
9	Extreme importance	The evidence favoring one activity over another is of the highest possible order of affirmation
Reciprocals of above	If activity i has one of the above nonzero numbers assigned to it when compared with activity j, then j has the reciprocal value when compared with i	A reasonable assumption
Rationals	Ratios arising from the scale	If consistency were to be forced by obtaining n numerical values to span the matrix

comparisons because there are n 1's on the diagonal for comparing elements with themselves and of the remaining judgments, half are reciprocals. Thus we have $(n^2-n)/2$ judgments. In some problems one may elicit only the minimum of $n-1$ judgments.

In a judgment matrix A, instead of assigning two numbers w_i and w_j (that generally we do not know), as one does with tangibles, and forming the ratio w_i/w_j we assign a single number drawn from the fundamental scale of absolute numbers shown in Table 9.1 to represent the ratio $(w_i/w_j)/1$. It is a nearest integer approximation to the ratio $(w_i/w_j)/1$. The ratio of two numbers from a ratio scale (invariant under multiplication by a positive constant) is an absolute

number (invariant under the identity transformation). The derived scale will reveal what w_i and w_i are.

This is a central fact about the relative measurement approach. It needs a fundamental scale to express numerically the relative dominance relationship.

If one wishes to use actual measurements or use fractional values for judgments one of course can. In the end one needs to justify with care what one does.

REMARK 32 *The reciprocal property plays an important role in combining the judgments of several individuals to obtain a judgment for a group. Judgments must be combined so that the reciprocal of the synthesized judgments must be equal to the syntheses of the reciprocals of these judgments. It has been proved that the geometric mean is the unique way to do that. If the individuals are experts, they my not wish to combine their judgments but only their final outcome from a hierarchy. In that case one takes the geometric mean of the final outcomes. If the individuals have different priorities of importance their judgments (final outcomes) are raised to the power of their priorities and then the geometric mean is formed [2].*

3.1   Validation Example

Here is an example (one of many) which shows that the scale works well on homogeneous elements of a real life problem. A matrix of paired comparison judgments is used to estimate relative drink consumption in the United States as shown in Table 9.2. To make the comparisons, the types of drinks are listed on the left and at the top, and judgment is made as to how strongly the consumption of a drink on the left dominates that of a drink at the top. For example, when coffee on the left is compared with wine at the top, it is thought that it is consumed extremely more and a 9 is entered in the first row and second column position. A 1/9 is automatically entered in the second row and first column position. If the consumption of a drink on the left does not dominate that of a drink at the top, the reciprocal value is entered. For example in comparing coffee and water in the first row and eighth column position, water is consumed more than coffee slightly and a 1/2 is entered. Correspondingly, a value of 2 is entered in the eighth row and first column position. At the bottom of Table 9.2, we see that the derived values and the actual values are close.

3.2   Clustering and Homogeneity; Using Pivots to Extend the Scale from 1-9 to 1-∞

Most real life decisions are not widely separated in ranges of criteria (one or two) because what is important to individuals or to groups to corporations and finally to governments needs to meet their most essential requirements. Note that the priorities in two adjacent categories would be sufficiently different, one

being an order of magnitude smaller than the other, that in the synthesis, the priorities of the elements in the smaller set would ordinarily have little effect on the decision.

We note that our ability to make accurate comparisons of widely disparate objects on a common property is limited. We cannot compare with any reliability the very small with the very large. However, we can do it in stages by comparing objects of relatively close magnitudes and gradually increase their sizes until we reach the desired object of large size (see example later). In this process, we can think of comparing several close or homogeneous objects for which we obtain a scale of relative values, and then again pairwise compare the next set of larger objects that includes for example the largest object from the previous already compared collection, and then derive a scale for this second set. We then divide all the measurements in the second set by the value of the common object and multiply all the resulting values by the weight of the common element in the first set, thus rendering the two sets to be measurable on the same scale and so on to a third collection of the objects using a common object from the second set.

Table 9.2. Which drink is consumed more in the U.S.? An example of estimation using judgments.

Drink Consumption in th U.S.	Coffee	Wine	Tea	Beer	Sodas	Milk	Water
Coffee	1	9	5	2	1	1	1/2
Wine	1/9	1	1/3	1/9	1/9	1/9	1/9
Tea	1/5	2	1	1/3	1/4	1/3	1/9
Beer	1/2	9	3	1	1/2	1	1/3
Sodas	1	9	4	2	1	2	1/2
Milk	1	9	3	1	1/2	1	1/3
Water	2	9	9	3	2	3	1

	The derived scale based on the judgments in the matrix is:						
	Coffee	Wine	Tea	Beer	Sodas	Milk	Water
	.177	.019	.042	.116	.190	.129	.327

with a consistency index of 0.022.
The actual consumption (from statistical sources) is:

	Coffee	Wine	Tea	Beer	Sodas	Milk	Water
	.180	.010	.040	.120	.180	.140	.330

In Figure 9.1 a cherry tomato is eventually and indirectly compared with a large watermelon by first comparing it with a small tomato and a lime, the lime is then used again in a second cluster with a grapefruit and a honey dew where we then divide by the weight of the lime and then multiply by its weight in the first cluster, and then use the honey dew again in a third cluster and so on. In

the end we have a comparison of the cherry tomato with the large watermelon and would accordingly extended the scale from 1-9 to 1-721.
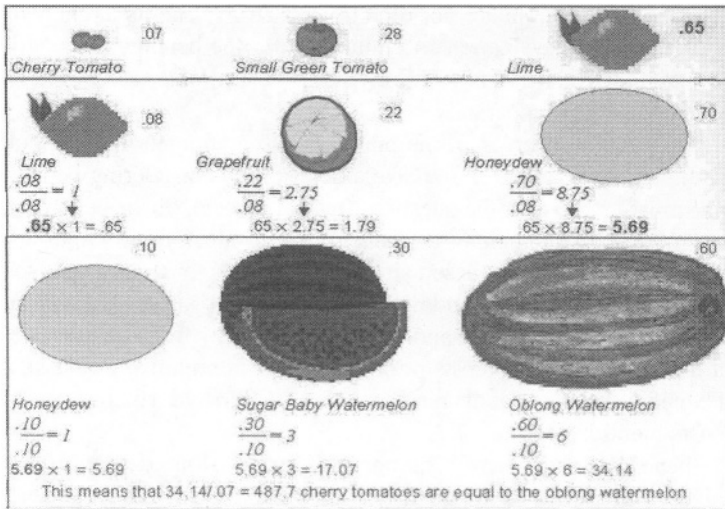
Figure 9.1. Comparisons according to volume.

4. Hospice Decision

Westmoreland County Hospital in Western Pennsylvania, like hospitals in many other counties around the United States, has been concerned with the costs of the facilities and manpower involved in taking care of terminally ill patients. Normally these patients do not need as much medical attention as do other patients. Those who best utilize the limited resources in a hospital are patients who require the medical attention of its specialists and advanced technology equipment, whose utilization depends on the demand of patients admitted into the hospital. The terminally ill need medical attention only episodically. Most of the time, such patients need psychological support. Such support is best given by the patient's family, whose members are able to supply the love and care the patients most need. For the mental health of the patient, home therapy is a benefit. From the medical standpoint, especially during a crisis, the hospital provides a greater benefit. Most patients need the help of medical professionals only during a crisis. Some will also need equipment and surgery. The planning association of the hospital wanted to develop alternatives and to choose the best

one considering various criteria from the standpoint of the patient, the hospital, the community, and society at large.

In this problem, we need to consider the costs and benefits of the decision. Costs include economic costs and all sorts of intangibles, such as inconvenience and pain. Such disbenefits are not directly related to benefits as their mathematical inverses, because patients infinitely prefer the benefits of good health to these intangible disbenefits. To study the problem, one needs to deal with benefits and with costs separately.

I met with representatives of the planning association for several hours to decide on the best alternative. To make a decision by considering benefits and costs, one must first answer the question: In this problem, do the benefits justify the costs? If they do, then either the benefits are so much more important than the costs that the decision is based simply on benefits, or the two are so close in value that both the benefits and the costs should be considered. Then we use two hierarchies for the purpose and make the choice by forming the ratio from them of the benefits priority/costs priority for each alternative. One asks which is most beneficial in the benefits hierarchy (Figure 9.2) and which is most costly in the costs hierarchy (Figure 9.3).

If the benefits do not justify the costs, the costs alone determine the best alternative, which is the least costly. In this example, we decided that both benefits and costs had to be considered in separate hierarchies. In a risk problem, a third hierarchy is used to determine the most desired alternative with respect to all three: benefits, costs, and risks. In this problem, we assumed risk to be the same for all contingencies.

The planning association thought the concepts of benefits and costs were too general to enable it to make a decision. Thus, the planners and I further subdivided each (benefits and costs) into detailed subcriteria to enable the group to develop alternatives and to evaluate the finer distinctions the members perceived between the three alternatives. The alternatives were to care for terminally ill patients at the hospital, at home, or partly at the hospital and partly at home.

The two hierarchies are fairly clear and straightforward in their description. They descend from the more general criteria in the second level to secondary subcriteria in the third level and then to tertiary subcriteria in the fourth level on to the alternatives at the bottom or fifth level. At the general criteria level, each of the hierarchies, benefits or costs, involved three major interests. The decision should benefit the recipient, the institution, and society, and their relative importance is the prime determinant as to which outcome is more likely to be preferred. We located these three elements on the second level of the benefits hierarchy. As the decision would benefit each party differently and the importance of the benefits to each recipient affects the outcome, the group thought that it was important to specify the types of benefit for the recipient and the institution. Recipients want physical, psycho-social and economic benefits, while the

Goal — **Choosing Best Hospice Benefits Hierarchy**

General criteria:

| Recipient benefits 0.64 | Institutional benefits 0.26 | Societal benefits 0.10 |

Secondary subcriteria:

| Physical 0.16 | Psycho-social 0.44 | Economic 0.04 | Psycho-social 0.23 | Economic 0.03 |

Tertiary subcriteria:

Physical 0.16
- direct care of patients 0.02
- palliative care 0.14

Psycho-social 0.44
- volunteer support 0.02
- networkig in families 0.06
- relief of post-death distress 0.12
- emotional support to family and patient 0.21
- alleviation of guilt 0.03

Economic 0.04
- reduced costs 0.01
- improved productivity 0.03

Psycho-social 0.23
- publicity and public relations 0.19
- volunteer recruitment 0.03
- professional recruitment and support 0.06

Economic 0.03
- reduced length of stay 0.006
- better utilization of resources 0.023
- professional recruitment and support 0.06

Societal benefits 0.10
- death as a social issue 0.02
- rehumanization of medical, professional and health institutions 0.08

—(each alternative model below is connected to every tertiary subcriterion)—

Alternatives:

Model 1
0.43
Unit of beds with team giving home care (as in hospital or nursing home)

Model 2
0.12
Mixed bed, contractual home care (partly in hospital for emergency care and partly in home when better -no nurses going to the house)

Model 3
0.45
Hospital and home care share case m anagement (with visiting nurses to the home; if extremely sick patient goes to the hospital)
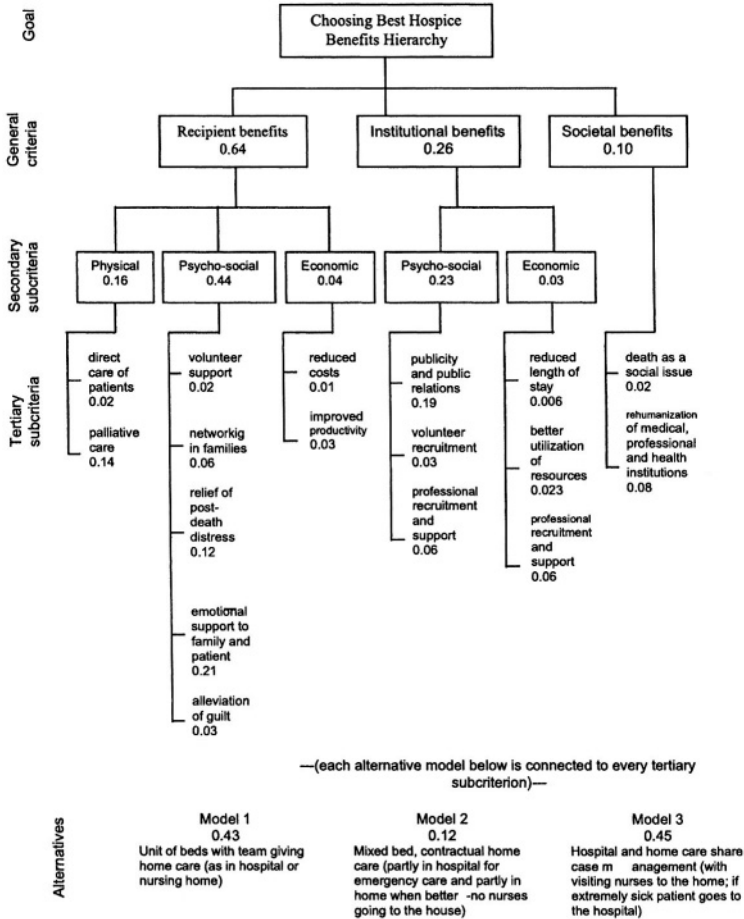
Figure 9.2. To choose the best hospice plan, one constructs a hierarchy modeling the benefits to the patient, to the institution, and to society. This is the benefits hierarchy of two separate hierarchies.
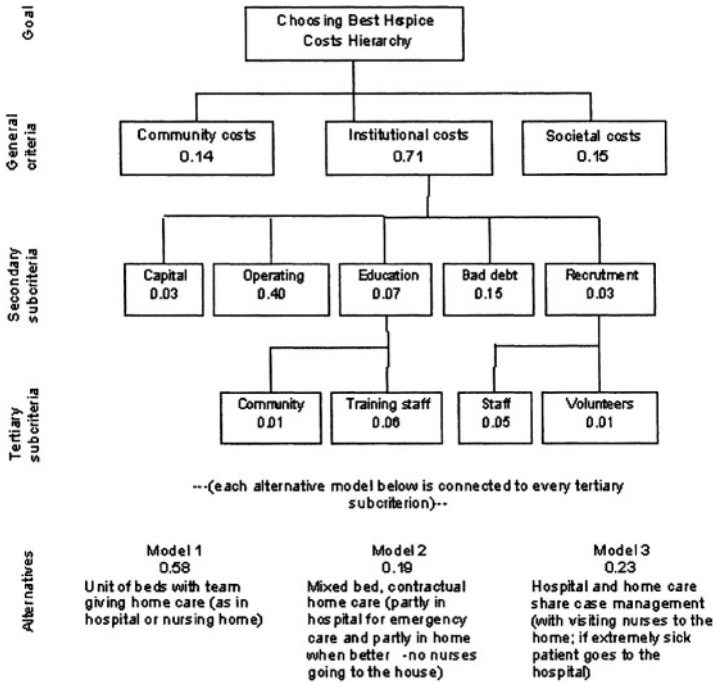
Figure 9.3. To choose the best hospice plan, one constructs a hierarchy modeling the community, institutional, and societal costs. This is the costs hierarchy of two separate hierarchies.

institution wants only psycho-social and economic benefits. We located these benefits in the third level of the hierarchy. Each of these in turn needed further decomposition into specific items in terms of which the alternatives could be evaluated. For example, while the recipient measures economic benefits in terms of reduced costs and improved productivity, the institution needed the more specific measurements of reduced length of stay, better utilization of resources, and increased financial support from the community. There was no reason to decompose the societal benefits into a third level subcriteria, hence societal benefits connects directly to the fourth level. The group considered three models for the alternatives, and they are at the bottom (or fifth level in this case) of the hierarchy: in Model 1, the hospital provided full care to the patients; in Model 2, the family cares for the patient at home, and the hospital provides only emergency treatment (no nurses go to the house); and in Model 3, the hospital and the home share patient care (with visiting nurses going to the home).

In the costs hierarchy there were also three major interests in the second level that would incur costs or pains: community, institution, and society. In this decision the costs incurred by the patient were not included as a separate factor. Patient and family could be thought of as part of the community. We thought decomposition was necessary only for institutional costs. We included five such costs in the third level: capital costs, operating costs, education costs, bad debt costs, and recruitment costs. Educational costs apply to educating the community and training the staff. Recruitment costs apply to staff and volunteers. Since both the costs hierarchy and the benefits hierarchy concern the same decision, they both have the same alternatives in their bottom levels, even though the costs hierarchy has fewer levels.

As usual with the AHP, in both the costs and the benefits models, we compared the criteria and subcriteria according to their relative importance with respect to the parent element in the adjacent upper level. For example, in the first matrix of comparisons of the three benefits criteria with respect to the goal of choosing the best hospice alternative, recipient benefits are moderately more important than institutional benefits and are assigned the absolute number 3 in the (1, 2) or first-row second-column position. Three signifies three times more. The reciprocal value is automatically entered in the (2, 1) position, where institutional benefits on the left are compared with recipient benefits at the top. Similarly a 5, corresponding to strong dominance or importance, is assigned to recipient benefits over social benefits in the (1, 3) position, and a 3, corresponding to moderate dominance, is assigned to institutional benefits over social benefits in the (2, 3) position with corresponding reciprocals in the transpose positions of the matrix.

REMARK 33 *In order to give the reader familiarity with the AHP without too much theory, we have delayed discussion of the measurement of the inconsistency and random inconsistency and of the ratio C.R. of the inconsistency of a given matrix and the corresponding random inconsistency to a later section. However, we have indicated the C.R. corresponding to each matrix immediately under that matrix.*

Judgments in a matrix may not be consistent. In eliciting judgments, one makes redundant comparisons to improve the validity of the answer, given that respondents may be uncertain or may make poor judgments in comparing some of the elements. Redundancy gives rise to multiple comparisons of an element with other elements and hence to numerical inconsistencies. For example, where we compare recipient benefits with institutional benefits and with societal benefits, we have the respective judgments 3 and 5. Now if $x = 3y$ and $x = 5z$ then $3y = 5z$ or $y = 5/3z$. If the judges were consistent, institutional benefits would be assigned the value 5/3 instead of the 3 given in the matrix. Thus the judgments are inconsistent. In fact, we are not sure which judgments are the

Table 9.3.   The entries in this matrix respond to the question: Which criterion is more important
with respect to choosing the best hospice alternative and how strongly?

Choosing best hospice	Recipient benefits	Institutional benefits	Social benefits	Priorities
Recipient benefits	1	3	5	.64
Institutional benefits	1/3	1	3	.26
Social benefits	1/5	1/3	1	.11
		C.R. = .033		

accurate ones and which are the cause of the inconsistency. Inconsistency is
inherent in the judgment process. Inconsistency may be considered a tolerable
error in measurement only when it is of a lower order of magnitude (10 %) than
the actual measurement itself; otherwise the inconsistency would bias the result
by a sizable error comparable to or exceeding the actual measurement itself.

When the judgments are inconsistent, the decision-maker may not know
where the greatest inconsistency is. The AHP can show one by one in sequential
order which judgments are the most inconsistent, and suggests the value that best
improves consistency. However, this recommendation may not necessarily lead
to a more accurate set of priorities that correspond to some underlying preference
of the decision-maker. Greater consistency does not imply greater accuracy and
one should go about improving consistency (if one can, given the available
knowledge) by making slight changes compatible with one's understanding. If
one cannot reach an acceptable level of consistency, one should gather more
information or reexamine the framework of the hierarchy. For a 3-by-3 matrix
this ratio should be about 5 %, for a 4-by-4 matrix about 8 %, and for larger
matrices, about 10 %.

The process is repeated in all the matrices by asking the appropriate domi-
nance or importance question. For example, for the matrix comparing the sub-
criteria of the parent criterion institutional benefits (Table 9.4), psycho-social
benefits are regarded as very strongly more important than economic benefits,
and 7 is entered in the (1, 2) position and 1/7 in the (2, 1) position.

In comparing the three models for patient care, we asked members of the
planning association which model they preferred with respect to each of the
covering or parent secondary criteria in level 3 or with respect to the tertiary
criteria in level 4. For example, for the subcriterion direct care (located on
the left-most branch in the benefits hierarchy), we obtained a matrix of paired
comparisons (Table 9.5) in which Model 1 is preferred over Models 2 and 3 by

Table 9.4. The entries in this matrix respond to the question: Which subcriterion yields the greater benefit with respect to institutional benefits and how strongly?

Institutional benefits	Psycho-social	Economic	Priorities
Psycho-social	1	7	.875
Economic	1/7	1	.125
	C.R. = .000		

5 and 3 respectively and Model 3 is preferred by 3 over Model 2. The group first made all the comparisons using semantic terms for the fundamental scale and then translated them to the corresponding numbers.

Table 9.5. The entries in this matrix respond to the question: Which model yields the greater benefit with respect to direct care and how strongly?

Direct care of patient	Model I	Model II	Model III	Priorities
Model I unit team	1	5	3	.64
Model II mixed/home care	1/5	1	1/3	.10
Model III case management	1/3	3	1	.26
	C.R. = .033			

For the costs hierarchy, I again illustrate with three matrices. First the group compared the three major cost criteria and provided judgments in response to the question: which criterion is a more important determinant of the cost of a hospice model (Table 9.6)?

Table 9.6. The entries in this matrix respond to the question: Which criterion is a greater determinant of cost with respect to the care method and how strongly?

Choosing best hospice (costs)	Community	Institutional	Societal	Priorites
Community costs	1	1/5	1	.14
Institutional costs	5	1	5	.71
Societal costs	1	1/5	1	.14
	C.R. = .000			

The group then compared the subcriteria under institutional costs and obtained the importance matrix shown in Table 9.7.

Table 9.7.   The entries in this matrix respond to the question: Which criterion incurs greater institutional costs and how strongly?

Institutional costs	Capital	Operating	Education	Bad debt	Recruit- ment	Priorities
Capital	1	1/7	1/4	1/7	1	.05
Operating	7	1	9	4	5	.57
Education	4	1/9	1	1/2	1	.01
Bad debt	7	1/4	2	1	3	.21
Recruitment	1	1/5	1	1/3	1	.07

C.R. = .08

Finally, we compared the three models to find out which incurs the highest cost for each criterion or subcriterion. Table 9.8 shows the results of comparing them with respect to the costs of recruiting staff.

Table 9.8.   The entries in this matrix respond to the question: Which model incurs greater cost with respect to institutional costs for recruiting staff and how strongly?

Institutional costs for recruiting staff	Model I	Model II	Model III	Priorities
Model I unit team	1	5	3	.64
Model II mixed/home care	1/5	1	1/3	.10
Model III case management	1/3	3	1	.26

C.R. = .000

As shown in Table 9.9 we divided the benefits priorities by the costs priorities for each alternative to obtain the best alternative, Model 3, that with the largest value for the ratio.

Table 9.9 shows two ways or modes of synthesizing the local priorities of the alternatives using the global priorities of their parent criteria: The distributive mode and the ideal mode. In the distributive mode, the weights of the alternatives sum to one. It is used when there is dependence among the alternatives and a unit priority is distributed among them. The ideal mode is used to obtain the single best alternative regardless of what other alternatives there are. In the ideal mode, the local priorities of the alternatives under each criterion are divided by the

largest value among them. This is done for each criterion; for each criterion one alternative becomes an ideal with value one. In both modes, the local priorities are weighted by the global priorities of the parent criteria and synthesized and the benefit-to-cost ratios formed. In Table 9.9 we rounded off the numbers to two decimal places. Unfortunately, that causes substantial difference from the actual results obtained in the AHP calculations. We request that the reader accept this as an illustration.

Table 9.9. Synthesis (P=Priorities, M=Model).

Benefits	P	Distributive Mode			Ideal Mode		
		M 1	M 2	M 3	M 1	M 2	M 3
Direct Care of Patient	.02	0.64	0.10	0.26	1.00	0.16	0.41
Palliative Care	.14	0.64	0.10	0.26	1.00	0.16	0.41
Volunteer Support	.02	0.09	0.17	0.74	0.12	0.23	1.00
Networking in Families	.06	0.46	0.22	0.32	1.00	0.48	0.70
Relief of Post Death Stress	.12	0.30	0.08	0.62	0.48	0.13	1.00
Emotional Support of Family and Patient	.21	0.30	0.08	0.62	0.48	0.13	1.00
Alleviation of Guilt	.03	0.30	0.08	0.62	0.48	0.13	1.00
Reduced Economic Costs for Patient	.01	0.12	0.65	0.23	0.18	1.00	0.35
Improved Productivity	.03	0.12	0.27	0.61	0.20	0.44	1.00
Publicity and Public Relations	.19	0.63	0.08	0.29	1.00	0.13	0.46
Volunteer Recruitment	.03	0.64	0.10	0.26	1.00	0.16	0.41
Professional Recruitment and Support	.06	0.65	0.23	0.12	1.00	0.35	0.18
Reduced Length of Stay	.01	0.26	0.10	0.64	0.41	0.41	1.00
Better Utilization of Resources	.02	0.09	0.22	0.69	0.13	0.13	1.00
Increased Monetary Support	.06	0.73	0.08	0.19	1.00	1.00	0.26
Death as a Social Issue	.02	0.20	0.20	0.60	0.33	0.33	1. 00
Rehumanization of Institutions	.08	0.24	0.14	0.62	0.39	0.23	1.00
Synthesis (taken from original AHP without approximations)		0.428	0.121	0.451	0.424	0.123	0.453

When the criteria priorities do not depend on the values of the alternatives with regard to those criteria, we need to derive their priorities by comparing them pairwise with each other with respect to higher-level criteria or goal. It is a process of trading off one unit of one criterion against a unit of another, an ideal alternative from one against an ideal alternative from another. To determine

Table 9.9 (continued)
Synthesis (P=Priorities, M=Model).

		Distributive Mode			Ideal Mode		
Costs	P	M 1	M 2	M 3	M 1	M 2	M 3
Community Costs	.14	0.33	0.33	0.33	1.00	1.00	1.00
Institutional Capital Costs	.03	0.76	0.09	0.15	1.00	0.12	0.20
Institutional Operating Costs	.40	0.73	0.08	0.19	1.00	0.11	0.26
Institutional Costs for Educating the Community	.01	0.65	0.24	0.11	1.00	0.37	0.17
Institutional Costs for Training Staff	.06	0.56	0.32	0.12	1.00	0.57	0.21
Institutional Bad Debt	.15	0.60	0.20	0.20	1.00	0.33	0.33
Institutional Costs of Recruiting Staff	.05	0.66	0.17	0.17	1.00	0.26	0.26
Institutional Costs of Recruiting Volunteers	.01	0.60	0.20	0.20	1.00	0.33	0.33
Societal Costs	.15	0.33	0.33	0.33	1.00	1.00	1.00
Synthesis (taken from original AHP without approximations)		0.583	0.192	0.224	0.523	0.229	0.249
Benefit/Cost Ratio		0.734	0.630	2.013	0.811	0.537	1.819

the ideal, the alternatives are divided by the largest value among them for each criterion. In that case, the process of weighting and adding assigns each of the remaining alternatives a value that is proportionate to the value 1 given to the highest rated alternative. In this way the alternatives are weighted by the priorities of the criteria and summed to obtain the weights of the alternatives. This is the ideal mode of the AHP.

The distributive mode is essential for synthesizing the weights of alternatives with respect to tangible criteria with the same scale of measurement into a single criterion for that scale and then they are treated as intangibles and compared pairwise and combined with other intangibles with the ideal mode. The dominant mode of synthesis in the AHP where the criteria are independent from the alternatives is the ideal mode. The standard mode for synthesizing in the ANP where criteria depend on alternatives and also alternatives may depend on other alternatives is the distributive mode.

In this case, both modes lead to the same outcome for hospice, which is Model 3. As we shall see below, we need both modes to deal with the effect of adding (or deleting) alternatives on an already ranked set. The priorities of

the alternatives in the benefits hierarchy belong to an absolute scale of relative numbers and the priorities of the alternatives in the costs hierarchy also belong to another absolute scale of relative numbers. These two relative scales cannot be arbitrarily combined. Later we provide another way to combine them. In this exercise they were assumed to be commensurate and were combined in the traditional way by forming benefit to cost ratios. To derive the answer we divide the benefits priority of each alternative by its costs priority. We then choose the alternative with the largest of these ratios.

Model 3 has the largest benefit to cost ratio in both the distributive and ideal modes, and the hospital selected it for treating terminal patients. This need not always be the case. In this case, there is dependence of the personnel resources allocated to the three models because some of these resources would be shifted based on the decision. Therefore the distributive mode is the appropriate method of synthesis. If the alternatives were sufficiently distinct with no dependence in their definition, the ideal mode would be the way to synthesize.

I also performed marginal analysis to determine where the hospital should allocate additional resources for the greatest marginal return. To perform marginal analysis, I first ordered the alternatives by increasing cost priorities and then formed the benefit-to-cost ratios corresponding to the smallest cost, followed by the ratios of the differences of successive benefits to differences in costs. If this difference in benefits is negative, the new alternative is dropped from consideration and the process continued. The alternative with the largest ratio is then chosen. For the costs and corresponding benefits from the synthesis rows in Table 9.9 one obtains:

- Benefits: .12, .45, .43;

- Costs: .20, .21, .59;

- Ratios: $.12/.20 = .60$, $(.45-.12)/(.21-.20) = 33$, $(.43-.45)/(.59-.21) = -0.051$.

The third alternative is not a contender for resources because its marginal return is negative. The second alternative is the best. In fact, in addition to adopting the third model, the hospital management chose the second model of hospice care for further development.

5.   Rating Alternatives One at a Time in the AHP – Absolute Measurement

The AHP has a second way to derive priorities known as absolute measurement. It involves making paired comparisons but the criteria just above the alternatives, known as the covering criteria, are assigned intensities that vary in number and type. For example they can simply be: high, medium and low; or they can be:

excellent, very good, good, average, poor and very poor; or for experience: more than 15 years, between 10 and 15, between 5 and 10 and less than 5 and so on. These intensities themselves are also compared pairwise to obtain their priorities as to importance, and they are then put in ideal form by dividing by the largest value. Finally each alternative is assigned an intensity, along with its accompanying priority, for each criterion. This process of assigning intensities is called rating the alternatives. The priority of each intensity is weighted by the priority of its criterion and summed over the weighted intensities for each alternative to obtain that alternative's final rating that also belongs to a ratio scale. It is often necessary to have categories of ratings for alternatives that are widely disparate so that one can rate the alternatives correctly. Ratings are useful when standards are established with which the alternatives must comply. They are also useful when the number of alternatives n is very large to perform pairwise comparisons on them for each criterion. In this case if the number of criteria is c, the number of rating operations in rating the alternatives is cn, whereas doing all the pairwise judgments involves $cn(n-1)/2$ comparisons. Here is an example of absolute measurement.

5.1   Evaluating Employees for Salary Raises

Employees are evaluated for raises. The criteria are Dependability, Education, Experience, and Quality. Each criterion is subdivided into intensities, standards, or discrimination categories as shown in Figure 9.4. Priorities are set for the criteria by comparing them in pairs. The intensities are then pairwise compared according to importance with respect to their parent criterion (example as in Table 9.10). Their priorities are often divided by the largest intensity for each criterion (second column of priorities in Figure 9.4) particularly useful in preserving the ranks of the alternatives from the addition or deletion of other alternatives. Finally, each individual is rated in Table 9.11 by assigning the intensity rating that applies to him or her under each criterion and adding. The score is obtained by weighting the intensities by the priority of their criteria and then summing over the criteria to derive a total score for each individual. This approach can be used whenever it is possible to set priorities for intensities of the criteria, which is usually possible when sufficient experience with a given operation has been accumulated. The raises can be made in proportion to the normalized values on the right.

   One needs to choose the intensities widely enough by putting them in different order-of- magnitude categories in which the elements can be compared with the fundamental scale, and then combine the categories with pivots as in the cherry with watermelon example. Any alternative can be appropriately rated and receives its correct final value no matter how large or how small. When rating widely contrasting alternatives and the rating of an alternative is exceed-
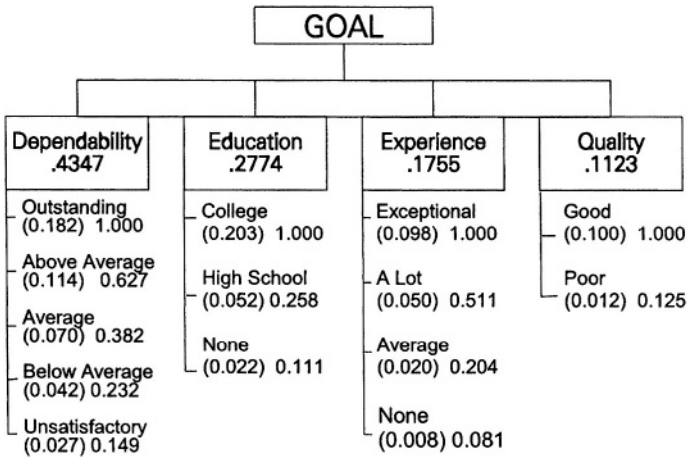
Figure 9.4. Employee evaluation hierarchy.

Table 9.10. **Ranking** intensities.

	Out-standing	Above Average	Average	Below Average	Unsatis-factory	Priorities
Outstanding	1.0	2.0	3.0	4.0	5.0	0.419
Above Average	1/2	1.0	2.0	3.0	4.0	0.263
Average	1/3	1/2	1.0	2.0	3.0	0.160
Below Average	1/4	1/3	1/2	1.0	2.0	0.097
Unsatisfactory	1/5	1/4	1/3	1/2	1.0	0.062

Inconsistency Ratio = 0.015

ingly small with respect to a certain criterion, a zero value can be assigned to that alternative.

In ratings, adding new alternatives has no effect on the rank of existing alternatives. In paired comparisons the alternatives depend on each other and a new alternative can affect the relative ranks of existing alternatives. Using the ideal mode each time a new alternative is added prevents rank reversal with respect to irrelevant alternatives. However, if it is done only the first time and new alternatives are only compared with the first ideal so their values go above that ideal (more than one when necessary) there can be no rank reversal. It is clear that when alternatives are independent they can be rated one at a time and there would be no rank reversal. But even with independence, how many

Table 9.11.   Ranking alternatives.

	Depend-ability	Education .2774	Experience .1775	Quality .1123	Total	Norm
1. Adams, V.	Outstand.	College	Excep.	Good	1.000	0.245
2. Becker, L.	Average	College	Average	Good	0.592	0.145
3. Hayat, F.	Average	College	A Lot	Good	0.645	0.158
4. Kesselman, S.	Above Av.	HighSch.	None	Poor	0.373	0.091
5. O'Shea, K.	Average	College	Average	Poor	0.493	0.121
6. Petres, T.	Average	College	None	Good	0.570	0.140
7. Tobias, K.	Average	None	A Lot	Poor	0.407	0.100

other alternatives of the same kind (sometimes also of a different kind) there are, can affect their rank. However, the number of alternatives cannot be used as a criterion for rating because it implies dependence of an alternative on how many others there are and a fortiori on their presence.

6.   Paired Comparisons Imply Dependence

In most multicriteria decision problems the criteria are assumed independent of the alternatives and the alternatives independent of other alternatives. Paired comparisons imply dependence of a different kind. The common understanding is that when alternatives depend on each other it is according to their function like the electric industry depending on the coal industry for its output. In paired comparisons, the importance assigned to an alternative depends on what other alternatives it is compared with and how many there are. This is dependence not according to function but according to structure. This dependence happens even when the alternatives may be independent of each other according to function. Independence means that the rank of an alternative does not depend on what other alternatives there are and how many of them there may be. The situation with pairwise comparisons is that it automatically implies structural dependence. When a new alternative is added or an old one deleted the ranks of the other alternatives relative to each other may change. However one can preserve rank from adding new but irrelevant alternatives by creating an ideal alternative each time alternatives are added or deleted, or preserve it from any new alternative by simply idealizing the first time but never after and only comparing new alternatives with the first ideal and allowing the priority value of the new alternative to exceed one. Rating alternatives one at a time with appropriate and exhaustive orders of intensities for each criterion always preserves rank from structural effects, but is not always the best way to prioritize alternatives that may depend on the number and quality of other alternatives.

As we increase the number of copies of an alternative, it often loses (or conversely increases) its importance. For example, if gold, which is important, were to increase in quantity to fill the universe, it could lose its importance. No new criterion is added and no judgment is changed but only the quantity of gold. Relative measurementmeasurement,relative takes quantity into consideration. We often need to consider this kind of dependence known as structural dependence. When we add more alternatives, the ranks among old ones may change and what was preferred to another now because of the presence of new ones may no longer be preferred to the other. Another example is that of a company that sells cars A and B. Car B is better than car A but it costs more to make. It is more desirable all around for people to buy car B but they buy A because it is cheaper. The company advertises that it is going to make car C that is similar to B but much more expensive. People are now observed more and more to buy car B. The company never makes car C. This is a real life example from marketing. However, in some decision problems we may want to treat by fiat the alternatives of a decision as completely independent both in property and in number and quality and want to preserve the ranks of existing alternatives when new ones are added or old ones deleted. The AHP allows for both these possibilities. Actually, change in rank in the presence of relevant alternatives is a fact of our world. It is also a fact that when the number of irrelevant alternatives is very large, they can cause rank to change. Viruses are irrelevant in most decisions but they can eventually cause the death of all decision makers and make mockery of the decisions they thought were so important. In essence reality is much more interdependent than we have allowed for in our limited ways of thinking. Admittedly there are times when we wish to preserve rank no matter what the situation may be. We need to allow for both in our decision theories and not take the simple way out by always assuming independence.

7.   When is a Positive Reciprocal Matrix Consistent?

In light of the foregoing, for the validity of the vector of priorities to describe response, we need greater redundancy and therefore also a large number of comparisons. Because of the reciprocal relation, in all we need $n(n-1)/2$ comparisons. An expert may provide $(n-1)$ comparisons to fill one row or a spanning tree from which the matrix is consistent and the priorities are easily obtained. Let us relate the psychological idea of the consistency of judgments and its measurement to a central concept in matrix theory and also to the size of our channel capacity to process information. Let $A = [a_{ij}]$ be an n-by-n positive reciprocal matrix, so all $a_{ii} = 1$ and $a_{ij} = 1/a_{ji}$ for all $i, j = 1, \cdots, n$. Let $w = [w_i]$ be the principal right eigenvector of A, let $D = diag(w_1, ..., w_n)$ be the n-by-n diagonal matrix whose main diagonal entries are the entries of w, and set $E = D^{-1}AD = [a_{ij}w_j/w_i] = [\varepsilon_{ij}]$. Then E is similar to A and is a

positive reciprocal matrix since $\varepsilon_{ji} = (a_{ji}w_i/w_j) = (a_{ij}w_j/w_i)^{-1} = 1/\varepsilon_{ij}$.
Moreover, all the row sums of E are equal to the principal eigenvalue of A:

$$\sum_{j=1}^{n} \varepsilon_{ij} = \sum_{j} \frac{a_{ij}w_j}{w_i} = \frac{[Aw]_i}{w_i} = \frac{\lambda_{max}w_i}{w_i} = \lambda_{max}.$$

The computation

$$n\lambda_{max} = \sum_{i=1}^{n}(\sum_{j=1}^{n} \varepsilon_{ij}) = \sum_{i=1}^{n} \varepsilon_{ii} + \sum_{i,j=1\ i\neq j}^{n} (\varepsilon_{ij} + \varepsilon_{ji})$$

$$= n + \sum_{i,j=1\ i\neq j}^{n} (\varepsilon_{ij} + \varepsilon_{ij}^{-1})$$

$$\geq n + (n^2 - n) = n^2$$

reveals that $\lambda_{max} \geq n$. Moreover, since $(x+1)/x \geq 2$ for all $x > 0$, with
equality if and only if $x = 1$, we see that $\lambda_{max} = n$ if and only if all $\varepsilon_{ij} = 1$,
which is equivalent to having all $a_{ij} = w_i/w_j$.

The foregoing arguments show that a positive reciprocal matrix A has $\lambda_{max} \geq n$, with equality if and only if A is consistent. When A is consistent we have $A^k = n^{k-1}A$. As our measure of deviation of A from consistency, we choose the *consistency index*

$$\mu = \frac{\lambda_{max} - n}{n - 1}.$$

We have seen that $\mu \geq 0$ and $\mu = 0$ if and only if A is consistent. We
can say that as $\mu \to 0$, $a_{ij} \to w_i/w_j$, or $\varepsilon_{ij} = a_{ij}(w_j/w_i) \to 1$. These two
desirable properties explain the term "n" in the numerator of μ; what about the
term "$n-1$" in the denominator? Since trace $(A) = n$ is the sum of all the
eigenvalues of A, if we denote the eigenvalues of A that are different from λ_{max}
by $\lambda_2, \cdots, \lambda_{n-1}$, we see that $n = \lambda_{max} + \sum_{i=2}^{n} \lambda_i$, so $n - \lambda_{max} = \sum_{i=2}^{n} \lambda_i$
and $\mu = -(1/(n-1)) \sum_{i=2}^{n} \lambda_i$ is the average of the non-principal eigenvalues
of A.

Table 9.12.   Random index.

Order	1	2	3	4	5	6	7	8	9	10
R.I.	0	0	0.52	0.89	1.11	1.25	1.35	1.40	1.45	1.49

In order to get some feel for what the consistency index might be telling us
about a positive *n*-by-*n* reciprocal matrix A, consider the following simulation:
choose the entries of A above the main diagonal at random from the 17 values

{1/9, 1/8, $\cdots$, 1, 2, $\cdots$, 8, 9}. Then fill in the entries of A below the diagonal by taking reciprocals. Put ones down the main diagonal and compute the consistency index. Do this many thousands of times and take the average, which we call the random index. Table 9.12 shows the values obtained from one set of such simulations and also their first order differences, for matrices of size 1, 2, $\cdots$, 10.

A plot of the first two rows of Table 9.12 shows the asymptotic nature of random inconsistency. We also have shown that one should not compare more than about seven elements because increase in inconsistency is so small that it becomes difficult to perceive the ensuing small changes in the judgments needed to improve consistency [7]. In passing we note that there are several algorithms to change judgment to improve consistency, the best known among them is the gradient method of Patrick Harker [1,3].

For a given positive reciprocal matrix $A = [a_{ij}]$ and a given pair of distinct indices $k > l$, define $A(t) = [a_{ij}(t)]$ by $a_{kl}(t) \equiv a_{kl} + t$, $a_{lk}(t) \equiv (a_{kl} + t)^{-1}$, and $a_{ij}(t) \equiv a_{ij}$ for all $i \neq k$, $j \neq l$, so $A(0) = A$. Let $\lambda_{\max}(t)$ denote the Perron eigenvalue of $A(t)$ for all t in a neighborhood of $t = 0$ that is small enough to ensure that all entries of the reciprocal matrix $A(t)$ are positive there. Finally, let $v = [v_i]$ be the unique positive eigenvector of the positive matrix A^T that is normalized so that $v^T w = 1$. Then a classical perturbation formula tells us that

$$\frac{d\lambda_{\max}(t)}{da_{ji}}\bigg|_{t=0} = \frac{v^T A'(0)w}{v^T w} = v^T A'(0)w = v_k w_l - \frac{1}{a_{kl}^2} v_l w_k.$$

We conclude that

$$\frac{\partial \lambda_{\max}}{\partial a_{ij}} = v_i w_j - a_{ij}^2 v_j w_i, \quad i, j = 1, \cdots, n.$$

Because we are operating within the set of positive reciprocal matrices, $\partial \lambda_{\max}/\partial a_{ji} = -\partial \lambda_{\max}/\partial a_{ij}$ for all i and j. Thus, to identify an entry of A whose adjustment within the class of reciprocal matrices would result in the largest rate of change in $\lambda_{\max}$ we should examine the $n(n-1)/2$ values $\{v_i w_j - a_{ji}^2 v_j w_i\}$, $i > j$ and select (any) one of largest absolute value.

8. In the Analytic Hierarchy Process Additive Composition is Necessary

Sometimes people have assigned criteria different weights when they are measured in the same unit. Others have used different ways of synthesis than multiplying and adding. An example should clarify what we must do. Synthesis in the AHP involves weighting the priorities of elements compared with respect to an element in the next higher level, called a parent element, by the priority

Table 9.13.   Calculating returns arithmetically.

Alternatives	Criterion C_1 Unnormalized weight = 1.0	Criterion C_2 Unnormalized weight = 1.0	Weighted Sum Unnormalized	Normalized or Relative values
A_1	200	150	350	350/1300=.269
A_2	300	50	350	350/1300=.269
A_3	500	100	600	600/1300=.462
Column totals	1000	300	1300	1

of that element and adding over all such parents for each element in the lower level. Consider the example of two criteria C_1 and C_2 and three alternatives A_1, A_2 and A_3 measured in the same scale such as dollars. If the criteria are each assigned the value 1, then the weighting and adding process produces the correct dollar value as in Table 9.13.

However, it does not give the correct outcome if the weights of the criteria are normalized, with each criterion having a weight of .5. Once the criteria are given in relative terms, so must the alternatives also be given in relative terms. A criterion that measures values in pennies cannot be as important as another measured in thousands of dollars. In this case, the only meaningful importance of a criterion is the ratio of the total money for the alternatives under it to the total money for the alternatives under both criteria. By using these weights for the criteria, rather than .5 and .5, one obtains the correct final relative values for the alternatives.

What is the relative importance of each criterion? Normalization indicates relative importance. Relative values require that criteria be examined as to their relative importance with respect to each other. What is the relative importance of a criterion, or what numbers should the criteria be assigned that reflect their relative importance? Weighting each criterion by the proportion of the resource under it, as shown in Table 9.14, and multiplying and adding as in the ad-ditive synthesis of the AHP, we get the same correct answer. For criterion C_1 we have (200+300+500)/[(200+300+500)+ (150+50+100)]=1000/1300 and for criterion C_2 we have (150+50+100)/[(200+300+500) + (150+50+100)] = 300/1300. Here the criteria are automatically in normalized form, and their weights sum to one. We see that when the criteria are normalized, the alterna-tives must also be normalized to get the right answer. For example, if we look in Table 9.13 we have 350/1300 for the priority of alternative A_1. Now if we simply weight and add the values for alternative A_1 in Table 9.14 we get for its final value (200/1000)(1000/1300) + (150/300)(300/1300) = 350/1300. It is clear that if the priorities of the alternatives are not normalized one does not get the desired outcome.

Table 9.14. Normalized criteria weights and normalized alternative weights from measurements in the same scale (additive synthesis).

Alternatives	Criterion C_1 Normalized weight = 1000/1300=0.7692	Criterion C_2 Normalized weight = 300/1300=0.2308	Weighted Sum
A_1	200/1000	150/300	350/1300 = .2692
A_2	300/1000	50/300	350/1300 = .2692
A_3	500/1000	100/300	600/1300 = .4615

We have seen in this example that in order to obtain the correct final relative values for the alternatives when measurements on a measurement scale are given, it is essential that the priorities of the criteria be derived from the priorities of the alternatives. Thus when the criteria depend on the alternatives we need to normalize the values of the alternatives to obtain the final result. This procedure is known as the distributive mode of the AHP. It is also used in case of functional dependence of the alternatives on the alternatives and of the criteria on the alternatives. The AHP is a special case of the Analytic Network Process. The dominant mode of synthesis in the ANP with all its interdependencies is the distributive mode. The ANP automatically assigns the criteria the correct weights, if one only uses the normalized values of the alternatives under each criterion and also the normalized values for each alternative under all the criteria without any special attention to weighting the criteria.

9.   Benefits, Opportunities, Costs and Risks

In many decision problems four kinds of concerns or merits are considered: benefits, opportunities, costs and risks, which we abbreviate as BOCR. The first two are advantageous and hence are positive and the second two are disadvantageous and are therefore negative [5, 6]. Later we show how to determine the relative importance of each of the BOCR.

There are two ways to combine BOCR priorities. The first is the traditional one (used by economists) in which one does not need the relative importance of the BOCR by simply forming their ratio *BO/CR* for each alternative obtained from a separate hierarchy for each of the four BOCR merits and selecting that alternative with the largest ratio. It is known as the ratio outcome. The second derives corresponding normalized weights b, o, c, and r obtained respectively by rating the best alternative (one at a time) for each of the BOCR with respect to strategic criteria illustrated with an example later. One then forms for the four values of each alternative the expression

$$bB + oO - cC - rR.$$

The first way is a tradeoff between a unit of *BO* against a unit of *CR*, a unit of the desirable against a unit of the undesirable. It may be advisable, for example, that if the costs are considered to be negligibly smaller than the benefits to use only the benefits for the best alternative of a decision and not form the ratio and vice versa. The second way simply subtracts the sum of the weighted undesirables from the sum of the weighted desirables to give the total gain or loss. It can give rise to negative priorities and when applied to measurements in dollars, for example, where the weights *b, o, c,* and *r* are the same, gives back the correct answer. We have seen examples in which numbers or differences of numbers are made so small that one faces the classical problem of dividing by zero or comparing things whose measurements are near zero.

Two other formulas have been considered and set aside. They are $bB + oO + c/C + r/R$, and $bB + oO + c(1 - C) + r(1 - R)$. The first with only $bB + c/C$ makes the benefits determine the outcome when the cost is very high, which is counter intuitive. The second is always positive and is equal to $bB + oO - cC - rR + (c + r)$, and adds a constant to the subtractive formula $bB + oO - cC - rR$.

Note that there is no advantage in using the weights *b, o, c* and *r* in the formula *BO/CR* because we would be multiplying the result for each alternative by the same constant bo/cr. Because all values lie between zero and one, we have from the series expansions of the exponential and logarithmic functions the approximation:

$$
\begin{aligned}
\frac{bBoO}{cCrR} &= \exp(\log bB + \log oO - \log cC - \log rR) \\
&= 1 + (\log bB + \log oO - \log cC - \log rR) + \dots \\
&\approx 1 + (bB - 1) + (oO - 1) - (cC - 1) - (rR - 1) \\
&= 1 + bB + oO - cC - rR.
\end{aligned}
$$

Because one is added to the overall value of each alternative we can eliminate it. The approximate result is that the ratio formula is similar to the total formula with equal weights assumed for the *B, O, C, R.*

10.   On the Admission of China to the World Trade Organization (WTO)

This section was taken from an analysis done in 2000 carried out before the US Congress acted favorably on China joining the WTO and was hand-delivered to many of the members of the committee including its Chairperson. Since 1986, China had been attempting to join the multilateral trade system, the General Agreement on Tariffs and Trade (GATT) and, its successor, the World Trade Organization (WTO). According to the rules of the 135-member nations of WTO, a candidate member must reach a trade agreement with any existing

member country that wishes to trade with it. By the time this analysis was done, China signed bilateral agreements with 30 countries – including the US (November 1999) – out of 37 members that had requested a trade deal with it [5].

As part of its negotiation deal with the US, China asked the US to remove its annual review of China's Normal Trade Relations (NTR) status, until 1998 called Most Favored Nation (MFN) status. In March 2000, President Clinton sent a bill to Congress requesting a Permanent Normal Trade Relations (PNTR) status for China. The analysis was done and copies sent to leaders and some members in both houses of Congress before the House of Representatives voted on the bill, May 24, 2000. The decision by the US Congress on China's trade-relations status will have an influence on US interests, in both direct and indirect ways. Direct impacts include changes in economic, security and political relations between the two countries as the trade deal is actualized. Indirect impacts will occur when China becomes a WTO member and adheres to WTO rules and principles. China has said that it would join the WTO only if the US gives it Permanent Normal Trade Relations status.

It is likely that Congress will consider four options. The least likely is that the US will deny China both PNTR and annual extension of NTR status. The other three options are:

1 **Passage of a clean PNTR bill:** Congress grants China Permanent Normal Trade Relations status with no conditions attached. This option would allow implementation of the November 1999 WTO trade deal between China and the Clinton administration. China would also carry out other WTO principles and trade conditions.

2 **Amendment of the current NTR status bill:** This option would give China the same trade position as other countries and disassociate trade from other issues. As a supplement, a separate bill may be enacted to address other matters, such as human rights, labor rights, and environmental issues.

3 **Annual extension of NTR status:** Congress extends China's Normal Trade Relations status for one more year, and, thus, maintains the status quo.

The conclusion of the study is that the best alternative is granting China PNTR status. China now has that status.

Our analysis involves four steps. First, we prioritize the criteria in each of the benefits, costs, opportunities and risks hierarchies with respect to the goal. Figure 9.5 shows the resulting prioritization of these criteria. The alternatives and their priorities are shown under each criterion both in the distributive and in

the ideal modes. The ideal priorities of the alternatives were used appropriately to synthesize their final values beneath each hierarchy.
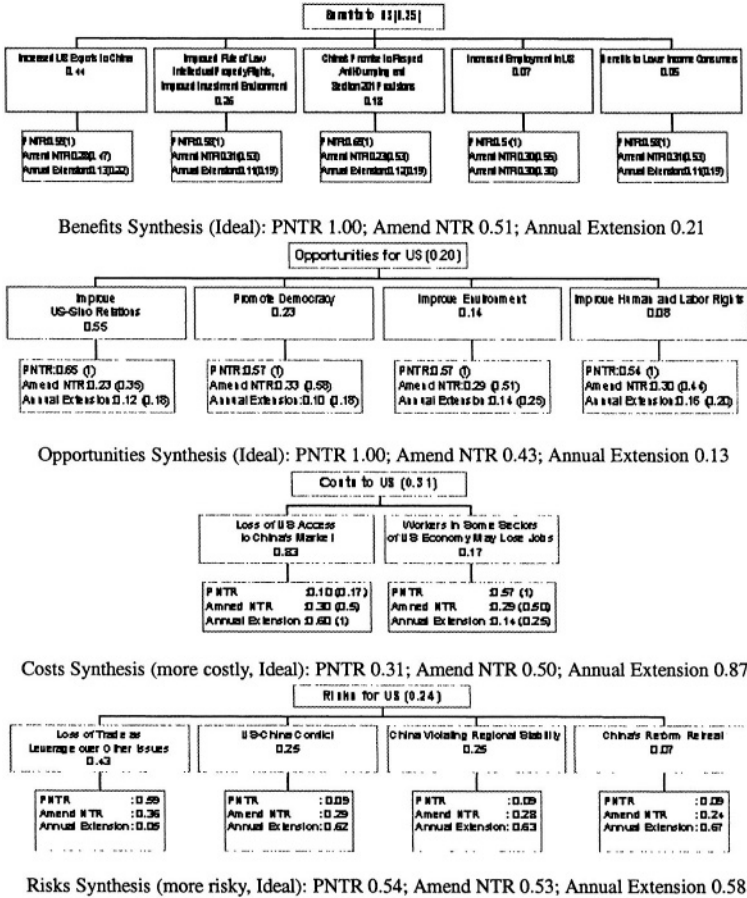
Figure 9.5.   Hierarchies for rating benefits, costs, opportunities, and risks.

The priorities shown in Figure 9.5 were derived from judgments that compared the elements involved in pairs. For readers to estimate the original pairwise judgments (not shown here) one forms the ratio of the corresponding two priorities shown, leave them as they are, or take the closest whole number, or its reciprocal if it is less than 1.0.

The idealized values are shown in parentheses after the original distributive priorities obtained from the eigenvector. The ideal values are obtained by dividing each of the distributive priorities by the largest one among them. For the
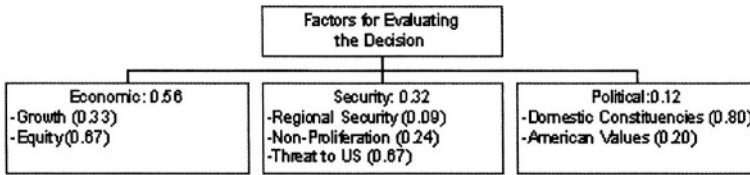
Figure 9.6. Prioritizing the strategic criteria to be used in rating the BOCR.

Costs and Risks structures, the question is framed as to which is the *most* costly or risky alternative. That is, the most costly alternative ends up with the highest priority.

Table 9.15. Priority Ratings for the Merits: Benefits, Costs, Opportunities, and Risks. Intensities: Very High (0.42), High (0.26), Medium (0.16), Low (0.1), Very Low (0.06).

		Benefits	Opportunities	Costs	Risks
Economic (0.56)	Growth (0.19)	High	Medium	Very Low	Very Low
	Equity (0.37)	Medium	Low	High	Low
Security (0.32)	Regional (0.03) (0.03)	Low	Medium	Medium	High
	Non-Proliferation (0.08)	Medium	High	Medium	High
	Threat to US (0.21)	High	High	Very High	Very High
Political (0.12)	Constituencies (0.1)	High	Medium	Very High	High
	American Values (0.02)	Very Low	Low	Low	Medium
Priorities		0.25	0.20	0.31	0.24

It is likely that, in a particular decision, the benefits, costs, opportunities and risks (BOCR) are not equally important, so we must also prioritize them. This is shown in Table 9.15. The priorities for the economic, security and political factors themselves were established as shown in Figure 9.6 and used to rate the importance of the top ideal alternative for each of the benefits, costs, opportunities and risks from Table 9.15. Finally, we used the priorities of the latter to combine the synthesized priorities of the alternatives in the four hierarchies, using both formulas BO/CR and $bB + oO - cC - rR$ to obtain their final ranking, as shown in Table 9.11.

How to derive the priority shown next to the goal of each of the four hierarchies in Figure 9.5 is outlined in Table 9.15. We rated each of the four merits: benefits, costs, opportunities and risks of the dominant PNTR alternative, as it happens to be in this case, in terms of intensities for each assessment criterion. The intensities, Very High, High, Medium, Low, and Very Low were themselves prioritized in the usual pairwise comparison matrix to determine their priorities. We then assigned the appropriate intensity for each merit on all assessment criteria. The outcome is as found in the bottom row of Table 9.15.

We are now able to obtain the overall priorities of the three major decision alternatives, given in the last two columns of Table 9.16. We see in bold that PNTR is the dominant alternative either way we synthesize as in the last two columns.

Table 9.16.   Four methods of synthesizing BOCR using the ideal mode.

Alternatives	Benefits (0.25)	Opportunities (0.20)	Costs (0.31)	Risks (0.24)	BO/CR	$bB + oO$ $-cC - rR$
PNTR	1	1	0.31	0.54	**5.97**	**0.22**
Amend NTR	0.51	0.43	0.50	0.53	0.83	-0.07
Annual Exten.	0.21	0.13	0.87	0.58	0.05	-0.33

We have laid the basic foundation with hierarchies for what we need to deal with networks involving interdependencies. Let us now turn to that subject.

11.   The Analytic Network Process (ANP)

To simplify and deal with complexity, people who work in decision-making use mostly very simple hierarchic structures consisting of a goal, criteria, and alternatives. Yet, not only are decisions obtained from a simple hierarchy of three levels different from those obtained from a multilevel hierarchy, but also decisions obtained from a network can be significantly different from those obtained from a multilevel hierarchy. We cannot collapse complexity artificially into a simplistic structure of two levels, criteria and alternatives, and hope to capture the outcome of interactions in the form of highly condensed judgments that correctly reflect all that goes on in the world. For 30 years we have worked with people to decompose these judgments through more elaborate structures to organize our reasoning and calculations in sophisticated but simple ways to serve our understanding of the complexity around us. Experience indicates that it is not very difficult to do this although it takes more time and effort, but not too much more. We have consulted and lectured on this subject in many countries: extensively in the US, in Brazil, Chile, the Czech Republic, Turkey, Poland, Indonesia, Switzerland, and soon in England and in China. There seems to be worldwide interest in decisions with dependence and feedback. My book on

this subject has been translated to two languages. *Indeed, we must use feedback networks to arrive at the kind of decisions needed to cope with the future.*

Many decision problems cannot be structured hierarchically because they involve the interaction and dependence of higher-level elements in a hierarchy on lower-level elements. Not only does the importance of the criteria determine the importance of the alternatives as in a hierarchy, but also the importance of the alternatives themselves determines the importance of the criteria. Two elephants chosen for work should have powerful trunks. One of them is slightly stronger but has only one ear. Strength alone would lead one to choose the strong but less attractive elephant unless the criteria of strength and attractiveness are evaluated in terms of the elephants, and strength receives a smaller value, and appearance a larger value because both elephants are strong. Feedback also enables us to factor the future into the present to determine what we have to do to attain a desired future. The Analytic Network Process is a generalization of the Analytic Hierarchy Process. The basic structures are networks. Priorities are established in the same way they are in the AHP using pairwise comparisons and judgments.

The feedback structure does not have the top-to-bottom form of a hierarchy but looks more like a network, with cycles connecting its components of elements, which we can no longer call levels, and with loops that connect a component to itself (see Figure 9.7). It also has sources and sinks. A **source** node is an origin of paths of influence (importance) and never a destination of such paths. A **sink** node is a destination of paths of influence and never an origin of such paths. A full network can include source nodes; intermediate nodes that fall on paths from source nodes, lie on cycles,cycle or fall on paths to sink nodes; and finally sink nodes. Some networks can contain only source and sink nodes. Still others can include only source and cycle nodes or cycle and sink nodes or only cycle nodes. A decision problem involving feedback arises often in practice. It can take on the form of any of the networks just described. The challenge is to determine the priorities of the elements in the network and in particular the alternatives of the decision and to justify the validity of the outcome. Because feedback involves cycles, and cycling is an infinite process, the operations needed to derive the priorities become more demanding than is familiar with hierarchies.

To obtain the overall dependence of elements such as the criteria, one proceeds as follows: Construct a zero-one matrix of criteria against criteria using the number one to signify dependence of one criterion on another, and zero otherwise. A criterion need not depend on itself as an industry, for example, may not use its own output. For each column of this matrix, construct a pairwise comparison matrix only for the dependent criteria, derive an eigenvector, and augment it with zeros for the excluded criteria. If a column is all zeros, then assign a zero vector to represent the priorities. The question in the comparison

would be: For a given criterion, which of two criteria depends more on that criterion with respect to the goal or with respect to a higher-order controlling criterion?

In Figure 9.7, a view is shown of a hierarchy and a network. A hierarchy is comprised of a goal, levels of elements and connections between the elements. These connections go only to elements in lower levels. A network has clusters of elements, with the elements being connected to elements in another cluster (outer dependence) or the same cluster (inner dependence). A hierarchy is a special case of a network with connections going only in one direction. In a view of a hierarchy, such as that shown in Figure 9.7, the levels in the hierarchy correspond to clusters in a network. One example of inner dependence in a component consisting of a father mother and baby is whom does the baby depend on more for its survival, its mother or itself. The baby depends more on its mother than on itself. Again suppose one makes advertising by newspaper and by television. It is clear that the two influence each other because the newspaper writers watch television and need to make their message unique in some way, and vice versa. If we think about it carefully everything can be seen to influence everything including itself according to many criteria. The world is far more interdependent than we know how to deal with using our existing ways of thinking and acting. We know it but how to deal with it. The ANP appears to be a plausible logical way to deal with dependence.
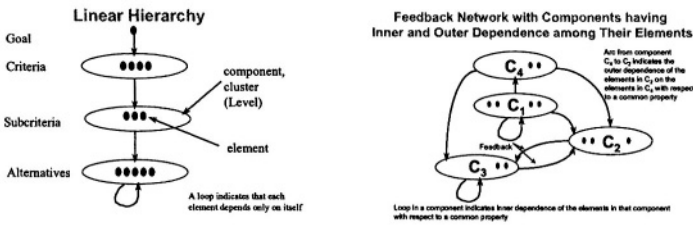
Figure 9.7.   How a hierarchy compares to a network.

The priorities derived from pairwise comparison matrices are entered as parts of the columns of a supermatrix. The supermatrix represents the influence priority of an element on the left of the matrix on an element at the top of the matrix. A supermatrix along with an example of one of its general entry matrices is shown in Figure 9.8. The component C_i in the supermatrix includes all the priority vectors derived for nodes that are "parent" nodes in the C_i cluster. Figure 9.9 gives the supermatrix of a hierarchy along with the kth power that yields the principle of hierarchic composition in its $(k, 1)$ position.

The Supermatrix of a Network

W $_{ij}$ Component of Supermatrix
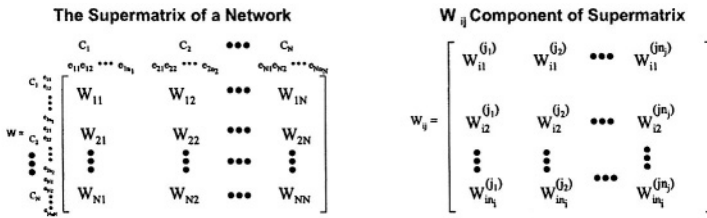
Figure 9.8. The supermatrix of a network and detail of a component in it.

Supermatrix of a Hierarchy

Supermatrix to nth Power Gives Hierarchical Synthesis

for k>n-1
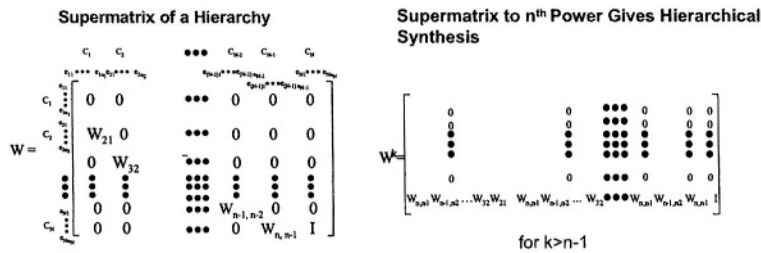
Figure 9.9. The supermatrix of a hierarchy with the resulting limit matrix corresponding to hierarchical composition.

Hierarchic composition yields multilinear forms that are of course nonlinear and have the form

$$\sum_{i_1,\cdots,i_p} x_1^{i_1} x_2^{i_2} \cdots x_p^{i_p}$$

where i_j indicates the jth level of the hierarchy and the x_j is the priority of an element in that level. The richer the structure of a hierarchy in breadth and depth, the more elaborate are the multilinear forms derived from it. There seems to be a good opportunity to investigate the relationship obtained by composition to covariant tensors and their algebraic properties. Powers of a variable allow for the possibility that the variable is repeated in the composition. Multilinear forms are related to polynomials and these by the Stone-Weierstrass theorem can be used to approximate arbitrarily close to continuous functions. Such functions may be assumed to underlie the representations of complex events in a decision. In this manner, mathematics and the apparent complicated use of numbers in decision-making can be related in a way that one can understand.

More concretely we have the covariant tensor

$$
w_i^h = \sum_{i_2,\cdots,i_{h-1}=1}^{N_{h-1},\cdots,N_1} w_{i_1,i_2}^{h-1} \cdots w_{i_{h-2},i_{h-1}}^2 w_{i_{h-1}}^1 \qquad i_1 \equiv i
$$

for the priority of the ith element in the hth level of the hierarchy. The composite vector W^h for the entire hth level is represented by the vector with covariant tensorial components. Similarly, the left eigenvector approach to a hierarchy gives rise to a vector with contravariant tensor components.

The classical problem of relating space (geometry) and time to subjective thought can perhaps be examined by showing that the functions of mathematical analysis (and hence also the laws of physics) are derivable as truncated series from the above tensors by composition in an appropriate hierarchy. The foregoing is reminiscent of the theorem in dimensional analysis that any physical variable is proportional to the product of powers of primary variables.

Multilinear forms are obviously nonlinear and are a powerful building stone to go from linearity to non-linearity through the use of complex structures (hierarchies and networks) and enable us to deal with the world according to our deepest ways of understanding and judgment.

In the ANP we look for steady state priorities from a limit supermatrix. To obtain the limit we must raise the matrix to powers. The reason for that is that to capture overall influence (dominance) one must consider all transitivities of different length. These are each represented by the corresponding power of the supermatrix. For each such matrix, the influence of an element on all others is obtained by taking the sum of its corresponding row. If we do that for all the elements, we obtain a vector of influence from that matrix. The sum of all such vectors gives the overall influence. Cesaro summability tells us that it is sufficient to obtain the outcome from the limiting power of the supermatrix.

The outcome of the ANP is nonlinear and rather complex. We know, from a theorem due to J.J. Sylvester that when the multiplicity of each eigenvalue of a matrix W is equal to one that an entire function $f(x)$ (power series expansion of $f(x)$ converges for all finite values of x) with x replaced by W, is given by

$$
f(W) = \sum_{i=1}^n f(\lambda_i) Z(\lambda_i),
$$

$$
Z(\lambda_i) = \frac{\prod_{j\neq i}(\lambda_j I - A)}{\prod_{j\neq i}(\lambda_j - \lambda_i)},
$$

$$
\sum_{i=1}^n Z(\lambda_i) = 1, \quad Z(\lambda_i)Z(\lambda_j) = 0, \quad Z^2(\lambda_i) = Z(\lambda_i),
$$

where I and 0 are the identity and null matrices respectively.

A similar expression is also available when some or all of the eigenvalues have multiplicities greater than one. We can easily see that if, as we need in our case, $f(W) = W^k$, then $f(\lambda_i) = \lambda_i^k$ and as $k \to \infty$ the only terms that give a finite nonzero value are those for which the modulus of λ_i is equal to one.

The fact that W is stochastic ensures this because

$$\max \sum_{j=1}^{n} a_{ij} \geq \sum_{j=1}^{n} a_{ij} \frac{w_j}{w_i} = \lambda_{\max} \text{ for } \max w_i$$

$$\min \sum_{j=1}^{n} a_{ij} \leq \sum_{j=1}^{n} a_{ij} \frac{w_j}{w_i} = \lambda_{\max} \text{ for } \min w_i$$

Thus for a row stochastic matrix we have

$$1 = \min \sum_{j=1}^{n} a_{ij} \leq \lambda_{\max} \leq \max \sum_{j=1}^{n} a_{ij} = 1,$$

and $\lambda_{\max} = 1$. See this author's 2001 book on the ANP [4], and also the manual for the ANP software [2]. Here are two examples that illustrate the validity of the supermatrix as a general framework for prioritization. The first as a generalization of hierarchies that gives back hierarchic answers, and the second as a method of computation and synthesis that carries the burden of computation with the user mostly providing judgments.

11.1   The Classic AHP School Example as an ANP Model

We show in Figures 9.10a and 9.10b below the hierarchy, and its corresponding supermatrix, and its limit supermatrix to obtain the priorities of three schools involved in a decision to choose one for the author's son. They are precisely what one obtains by hierarchic composition using the AHP. Figure 9. 10a shows the priorities of the criteria with respect to the goal and those of the alternatives with respect to each criterion. There is an identity submatrix for the alternatives with respect to the alternatives in the lower right hand part of the matrix, because each alternative depends on itself. The level of alternatives in a hierarchy is a sink cluster of nodes that absorbs priorities but does not pass them on. This calls for using an identity submatrix for them in the supermatrix. The last three entries of column one of Figure 9. 10b give the overall priorities of the alternatives with respect to the goal.

11.2   Criteria Weights Automatically Derived from Supermatrix

Let us revisit the example we gave earlier in Table 9.13 of three alternatives and two criteria measured in the same unit. We use interdependence to determine
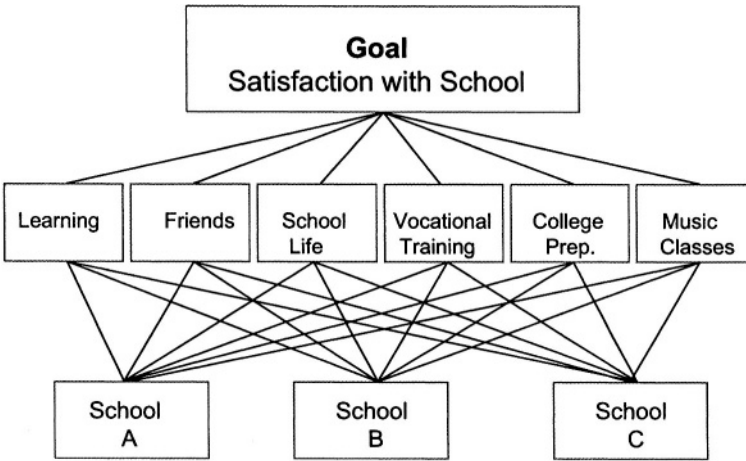
Figure 9.10a.   School choice hierarchy composition.

The School Hierarchy as Supermatrix

	Goal	Learning	Friends	School life	Vocational training	College preparation	Music classes	A	B	C
Goal	0	0	0	0	0	0	0	0	0	0
Learning	0.32	0	0	0	0	0	0	0	0	0
Friends	0.14	0	0	0	0	0	0	0	0	0
School life	0.03	0	0	0	0	0	0	0	0	0
Vocational training	0.13	0	0	0	0	0	0	0	0	0
College preparation	0.24	0	0	0	0	0	0	0	0	0
Music classes	0.14	0	0	0	0	0	0	0	0	0
Alternative A	0	0.16	0.33	0.45	0.77	0.25	0.69	1	0	0
Alternative B	0	0.59	0.33	0.09	0.06	0.50	0.09	0	1	0
Alternative C	0	0.25	0.34	0.46	0.17	0.25	0.22	0	0	1

Limiting Supermatrix & Hierarchic Composition

	Goal	Learning	Friends	School life	Vocational training	College preparation	Music classes	A	B	C
Goal	0	0	0	0	0	0	0	0	0	0
Learning	0	0	0	0	0	0	0	0	0	0
Friends	0	0	0	0	0	0	0	0	0	0
School life	0	0	0	0	0	0	0	0	0	0
Vocational training	0	0	0	0	0	0	0	0	0	0
College preparation	0	0	0	0	0	0	0	0	0	0
Music classes	0	0	0	0	0	0	0	0	0	0
Alternative A	0.3676	0.16	0.33	0.45	0.77	0.25	0.69	1	0	0
Alternative B	0.3781	0.59	0.33	0.09	0.06	0.50	0.09	0	1	0
Alternative C	0.2543	0.25	0.34	0.46	0.17	0.25	0.22	0	0	1

Figure 9.10b.   Supermatrix of school choice hierarchy gives same results as hierarchic composition.

what overall weight the criteria should have without computing the relative sum of the alternatives under each criterion to the total. Since we are dealing with tangibles we normalize each column to obtain the priorities for the alternatives under each criterion. We also normalize each row to obtain the priorities of the criteria with respect to each alternative. We enter these in a supermatrix as

Table 9.17. The supermatrix.

		Alternatives			Criteria	
		A_1	A_2	A_3	C_1	C_2
Alternatives	A_1	0.000	0.000	0.000	0.200	0.500
	A_2	0.000	0.000	0.000	0.300	0.167
	A_3	0.000	0.000	0.000	0.500	0.333
Criteria	C_1	0.571	0.857	0.833	0.000	0.000
	C_2	0.429	0.143	0.167	0.000	0.000

Table 9.18. The limit supermatrix.

		Alternatives			Criteria	
		A_1	A_2	A_3	C_1	C_2
Alternatives	A_1	0.135	0.135	0.135	0.135	0.135
	A_2	0.135	0.135	0.135	0.135	0.135
	A_3	0.231	0.231	0.231	0.231	0.231
Criteria	C_1	0.385	0.385	0.385	0.385	0.385
	C_2	0.115	0.115	0.115	0.115	0.115

shown in Table 9.17; there is no need to weight the supermatrix because it is already column stochastic, so we can raise it to limiting powers right away and obtain the limit supermatrix in Table 9.18 in which, in this case it turns out that, all the columns are identical.

12. Two Examples of Estimating Market Share – The ANP with a Single Benefits Control Criterion

A market share estimation model is structured as a network of clusters and nodes. The object is to determine the relative market share of competitors in a particular business, or endeavor, by considering what affects market share in that business and introducing them as clusters, nodes and influence links in a network. No actual statistics are used in these examples, but only judgments by experts about relative influence. The decision alternatives are the competitors and the synthesized results are their relative dominance. The relative dominance results can then be compared against some outside measure such as dollars. If dollar income is the measure being used, the incomes of the competitors must be normalized to get it in terms of relative market share.

The clusters might include customers, service, economics, advertising, and quality of goods. The customers cluster might then include nodes for the age groups of the people that buy from the business: teenagers, 20-33 year olds,

34-55 year olds, 55-70 year olds, and over 70. The advertising cluster might include newspapers, TV, Radio, and Fliers. After all the nodes are created start by picking a node and linking it to the other nodes in the model that influence it. The "children" nodes will then be pairwise compared with respect to that node as a "parent" node. An arrow will automatically appear going from the cluster the parent node is in to the cluster with its children nodes. When a node is linked to nodes in its own cluster, the arrow becomes a loop on that cluster and we say there is inner dependence.

The linked nodes in a given cluster are pairwise compared for their influence on the node they are linked from (the parent node) to determine the priority of their influence on the parent node. Comparisons are made as to which is more important to the parent node in capturing "market share". These priorities are then entered in the supermatrix.

The clusters are also pairwise compared to establish their importance with respect to each cluster they are linked from, and the resulting matrix of numbers is used to weight the components of the original unweighted supermatrix to give the weighted supermatrix. This matrix is then raised to powers until it converges to give the limit supermatrix. The relative values for the companies are obtained from the columns of the limit supermatrix that in this case, with the help of Cesaro summability, are reduced in the software to be all the same. Normalizing these numbers yields the relative market share.

If comparison data in terms of sales in dollars, or number of members, or some other known measures are available, one can use their relative values to validate the outcome. The AHP/ANP has a compatibility metric to determine how close the ANP result is to the known measure. It involves taking the Hadamard product of the matrix of ratios of the ANP outcome and the transform of the matrix of ratios of the actual outcome summing all the coefficients and dividing by n^2. The requirement is that the value should be close to 1 and certainly not much more than 1.1.

We will give two examples of market share estimation showing details of the process in the first example and showing only the models and results in the second.

12.1   Example 1. Estimating the Relative Market Share of Walmart, Kmart and Target

The network for the ANP model shown in Figure 9.11 describes quite well the influences that determine the market share of these companies. We will not use space in this chapter to describe the clusters and their nodes in greater detail.

12.1.1   The Unweighted Supermatrix.   The unweighted supermatrix is constructed from the priorities derived from the different pairwise comparisons.
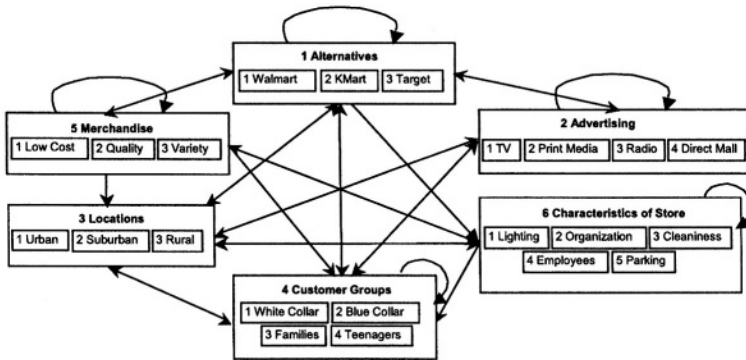
Figure 9.11. The clusters and nodes of a model to estimate the relative market share of Walmart, Kmart and Target.

The nodes, grouped by the clusters they belong to, are the labels of the rows and columns of the supermatrix. The column for a node a contains the priorities of the nodes that have been pairwise compared with respect to a. The supermatrix for the network in Figure 9.11 is shown in Table 9.19. In Tables 9.19 – 9.21 the following abbreviations have been used:

- Al - Alternatives, WM - Walmart, KM - KMart, Ta - Target;

- Ad - Advertising, TV, PM - Print Media, Ra - Radio, DM - Direct Mail;

- Lo - Location, Ur - Urban, Su - Suburban, Ru - Rural;

- CG - Custommer Groups, WC - White Collar, BC - Blue Collar, Fa - Families, Te - Teenagers;

- Me - Merchandise, LC - Low Cost, Qu - Quality, Va - Variety;

- CS - Characteristics of Store, Li - Lighting, Or - Organization, Cl - Cleanliness, Em - Employees, Pa - Parking.

Table 9.19. The unweighted supermatrix.

		1 Al			2 Ad				3 Lo		
		1 WM	2 KM	3 Ta	1 TV	2 PM	3 Ra	4 DM	1 Ur	2 Su	3 Ru
1 Al	1 WM	0.000	0.833	0.833	0.687	0.540	0.634	0.661	0.614	0.652	0.683
	2 KM	0.750	0.000	0.167	0.186	0.297	0.174	0.208	0.268	0.235	0.200
	3 Ta	0.250	0.167	0.000	0.127	0.163	0.192	0.131	0.117	0.113	0.117
2 Ad	1 TV	0.553	0.176	0.188	0.000	0.000	0.000	0.000	0.288	0.543	0.558
	2 PM	0.202	0.349	0.428	0.750	0.000	0.800	0.000	0.381	0.231	0.175
	3 Ra	0.062	0.056	0.055	0.000	0.000	0.000	0.000	0.059	0.053	0.048
	4 DM	0.183	0.420	0.330	0.250	0.000	0.200	0.000	0.273	0.173	0.219
3 Lo	1 Ur	0.114	0.084	0.086	0.443	0.126	0.080	0.099	0.000	0.000	0.000
	2 Su	0.405	0.444	0.628	0.387	0.416	0.609	0.537	0.000	0.000	0.000
	3 Ru	0.481	0.472	0.285	0.169	0.458	0.311	0.364	0.000	0.000	0.000
4 CG	1 WC	0.141	0.114	0.208	0.165	0.155	0.116	0.120	0.078	0.198	0.092
	2 BC	0.217	0.214	0.117	0.165	0.155	0.198	0.203	0.223	0.116	0.224
	3 Fa	0.579	0.623	0.620	0.621	0.646	0.641	0.635	0.656	0.641	0.645
	4 Te	0.063	0.049	0.055	0.048	0.043	0.045	0.041	0.043	0.045	0.038
5 Me	1 LC	0.362	0.333	0.168	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	2 Qu	0.261	0.140	0.484	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	3 Va	0.377	0.528	0.349	0.000	0.000	0.000	0.000	0.000	0.000	0.000
6 CS	1 Li	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	2 Or	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	3 Cl	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	4 Em	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	5 Pa	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Table 9.19 (continued)
The unweighted supermatrix.

		4 CG				5 Me			6 CS				
		1 WC	2 BC	3 Fa	4 Te	1 LC	2 Qu	3 Va	1 Li	2 Or	3 Cl	4 Em	5 Pa
1 Al	1 WM	0.637	0.661	0.630	0.691	0.661	0.614	0.648	0.667	0.655	0.570	0.644	0.558
	2 KM	0.105	0.208	0.218	0.149	0.208	0.117	0.122	0.111	0.095	0.097	0.085	0.122
	3 Ta	0.258	0.131	0.151	0.160	0.131	0.268	0.230	0.222	0.250	0.333	0.271	0.320
2 Ad	1 TV	0.323	0.510	0.508	0.634	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	2 PM	0.214	0.221	0.270	0.170	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	3 Ra	0.059	0.063	0.049	0.096	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	4 DM	0.404	0.206	0.173	0.100	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
3 Lo	1 Ur	0.167	0.094	0.096	0.109	0.268	0.105	0.094	0.100	0.091	0.091	0.111	0.067
	2 Su	0.833	0.280	0.308	0.309	0.117	0.605	0.627	0.433	0.455	0.455	0.444	0.293
	3 Ru	0.000	0.627	0.596	0.582	0.614	0.291	0.280	0.466	0.455	0.455	0.444	0.641
4 CG	1 WC	0.000	0.000	0.279	0.085	0.051	0.222	0.165	0.383	0.187	0.242	0.165	0.000
	2 BC	0.000	0.000	0.649	0.177	0.112	0.159	0.165	0.383	0.187	0.208	0.165	0.000
	3 Fa	0.857	0.857	0.000	0.737	0.618	0.566	0.621	0.185	0.583	0.494	0.621	0.000
	4 Te	0.143	0.143	0.072	0.000	0.219	0.053	0.048	0.048	0.043	0.056	0.048	0.000
5 Me	1 LC	0.000	0.000	0.000	0.000	0.000	0.800	0.800	0.000	0.000	0.000	0.000	0.000
	2 Quality	0.000	0.000	0.000	0.000	0.750	0.000	0.200	0.000	0.000	0.000	0.000	0.000
	3 Variety	0.000	0.000	0.000	0.000	0.250	0.200	0.000	0.000	1.000	0.000	0.000	0.000
6 CS	1 Li	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.251	0.169	0.121	0.000	0.250
	2 Or	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.575	0.200	0.750
	3 Cl	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.673	0.469	0.000	0.800	0.000
	4 Em	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.308	0.304	0.000	0.000
	5 Pa	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.075	0.055	0.000	0.000	0.000

Table 9.20.   The cluster matrix.

	1. Al	2. Ad	3. Li	4. CG	5. Me	6. CG
1. Alternatives (Al)	0.137	0.174	0.094	0.057	0.049	0.037
2. Advertising (Ad)	0.091	0.220	0.280	0.234	0.000	0.000
3. Locations (Li)	0.276	0.176	0.000	0.169	0.102	0.112
4. Customer						
Groups (CG)	0.054	0.429	0.627	0.540	0.252	0.441
5. Merchandise (Me)	0.442	0.000	0.000	0.000	0.596	0.316
6. Characteristics						
of Store (CS)	0.000	0.000	0.000	0.000	0.000	0.094

12.1.2   The Cluster Matrix.   The cluster themselves must be compared to establish their relative importance and use it to weight the supermatrix to make it column stochastic. A cluster impacts another cluster when it is linked from it, that is, when at least one node in the source cluster is linked to nodes in the target cluster. The clusters linked from the source cluster are pairwise compared for the importance of their impact on it with respect to market share, resulting in the column of priorities for that cluster in the cluster matrix. The process is repeated for each cluster in the network to obtain the matrix shown in Table 9.20. An interpretation of the priorities in the first column is that Merchandise (0.442) and Locations (0.276) have the most impact on Alternatives, the three competitors.

12.1.3   The Weighted Supermatrix.   The weighted supermatrix shown in Table 9.21 is obtained by multiplying each entry in a block of the component at the top of the supermatrix by the priority of influence of the component on the left from the cluster matrix in Table 9.20. For example, the first entry, 0.137, in Table 9.20 is used to multiply each of the nine entries in the block (Alternatives, Alternatives) in the unweighted supermatrix shown in Table 9.19. This gives the entries for the (Alternatives, Alternatives) component in the weighted supermatrix of Table 9.21. Each column in the weighted supermatrix has a sum of 1, and thus the matrix is stochastic.

Table 9.21. The weighted supermatrix.

		1 AI			2 Ad				3 Lo		
		1 WM	2 KM	3 Ta	1 TV	2 PM	3 Ra	4 DM	1 Ur	2 Su	3 Ru
1 AI	1 WM	0.000	0.114	0.114	0.120	0.121	0.110	0.148	0.058	0.061	0.064
	2 KM	0.103	0.000	0.023	0.033	0.066	0.030	0.047	0.025	0.022	0.019
	3 Ta	0.034	0.023	0.000	0.022	0.037	0.033	0.029	0.011	0.011	0.011
2 Ad	1 TV	0.050	0.016	0.017	0.000	0.000	0.000	0.000	0.080	0.152	0.156
	2 PM	0.018	0.032	0.039	0.165	0.000	0.176	0.000	0.106	0.064	0.049
	3 Ra	0.006	0.005	0.005	0.000	0.000	0.000	0.000	0.016	0.015	0.014
	4 DM	0.017	0.038	0.030	0.055	0.000	0.044	0.000	0.076	0.048	0.061
3 Lo	1 Ur	0.031	0.023	0.024	0.078	0.028	0.014	0.022	0.000	0.000	0.000
	2 Su	0.112	0.123	0.174	0.068	0.094	0.107	0.121	0.000	0.000	0.000
	3 Ru	0.133	0.130	0.079	0.030	0.103	0.055	0.082	0.000	0.000	0.000
4 CG	1 WC	0.008	0.006	0.011	0.071	0.086	0.050	0.066	0.049	0.124	0.058
	2 BC	0.012	0.011	0.006	0.071	0.086	0.085	0.112	0.140	0.073	0.141
	3 Fa	0.031	0.033	0.033	0.267	0.356	0.275	0.350	0.411	0.402	0.404
	4 Te	0.003	0.003	0.003	0.021	0.024	0.019	0.023	0.027	0.028	0.024
5 Me	1 LC	0.160	0.147	0.074	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	2 Qu	0.115	0.062	0.214	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	3 Va	0.166	0.233	0.154	0.000	0.000	0.000	0.000	0.000	0.000	0.000
6 CS	1 Li	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	2 Or	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	3 Cl	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	4 Em	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	5 Pa	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Table 9.21 (continued)
The weighted supermatrix.

| | | 4 CG | | | | 1 LC | 5 Me | | 1 Li | 2 Or | 6 CS | | |
		1 WC	2 BC	3 Fa	4 Te		2 Qu	3 Va			3 Cl	4 Em	5 Pa
1 AI	1 WM	0.036	0.038	0.036	0.040	0.033	0.030	0.032	0.036	0.024	0.031	0.035	0.086
	2 KM	0.006	0.012	0.012	0.009	0.010	0.006	0.006	0.006	0.004	0.005	0.005	0.019
	3 Ta	0.015	0.007	0.009	0.009	0.006	0.013	0.011	0.012	0.009	0.018	0.015	0.049
2 Ad	1 TV	0.076	0.119	0.119	0.148	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	2 PM	0.050	0.052	0.063	0.040	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	3 Ra	0.014	0.015	0.012	0.023	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	4 DM	0.095	0.048	0.040	0.023	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
3 Lo	1 Ur	0.028	0.016	0.016	0.018	0.027	0.011	0.010	0.016	0.010	0.015	0.018	0.031
	2 Su	0.141	0.047	0.052	0.052	0.012	0.062	0.064	0.071	0.051	0.074	0.073	0.135
	3 Ru	0.000	0.106	0.101	0.098	0.063	0.030	0.029	0.076	0.051	0.074	0.073	0.295
4 CG	1 WC	0.000	0.000	0.151	0.046	0.013	0.056	0.042	0.247	0.082	0.156	0.107	0.000
	2 BC	0.000	0.000	0.350	0.096	0.028	0.040	0.042	0.247	0.082	0.134	0.107	0.000
	3 Fa	0.463	0.463	0.000	0.398	0.156	0.143	0.157	0.119	0.257	0.318	0.400	0.000
	4 Te	0.077	0.077	0.039	0.000	0.055	0.013	0.012	0.031	0.019	0.036	0.031	0.000
5 Me	1 LC	0.000	0.000	0.000	0.000	0.000	0.477	0.477	0.000	0.000	0.000	0.000	0.000
	2 Qu	0.000	0.000	0.000	0.000	0.447	0.000	0.119	0.000	0.000	0.000	0.000	0.000
	3 Va	0.000	0.000	0.000	0.000	0.149	0.119	0.000	0.000	0.316	0.000	0.000	0.000
6 CS	1 Li	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.035	0.016	0.017	0.000	0.097
	2 Or	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.092	0.000	0.079	0.027	0.290
	3 Cl	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.044	0.000	0.000	0.000
	4 Em	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.029	0.042	0.110	0.000
	5 Pa	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.010	0.005	0.000	0.000	0.000

The limit supermatrix is not shown here to save space. It is obtained from the weighted supermatrix by raising it to powers until it converges so that all columns are identical. From the top part of the first column of the limit supermatrix we get the priorities we seek and normalize. We show what they are in Table 9.22.

Table 9.22. The synthesized results for the alternatives.

Alternatives	Values from Limit Supermatrix	Actual Values July13, 1998	Normalized Values from Supermatrix	Actual Market Share as Dollar Sales Normalized
Walmart	0.057	58 billion $	0.599	54.8
KMart	0.024	27.5 billion $	0.248	25.9
Target	0.015	20.3 billion $	0.254	19.2

12.1.4 Synthesized Results from the Limit Supermatrix. The relative market share of the alternatives Walmart, Kmart and Target from the limit supermatrix are: 0.057, 0.024 and 0.015. When normalized they are 0.599, 0.248 and 0.154.

The relative market share values obtained from the model were compared with the actual sales values by computing the compatibility index. The Compatibility Index, illustrated in the next example, is used to determine how close two sets of numbers from a ratio scale or an absolute scale are to each other. We form the matrix of ratios of each set and multiply element-wise one matrix by the transpose of the other (the Hadamard product), add all the entries of the resulting matrix and divide the outcome by n^2, where n is the order of the matrix which is the number of entries in each vector. The outcome should not exceed the value of 1.1. In this example the result is equal to 1.016 and falls below 1.1 and therefore is an acceptable outcome.

12.2 Example 2: US Athletic Footwear Market in 2000

My student Maria Lagasca has studied the US Athletic Footwear market. That market has seen tremendous growth over the years. Not only are these products used for specific athletic purposes but also they have been used as casual wear because of its ability to provide comfort and agility to consumers. Interest in the industry has grown to a large extent because of advances in research and development for durable yet comfortable materials. The industry is also considered as one of the heaviest advertisers based on a study made last year along with other industries such as apparel, beer/wine/liquor, computers and electronics. The study illustrated in Figure 9.12 aims at estimating the market share using the ANP with the aid of SuperDecisions software. The estimates

are then compared against the actual market share of various manufacturers in the year 2000. As the industry is fragmented (with many players holding fewer shares of the market), the other manufacturers have been lumped under the "Others" category as they are considered as homogeneous given the factors used in the analysis.
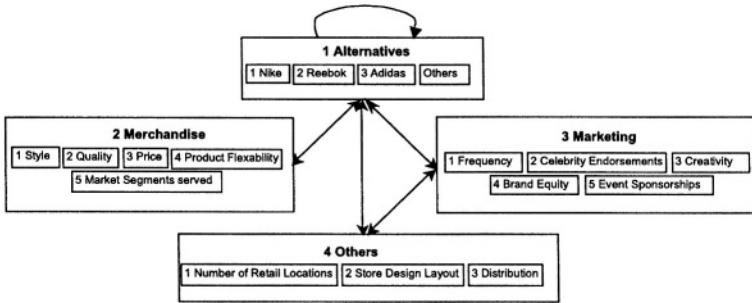

Figure 9.12.   The clusters and nodes of a model to estimate the relative market share of footware.

12.2.1   Clusters and Elements (Nodes).

1 Alternatives (brands competing against each other in the market)

   (a) Nike - Nike as an alternative brand for athletic footwear.

   (b) Reebok - Reebok as an alternative brand for athletic footwear.

   (c) Adidas - Adidas as an alternative brand for athletic footwear.

   (d) Others - Other alternative brands (And1, Skechers, New Balance, Timberland, etc) for athletic footwear.

2 Merchandise (affects each brand and each brand affects the type of merchandising strategy)

   (a) Style – the ability of a manufacturer to immediately respond to customers tastes and needs or create demand by introducing new products to the market.

   (b) Quality – Quality includes the reliability / durability of products including the ability to withstand pressure and frequent use.

   (c) Price – defined as value for money.

   (d) Product Flexibility – Ability of the product to substitute for other footwear, i.e. Running shoes can be used for casual wear and other purposes.

(e) Market Segments Served – Ability of the manufacturer to cover various target segments through their different product lines, i.e. men, women, children, basketball players, soccer players, etc.

3 Marketing (Marketing affects each of the brands and each brand affects the type of marketing strategy)

(a) Frequency – frequency of advertising regardless of media.

(b) Celebrity Endorsements – endorsement by a well-known popular sports celebrity.

(c) Creativity – Creativity of marketing advertisements regardless of length

(d) Brand Equity – Ability to create brand awareness and recognition among various segments of the market.

(e) Event Sponsorships-a marketing tool to advertise and create awareness for brand.

4 Others (Other factors affect the brand and each brand affects the type of strategy for these factors; also the Marketing strategy affects these factors)

(a) Number of retail locations – The number and the coverage of retail locations across the United States.

(b) Store design and layout – includes placement and effective layout of merchandise vis-à-vis competitors.

(c) Distribution – shelf space and coverage of merchandise across the United States. Includes relationships with distributors and even with own distribution chain.

Comparisons were done based on information gathered for each individual manufacturer. Advertising was determined due to factors such as each manufacturer's relative selling, general, and administrative expenses from their annual reports. Advertisements (mostly in print) were also viewed and use of celebrity endorsements in the same period were also assessed relative to each brand to measure creativity as well as frequency. Brand equity was measured on more intuitive terms i.e. Nike's Swoosh logo is considered as one of the most recognized logos and brands, which gave them an advantage over the other brands.

Other factors, such as the number of retail locations, were assessed by counting the total number of such locations (from individual websites). Store layout and distribution information were gathered from the websites as well to assess the relative effectiveness of each factor. For instance, Reebok and Adidas, have fewer individual stores than Nike (Factory outlets and Niketown) and tend to

be distributed in department stores or sporting goods stores facing more competition from other brands because of less exclusivity.

In terms of merchandise, prices are relatively the same for all brands although some like Adidas and Reebok may seem to be higher than other brands because of quality. Nike and other athletic footwear products tend to be more flexible in terms of how consumers use the products i.e. their basketball shoes are often substituted for casual wear and running shoes, which leads to a broader target segment. Also, Nike and the other brands seem to serve broader market segments specifically women and children. Their line extensions, e.g. Michael Jordan for men have been extended to children.

As more and more people substitute athletic footwear for everyday use, Nike and the other brands seem to be stronger in catering to this need thereby leading to more market share

Table 9.23 gives the actual and the estimated market share for each brand. They are surprisingly close. This example was done as a take home exercise. In this case the compatibility index obtained from the study is 1.001428, which is very small. We would be glad to provide the interested reader with at least a dozen such market share examples often worked out in class in about one hour without prior preparation or looking at numbers. They all have such close outcomes, because students, interested in the example, provided the judgments.

We now look at full blown decisions with their BOCR. First we give an outline of the steps recommended in applying the ANP.

13.   Outline of the Steps of the ANP

  1 Describe the decision problem in detail including its objectives, criteria and subcriteria, actors and their objectives and the possible outcomes of that decision. Give details of influences that determine how that decision may come out.

  2 Determine the control criteria and subcriteria in the four control hierarchies one each for the benefits, opportunities, costs and risks of that decision and obtain their priorities from paired comparisons matrices. If a control criterion or subcriterion has a global priority of 3% or less, you may consider carefully eliminating it from further consideration. The software automatically deals only with those criteria or subcriteria that have subnets under them. For benefits and opportunities, ask what gives the most benefits or presents the greatest opportunity to influence fulfillment of that control criterion. For costs and risks, ask what incurs the most cost or faces the greatest risk. Sometimes (very rarely), the comparisons are made simply in terms of benefits, opportunities, costs, and risks in the aggregate without using control criteria and subcriteria.

Table 9.23. Footwear actual statistics and model results along with the compatibility index.

Alternatives	A1	A2	A3	A4
Actual Market Share	39.200	15.100	10.900	34.800
Estimated Market Share from ANP Model	40.670	15.040	11.330	32.970

Pairwise Comparison Matrix from Actual Market Share Data

	A1	A2	A3	A4
A1	1	2.596026	3.59633	1.12643678
A2	0.385204082	1	1.385321	0.43390805
A3	0.278061224	0.721854	1	0.31321839
A4	0.887755102	2.304636	3.192661	1

Transpose of Comparison Matrix from Estimated Market Share

	A1	A2	A3	A4
A1	1	0.369806	0.278584	0.81067126
A2	2.70412234	1	0.753324	2.19215426
A3	3.589585172	1.327449	1	2.90997352
A4	1.233545648	0.456172	0.343646	1

Result of Hadamard (Element-wise) Multiplication of Previous Two Matrices

	A1	A2	A3	A4	Row Sums
A1	1	0.960026	1.001879	0.91316992	3.875075
A2	1.041638963	1	1.043596	0.95119337	4.036429
A3	0.998124448	0.958225	1	0.91145722	3.867807
A4	1.095086442	1.051311	1.097144	1	4.243542
				SUM =	16.02285

Number of Alternatives: $n = 4$
Compatibility Index = $(\text{SUM}/n^2) = 1.001428$

3 Determine the most general network of clusters (or components) and their elements that applies to all the control criteria. To better organize the development of the model as well as you can, number and arrange the clusters and their elements in a convenient way (perhaps in a column). Use the identical label to represent the same cluster and the same elements for all the control criteria.

4 For each control criterion or subcriterion, determine the clusters of the general feedback system with their elements and connect them according to their outer and inner dependence influences. An arrow is drawn from a cluster to any cluster whose elements influence it.

5 Determine the approach you want to follow in the analysis of each cluster or element, influencing (the preferred approach) other clusters and elements with respect to a criterion, or being influenced by other clusters

and elements. The sense (being influenced or influencing) must apply to all the criteria for the four control hierarchies for the entire decision.

6 For each control criterion, construct the supermatrix by laying out the clusters in the order they are numbered and all the elements in each cluster both vertically on the left and horizontally at the top. Enter in the appropriate position the priorities derived from the paired comparisons as subcolumns of the corresponding column of the supermatrix.

7 Perform paired comparisons on the elements within the clusters themselves according to their influence on each element in another cluster they are connected to (outer dependence) or on elements in their own cluster (inner dependence). In making comparisons, you must always have a criterion in mind. Comparisons of elements according to which element influences a given element more and how strongly more than another element it is compared with are made with a control criterion or subcriterion of the control hierarchy in mind.

8 Perform paired comparisons on the clusters as they influence each cluster to which they are connected with respect to the given control criterion. The derived weights are used to weight the elements of the corresponding column blocks of the supermatrix. Assign a zero when there is no influence. Thus obtain the weighted column stochastic supermatrix.

9 Compute the limit priorities of the stochastic supermatrix according to whether it is irreducible (primitive or imprimitive [cyclic]) or it is reducible with one being a simple or a multiple root and whether the system is cyclic or not. Two kinds of outcomes are possible. In the first all the columns of the matrix are identical and each gives the relative priorities of the elements from which the priorities of the elements in each cluster are normalized to one. In the second the limit cycles in blocks and the different limits are summed and averaged and again normalized to one for each cluster. Although the priority vectors are entered in the supermatrix in normalized form, the limit priorities are put in idealized form because the control criteria do not depend on the alternatives.

10 Synthesize the limiting priorities by weighting each idealized limit vector by the weight of its control criterion and adding the resulting vectors for each of the four merits: Benefits (B), Opportunities (O), Costs (C) and Risks (R). There are now four vectors, one for each of the four merits. An answer involving ratio values of the merits is obtained by forming the ratio BO/CR for each alternative from the four vectors. The alternative with the largest ratio is chosen for some decisions. Companies and individuals with limited resources often prefer this type of synthesis.

11 Determine strategic criteria and their priorities to rate the top ranked (ideal) alternative for each of the four merits one at a time. The synthesized ideals for all the control criteria under each merit may result in an ideal whose priority is less than one for that merit. Only an alternative that is ideal for all the control criteria under a merit receives the value one after synthesis for that merit. Normalize the four ratings thus obtained and use them to calculate the overall synthesis of the four vectors. For each alternative, subtract the sum of the weighted costs and risks from the sum of the weighted benefits and opportunities.

12 Perform sensitivity analysis on the final outcome. Sensitivity analysis is concerned with "what if kind of question to see if the final answer is stable to changes in the inputs whether judgments or priorities. Of special interest is to see if these changes change the order of the alternatives. How significant the change is can be measured with the Compatibility Index of the original outcome and each new outcome.

14.   Complex Decisions with Dependence and Feedback

With the China example for hierarchies and with the market share examples it is now easier to deal with complex decisions involving networks. For each of the four BOCR merits we have criteria (and subcriteria where relevant) called control criteria that are prioritized under that merit through paired comparisons. For each of the control criteria we create a network of influences with respect to that control criterion as we did in the market share examples. We obtain the ideal outcome ranking for each control criterion and then synthesize these outcomes by weighting by the importance of the control criteria for each merit. We then rate the top alternative under each merit to obtain the weights b,o,c and r for the BOCR and use them to synthesize and obtain the final weights for the alternatives using the two formulas BO/CR and more importantly, $bB + oO - cC - rR$. Let us sketch out an example using as little space as possible.

14.1   The National Missile Defense (NMD) Example

Not long ago, the United States government faced the crucial decision of whether or not to commit itself to the deployment of a National Missile Defense (NMD) system. Many experts in politics, the military, and academia had expressed different views regarding this decision. The most important rationale behind supporters of the NMD system was protecting the U.S. from potential threats said to come from countries such as North Korea, Iran and Iraq. According to the Central Intelligence Agency, North Korea's Taepo Dong long-range missile tests were successful, and it has been developing a second generation capable of reaching the U.S. Iran also tested its medium-range missile Shahab-3 in July 2000. Opponents expressed doubts about the technical feasibility, high

costs (estimated at $60 billion), political damage, possible arms race, and the exacerbation of foreign relations. The idea for the deployment of a ballistic missile defense system has been around since the late 1960s but the current plan for NMD originated with President Reagan's Strategic Defense Initiative (SDI) in the 1980s. SDI investigated technologies for destroying incoming missiles. The controversies surrounding the project were intensified with the National Missile Defense Act of 1996, introduced by Senator Sam Nunn (D-GA) in June 25, 1996. The bill required Congress to make a decision on whether the U.S. should deploy the NMD system by 2000. The bill also targeted the end of 2003 as the time for the U.S. to be capable of deploying NMD.

The ANP was applied to analyze this decision. It was done in the usual three steps of the ANP process: 1) the BOCR merits and their control criteria and subcriteria prioritized with respect to each merit, 2) the network of influence for each control criterion from which priorities for the alternatives are derived as in the market share examples and then synthesized using the weights of the control criteria for each merit and finally, 3) the use of strategic criteria as in Figure 9.13 to rate the merits one at a time as in Table 9.24 through their top alternative and use the resulting normalized ratings as priorities to weight and combine the priorities of each alternative with respect to the four merits to get the final answer.

On February 21, 2002 this author gave a half-day presentation on the subject to the National Defense University in Washington. In December 2002, President George W. Bush and his advisors decided to build the NMD. This study may have had no influence on the decision but still two years earlier (September 2000) it had arrived at the same decision produced by this analysis. The alternatives we considered for this analysis are: Deploy NMD, Global defense, R&D, Termination of the NMD program. Complete analysis of this example is given in the author's book on the ANP published in 2001. There were 23 criteria under the BOCR merits, including economic, terrorism, technological progress and everything else people were thinking about as important to develop or not to develop the NMD. After prioritization they were reduced to 9 control criteria for all four merits. Each criterion was treated in a very similar way to the single market share (essentially economic benefits) examples. Table 9.25 gives the final outcome. Here we see that the two formulas give the same outcome to deploy as the best alternative. The conclusion of this analysis is that pursuing the deployment of NMD is the best alternative. Sensitivity analysis indicates that the final ranks of the alternatives might change, but such change requires making extreme assumptions on the priorities of BOCR and of their corresponding control criteria.
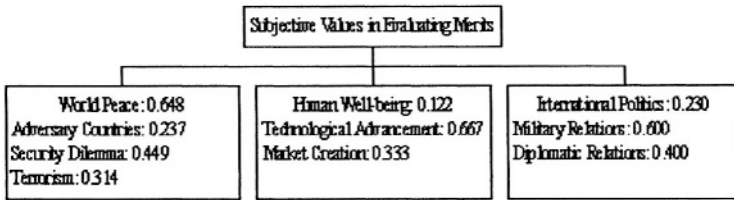
Figure 9.13. Hierarchy for rating benefits, opportunities, costs and risks.

Table 9.24. Priority ratings for the merits: Benefits, opportunities, costs and risks.
Very High (0.419), High (0.263), Medium (0.160), Low (0.097), Very Low (0.061)

		Benefits	Opportunities	Costs	Risks
World Peace	Adversary Countries	Very High	Medium	High	Very Low
	Security Dilemma	Very Low	Very Low	Very High	Very Low
	Terrorism	Medium	Very Low	High	High
Human Well-Being	Technological Advancement	High	High	Low	Very Low
	Market Creation	Medium	High	Very Low	Very Low
International Politics	Military Relations	High	High	Medium	Very Low
	Diplomatic Relations	Low	Low	Low	Very High
Priorities		b=0.264	o=0.185	c=0.361	r=0.190

15. Conclusions

Numerous other examples along with the software Super Decisions for the ANP can be obtained from www.superdecisions.com. We hope that the reader now has a good idea as to how to use the AHP/ANP in making a complex decision. The AHP and ANP have found application in practice by many companies and governments. My book Decision Making for Leaders is now in nearly 10 languages. Another recent policy study was done regarding whether the US should go to war with Iraq directly or through the UN done in September 2002. The analysis found that the US should go with the UN with priority more than double those of going alone or of going with a coalition. There is also the ongoing Middle East conflict. An ANP analysis showed that the best option is

Table 9.25.   Overall syntheses of the alternatives.

Alternatives	BO/CR (from unweighted columnns in Table 9.24)	Normalized	$bB + oO - cC - rR$ (from weighted columns in Table 9.24	Unitized*
Deploy	2.504	0.493	0.111	1.891
Global	1.921	0.379	0.059	1.000
R&D	0.560	0.110	-0.108	-1.830
Terminate	0.090	0.018	-0.278	-4.736

*Unitized means to divide each number in the column by the number with the smallest absolute value (it is recommended that one not unitize when such a number is close to zero).

for Israel and the US to help the Palestinians both set up a state and in particular achieve a viable economy. My forthcoming book The Encyclicon has about 100 summarized examples of applications of the ANP. A list of more than a thousand references until the early 1990's on the AHP appears in reference [3].

Acknowledgments

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