Chapter 19

MCDM LOCATION PROBLEMS

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Abstract In this chapter, we provide a broad overview of the most representative multi-criteria location problems as well as of the most relevant achievements in this field, indicating the relationship between them whenever possible. We consider a large number of references which have been classified in three sections depending on the type of decision space where the analyzed models are stated. Therefore, we distinguish between continuous, network, and discrete multicriteria location problems.

Keywords: Locational Analysis, multicriteria location problems, point-objective location problems, multiobjective location problems.

1. Introduction

Locational Analysis has become a very active field of research in the last decades. Since the seminal papers by Hakimi [68] and Witzgall [156] in the early sixties the number of researchers and publications have grown and Locational Analysis has become very popular among both practitioners and academia. The interested reader can find excellent surveys in the literature providing reviews of location references (among others [15, 42, 44, 63, 64, 93, 123]).

In this chapter, we present a survey of the most representative multicriteria location problems. Our goal is to give a broad overview of the different models and resolution procedures used in this field as well as to indicate how they relate to one another. Although we have not been exhaustive, we have tried to cover the most fruitful lines of research of Multicriteria Locational Analysis. Our hope is that this chapter will provide the readers with a helpful tool: the location analysts may complete their knowledge about the state of the art of their research field while the rest of the readers can find a comprehensive overview to introduce them to the main streams of this area.

Since our study focuses on multicriteria location problems, we proceed by giving a general formulation of this type of problems. To do so, we consider a family of possibly conflicting objective functions $F_i(\cdot)$ with $i \in I$. These functions represent different criteria to locate one or several new facilities and depend on the distances from these facilities to the set of fixed or demand facilities, usually called *A*. There are at least two natural ways of deriving the different F_i. First, a decision about a new facility to be located is typically a group decision and each decision maker i will have his own preferences, which my be expressed by F_i. Secondly, the functions F_i may represent different quality criteria for the new facility to be located, like cost, reachability, risk, etc. The general formulation is given by

$$\underset{X_p \subseteq S \subseteq X, |X_p|=p}{v-\min} (F_i(X_p, A))_{i \in I}, \qquad (19.1)$$

where $v-\min$ stands for vector minimization, *X* is the decision space, X_p is the finite set of service facilities, $|X_p|$ its cardinality, and *S* is the feasible region (see [51] for a survey of multicriteria optimization). The reader may note that the classical median and center problem in the literature of Locational Analysis are just particular aggregation procedures of the multiple criteria formulation in (19.1).

Problem (19.1) is a valid formulation for the general multifacility multicriteria problem. Nevertheless, a majority of the results published in the literature refers to the single facility case, $p = 1$. Therefore, in this chapter, the results will be generally referred to single facility models, unless the multifacility character is stated explicitly.

Formulation (19.1) corresponds to a multicriteria problem. Therefore, it is common to propose as solution concept different sets of feasible points that correspond to different levels of exigency regarding the ordering relationship between vectors. The most classical solution sets used in the literature are included in the following definition (for more details see for instance [50]).

DEFINITION 74 *Let X_p and X_p^* denote sets with cardinality p.*

i) $X_p^* \subseteq S$ is *a weakly efficient solution for Problem (19.1) if there is no $X_p \subseteq S$ such that $F_i(X_p, A) < F_i(X_p^*, A)$ for all $i \in I$.*

ii) $X_p^* \subseteq S$ *is an efficient or Pareto solution for Problem (19.1) if there is no $X_p \subseteq S$ such that $F_i(X_p, A) \leq F_i(X_p^*, A)$ for all $i \in I$ and it holds that $F_{i_o}(X_p, A) < F_{i_o}(X_p^*, A)$ for some $i_o \in I$.*

iii) $X_p^* \subseteq S$ *is a strictly efficient solution for Problem (19.1) if there is no $X_p^* \neq X_p \subseteq S$ such that $F_i(X_p, A) \leq F_i(X_p^*, A)$ for all $i \in I$.*

iv) $X_p^* \subseteq S$ *is a properly efficient solution for Problem (19.1) if it is an efficient solution and if there is a number $M > 0$ such that for all $i \in I$ and $X_p \subseteq S$ satisfying $F_i(X_p, A) \leq F_i(X_p^*, A)$ there exist $i_o \in I$ such that $F_{i_o}(X_p^*, A) < F_{i_o}(X_p, A)$ and moreover*

$$\frac{F_i(X_p^*, A) - F_i(X_p, A)}{F_{i_o}(X_p, A) - F_{i_o}(X_p^*, A)} \leq M.$$

Specific choices of solutions among the solution sets defined above have been suggested in the literature of Location Analysis. In the following definition we recall two of them that will be used later.

DEFINITION 75 *Let X_p and X_p^* denote sets with cardinality p.*

i) $X_p^* \subseteq S$ *is a lexicographic solution (or lex-optimal) if there exists a permutation π of the set I such that*

$$(F_{\pi(1)}(X_p^*, A), \ldots, F_{\pi(|I|)}(X_p^*, A)) \leq_{\text{lex}} (F_{\pi(1)}(X_p, A), \ldots, F_{\pi(|I|)}(X_p, A))$$

for all $X_p \subseteq S$, where $|I|$ is the cardinality of the set I and

$$z \leq_{\text{lex}} \bar{z} \quad :\Leftrightarrow \quad z = \bar{z} \textit{ or } z_{i_0} < \bar{z}_{i_0} \textit{ for } i_0 := \min\{i \in I \ : \ z_i \neq \bar{z}_i\}.$$

ii) $X_p^* \subseteq S$ *is a max-ordering solution if*

$$\max_{i \in I} F_i(X_p^*, A) \leq \max_{i \in I} F_i(X_p, A) \quad \textit{for all } X_p \subseteq S.$$

Our chapter is organized in five sections. After the introduction we present the standard models of location theory and describe their inherent multicriteria nature. The third, fourth and fifth sections are devoted to analyze the main models and results of continuous, network and discrete multicriteria location analysis, respectively. The chapter ends with the list of references cited in the text.

2. Location Problems

In order to establish a classification of the different problems of Locational Analysis, it is assumed that they can be divided into three branches: continuous, network and discrete location problems. Within each of these branches a further distinction can be made with respect to the number of new facilities, the distances used and pecularities such as forbidden regions. For more advanced classification schemes the reader is referred to [73].

An important characteristic of location models is their intrinsic multicriteria behavior. In any location problem with attractive criteria, every user wants to have the service as close as possible. Therefore, this behavior gives rise to a trade-off among users that leads to a multicritera formulation:

$$v\text{-}\min_{X_p \subseteq X,\, |X_p|=p} \Big(d(a, X_p) \Big)_{a\in A}, \tag{19.2}$$

being X the decision space, X_p the finite set of service facilities, $|X_p|$ its cardinality, $A \subset X$ the set of demand facilities, $\bar{d} : X \times X \longrightarrow \mathbb{R}_+$ the function used to measure the distances and

$$d(a, X_p) = \min_{x\in X_p} \bar{d}(a, x).$$

The formulation in (19.2) represents a new general trend in Operations Research. Considering more than one objective reflects better the actual world where usually several objectives, some of them in conflict, must be considered to model a problem. The reader can find excellent arguments justifying the multicriteria character of Locational Analysis and a detailed presentation of several aspects in [39].

In location theory many criteria have been used to locate one or several new facilities. However, median and center problems have attracted special attention of researchers for many years. The median problem has received different names in these years, as for instance, Fermat-Weber, Weber, Steiner or minisum problem, among others (see for instance [154] or Chapter 1 in [44]). This model uses as criterion to locate a new service the minimization of the average distances to all the users. The formulation of this problem, with the notation of Problem (19.2), is given by

$$\min_{X_p \subseteq X,\, |X_p|=p} \sum_{a\in A} w_a d(a, X_p), \tag{19.3}$$

where w_a is the weight associated to a. The median objective function is used in real world situations to locate a new facility minimizing the transportation costs. In a practical setting, the demand facilities represent customers or demands, the new facilities denote the unknown location of the servers, and the weighted distances are cost components associated with the interactions of flows between each new facility and its customers. For this model, we can find many applications cited in the literature involving communication network design, distribution centers, location and routing of robots, or the optimal location of utility and manufacturing plants, among others.

The center problem, which is also called the minimax problem or the problem of minimizing the eccentricity, uses as criterion to locate the new facilities the minimization of the largest distance supported by the users (see for instance [55]). Therefore, with the conventions of Problem (19.2), the center problem can be stated as

$$\min_{X_p \subseteq X, |X_p|=p} \max_{a \in A} u_a d(a, X_p), \tag{19.4}$$

where u_a is the weight associated to a. The minimax models may correspond to the social oriented notion of justice.

The median and center are the most frequently used criteria to locate new facilities. However, many real-world situations cannot be exactly modelled with one of these criteria. Indeed, since the median approach is based on averaging, it often provides solutions in which remote low-population density areas are discriminated in terms of accessibility. In the same sense, the center approach provides solutions where there may exist high population density areas with central locations, which have not been taken into account when locating the new facilities. However, to locate, for instance, a fire station, one goal may be to locate the station as close as possible to the farthest potential customer, while another goal would be to locate the station as close as possible to a majority of customers. Therefore, a possible approach to study this kind of situations is the cent-dian problem problem which consists of minimizing the convex combination of median and center objective functions, i.e.,

$$\min_{X_p \subseteq X, |X_p|=p} \lambda \sum_{a \in A} w_a d(a, X_p) + (1-\lambda) \max_{a \in A} u_a d(a, X_p) \tag{19.5}$$

with $\lambda \in [0, 1]$.

Notice that depending on the choice of λ, we are considering criteria more similar to the median objective function or to the center one, i.e., for λ close to 1 we are giving more importance to the averaged distances while for λ close to 0 we are giving more importance to the largest distance. Once more, we find in the cent-dian problem the intrinsic multicriteria nature of location problems. The analysis of the optimal solutions for varying λ coincides with the trade-off

Table 19.1. Summary of references.

	Continuous	*Networks*	*Discrete*
Surveys & Textbooks	[15, 39, 42, 44, 50, 51, 63, 64, 68, 93, 123, 154, 156]		
Point-Objective	[2, 26, 27, 28] [34, 37, 41, 43] [47, 48, 49, 61] [62, 79, 84, 86] [95, 101, 117] [118, 121, 122] [132, 140, 149] [150, 151, 152] [153, 155] *Constrained case:* [20, 22, 29] [105, 130, 131] *Majority rules:* [6, 19, 23, 24, 46]	[80]	[32]
Bicriteria Median & Center	[3, 21, 65] [81, 115, 129]	[69, 70, 71] [77, 78, 110] [119, 120, 136] *Extensive facility:* [4, 92, 100, 137]	[18]
Bicriteria Other	[10, 11, 88, 107]	[38, 60, 83] [87, 128, 139]	[104, 135]
Semiobnoxious	[14, 16, 25, 30] [31, 116, 134]	[75, 134]	[57]
Multicriteria Problems	[17, 33, 36, 35] [52, 59, 76, 106] [108, 109, 124] [125, 126] *Best aproximation:* [45, 54, 82, 138] [142, 143, 144] [145, 146, 147] [148] *Equity measurement:* [99, 113]	[74, 94, 127]	[13, 58, 90, 91] [111, 112, 114] [141, 157, 158] *Applications:* [1, 5, 7, 12] [53, 66, 67] [85, 89, 133]

analysis between the minisum and minimax solutions: the multicriteria analysis of the problem.

Although, many other different models have been considered in the literature, we have only described some of them because our purpose is to provide the reader a general overview that illustrates the use of different criteria to locate new facilities. For readers interested in location software a possibility is the public domain software LoLA (Library of Location Algorithms) [72]. LoLA contains several algorithms for multicriteria location problems in the plane and on networks.

We have summarized in Table 19.1 the references reviewed in this chapter.

3. Continuous Multicriteria Location Problems

In this section we give an annotated exposition of the literature on continuous multicriteria location problems. Before dealing with the references of this area, we recall the concept of a gauge which is a general function used to measure distances in continuous models. A gauge is a function defined with respect to a compact, convex set B containing the origin in its interior as:

$$\gamma_B(x) := \inf\{ r > 0 \, : \, x \in rB \}.$$

For instance, when B is the unit disk (ball) centered at 0, we have that $\gamma_B(\cdot) = \|\cdot\|_2$ (the Euclidean norm) or when B is a square of side two and centered at 0, we have that $\gamma_B(\cdot) = \|\cdot\|_\infty$ (the Tchebychev norm). We say that γ_B is: 1) a polyhedral or block gauge if B is a polytope, 2) a strictly convex gauge if B is a strictly convex set, 3) a norm if B is symmetric with respect to 0 and 4) a round norm if B is in addition a smooth set. Moreover, we denote by $co(A)$ to the convex hull of the set A, by $\bar{A}$ its topological closure and by $\mathrm{ri}(A)$ its relative interior.

The models analyzed in this section are organized in three subsections. The first one is devoted to study the point-objective location problem. The second subsection analyzes continuous bicriteria location problems and the last one considers multicriteria problems with more than two objective functions.

3.1 Point-objective Location Problems

The problem of locating one or several facilities to serve a certain number of demand facilities depends strongly on the criteria used to place such services. In order to obtain a general approach to this problem independently of the criterion, and having in mind that each demand facility wants to have the service as close as possible, the location problem is stated as follows:

$$\underset{X_p \subseteq S \subseteq X,\, |X_p| = p}{v-\min} \left(\min_{x \in X_p} \gamma_a(x - a) \right)_{a \in A} \tag{19.6}$$

where *S* is the feasible region, *A* is the set of demand facilities and $\gamma_a(\cdot)$ is the gauge associated to the demand facility a.

In location theory, Problem (19.6) is called Point-Objective location problem. This problem may be considered the first multicriteria model in location theory. The demand facilities may be communities that have to be served by some other facilities (fire houses, schools, hospitals, etc.) which have to be as close as possible. The distance to each demand facility a is measured by its corresponding gauge $\gamma_a(\cdot)$.

Because of the multiple objective nature of this problem, we are interested in the solution sets introduced in Definition 74. The final location is usually chosen from these sets in conjunction with other non-quantifiable criteria that the decision maker may have.

In this case, the different sets of efficient solutions of Definition 74 correspond to different level of exigency regarding the proximity to each demand facility. For Problem (19.6), the weakly efficient, efficient, strictly efficient and properly efficient sets are denoted by *WE*(*A, S*), *E*(*A, S*), *SE*(*A, S*) and *PE*(*A, S*) respectively. In the unconstrained case, i.e., *S* = *X*, these sets are denoted by *WE*(*A*), *E*(*A*), *SE*(*A*) and *PE*(*A*) respectively.

It is worth noting that the different solution sets, in addition of being considered as solution of the point-objective location problem, can be regarded as a global sensitivity analysis onto the weights of the solution set of the median problems with the same demand set. Hence, the first references that we overview do no state properly the formulation of the point-objective problem but the parametric analysis of weighted minisum problems. This fact is due to the scalarization results that establishes the relationship between the solution sets of a multicriteria problem and the set of minimizers of the weighted sum of their corresponding functions. In particular, if we denote by $M_\lambda(A)$, $\lambda = (\lambda_1, \dots, \lambda_{|A|}) \in \mathbb{R}^{|A|}$, the set of minimizers of the function $\sum_{a \in A} \lambda_a \gamma_a(x - a)$, we have (see [65] for the second statement)

$$x^* \in WE(A) \quad \text{if and only if} \quad \exists\, \lambda \in \mathbb{R}_+^{|A|} \text{ such that } x^* \in M_\lambda(A)$$
$$x^* \in PE(A) \quad \text{if and only if} \quad \exists\, \lambda \in \operatorname{int}(\mathbb{R}_+^{|A|}) \text{ such that } x^* \in M_\lambda(A).$$

In two dimensional space [49, 151] prove that $co(A) \cap M_\lambda(A) \neq \emptyset$, which implies that there exists at least one weakly efficient solution in the convex hull of A ($WE(A) \cap co(A) \neq \emptyset$). If a single block norm is used, [140] obtains that $E(A) \cap co(A) \cap IP \cap M_\lambda(A) \neq \emptyset$, where *IP* is the set of intersection points defined by the fundamental directions of the unit ball associated to the block norm starting at each demand point. In the case of mixed l_p**-norms** (different norms associated to each demand point), [79] obtains that the octogonal hull of the demand points has nonempty intersection with $M_\lambda(A)$. Later, [122] shows that this result fails for general norms as soon as the dimension of the space is at least three. [121] obtains that $E(A) = co(A)$ for: 1) any norm on the line, 2) any round norm on the plane and 3) any norm derived from inner product in spaces with finite dimension greater than two. [43] obtains that the smallest set which includes at least one point of $M_\lambda(A)$ is $\mathrm{int}(\mathrm{co}(A)) \cup A$ for l_p−norms with $1 < p < \infty$ and for $p = 1$ or ∞, this set is $SE(A) \cap IP$.

Compared with location problems in the plane, location problems on the sphere have received little attention in the literature. However, to model situations where the distances between the demand facilities and their corresponding servers are too long, it is necessary to take into account the spherical surface of the Earth. [2] extends the results of [151] to location problems on the surface of a sphere. In fact, they show that we can restrict ourselves to the spherically convex hull of the demand points to search a solution of the single facility median problem on the sphere if the demand points are not located entirely on a great circle arc. In addition, [41] obtains that if the demand points are located on a great circle arc, then the optimal solution is in this arc and some demand point is optimal.

For the multifacility case in a two dimensional space (with interaction), whatever the norm is, it holds that the optimal locations for all the new facilities can be found in *WE*(*A*), [101], and they belong to the convex hull of the existing facilities when l_p−norms $(1 < p < +\infty)$ are used, [62, 86].

In addition to the weighted sum approach, an alternative procedure to deal with the point-objective location problem and, in general, with a multicriteria problem is the ε−constrained method. Probably, this is among the most well-known techniques to solve multicriteria problems and it consists of the minimization of one of the original objective functions while the others are transformed to constraints, representing security or satisfaction levels that must be fulfilled by these criteria. For Problem (19.1), [84] studies properties of the optimal solutions for this kind of constrained problems involving generalizations of the median objective function.

Concerning the relationship between the different solution sets previously defined, we have that in general it holds that $SE(A,S) \subseteq E(A,S) \subseteq WE(A,S)$. In what follows we analyze these relationships for the unconstrained case and latter we will study the constrained one. In the case of a single gauge, that is, $\gamma_a = \gamma \; \forall a \in A$, it holds that $WE(A) = E(A) = SE(A) = \overline{co}(A)$ when γ is: 1) a round norm, [140]; 2) generated by a scalar product, [48]; or 3) strictly convex norm in a two dimensional real space with *A* being a bounded set, [48]. Besides, when γ is a strictly convex norm in a general real space, [48] proves that $WE(A) = E(A) = SE(A) = SE(\bar{A})$. In addition, when *A* is finite in $\mathbb{R}^n$, [95] proves that $WE(A) = \cup_{B \subseteq A} E(B)$. In the case of the Euclidean norm, [155] obtains that $E(A) = E(co(A)) = co(A)$ and for the l_1-norm that $E(A) = E(co(A))$.

Problem (19.6) also has some limit properties under particular hypotheses. In particular, when $X = \mathbb{R}^2$ and $\gamma_a = \gamma_n \; \forall a$ where $\{\gamma_n\}_{n\in\mathbb{N}}$ is a sequence of block norms approaching a round norm, we have that $WE_n(A)$, $E_n(A)$ and $SE_n(A)$ (the corresponding sets using γ_n) approach the convex hull of the demand facilities, [140].

Concerning topological properties, if A is bounded then $WE(A)$, $E(A)$ and $SE(A)$ are bounded. Moreover, $WE(A) = WE(\bar{A})$ and $E(A) = E(\bar{A})$ but not necessarily $SE(A) = SE(\bar{A})$. If A is compact then $SE(A)$ is closed. It holds that $WE(A)$, $SE(A)$ and $E(A)$ are weakly compact when X is an infinite dimensional, reflexive and strictly convex normed space with A being a compact set, [48].

Concerning geometrical characterizations, [48] gives a description of $WE(A)$, $E(A)$ and $SE(A)$ for A being compact, using recession cones in any arbitrary normed space. If we impose further A to be finite, in the rectilinear case the set of efficient solutions is a region enclosed by a boundary defined by horizontal and vertical lines throug each demand point, [37, 152, 153]. In $\mathbb{R}^n$ and mixed gauges (different gauges associated to each demand point) the set $WE(A)$ is characterized as the region enclosed by $\cup_{B\subseteq A, |B|\leq n} WE(B)$, [149], a similar result for the planar case is obtained in [132]. In finite dimension and mixed l_p−norms, [26, 27] show that the efficient set is a subset of the octogonal hull defined by the demand points (this result was also proved by [79] using a different methodology). In addition, they prove that under certain conditions these sets coincide. [28] obtains a similar result to those above for problems with polyhedral mixed norms. In fact, they propose a procedure to obtain a set containing the efficient solution set with certain properties of minimality, called pseudoefficient set.

Apart from the unconstrained case ($S = X$), there are also some results for the constrained models. The following references correspond to those models where the location decisions are restricted to a given set S. We assume in the following unless stated otherwise that S is convex and closed. This model is usually called constrained point-objective location problem. In this case, it holds that $WE(A, S) = E(A, S) = \text{proj}_S(\overline{co}(A))$, being A compact and $\text{proj}_S(\cdot)$ the projection operator using $\gamma(\cdot)$ onto S, whenever γ is: 1) strictly convex in a two dimensional space, [29]; 2) the Euclidean norm in $\mathbb{R}^n$ and A finite, [20]; or 3) generated by a scalar product in a two dimensional space, [105]. We can also find geometrical characterizations of constrained solution sets, using recession cones, in [105]; and a theoretical characterization of $WE(A, S)$ using the convex hull of subdifferentials in [29]. In addition, $PE(A, S) = A \cap S \cap \text{proj}_S(\text{ri}(co(A)))$ when a Euclidean norm in $\mathbb{R}^n$ is used and A is finite, [20]. The case where S is not necessarily convex but can be decomposed into a finite number of polyhedra was studied by [22]. On the plane with mixed gauges (different gauges associated to each demand point) one can find a complete geometrical description of the weakly efficient and efficient solution sets in [130, 131].

In addition to the theoretical results already presented there also exist several algorithms to compute some of these sets. In general the problem is very difficult and in many cases of enumerative nature. When total polyhedrality is given through block norms and linear constraints, the problem reduces to a multicriteria linear problem. Notice that even in this very easy case the general problem is already NP-hard. However, there are some particular cases where efficient algorithms exist. The set of efficient solutions using $l_1-\mathbf{norm}$ in two dimensional spaces was obtained by [152] with an algorithm based on generating the boundary of the set of efficient solutions. [34] presents a simple row algorithm based entirely on a geometrical analysis, that constructs all efficient solutions with complexity $O(|A| \log |A|)$. They also prove that no alternative algorithm can be of a lower order. [150] considers a simple schematic algorithm for characterizing the efficient solution set for the one-infinity norm. [117, 118] propose a polynomial algorithm for the case of polyhedral norms in the plane. Finally, for the case of polyhedral norms in $\mathbb{R}^n$, [47] presents a general method for determining, in a finite number of steps, the set of all efficient solutions. Besides, he states a geometrical characterization of properly efficient points which later is proved by [61] that only works on dimension one and two.

A different line of research is concerned with the use of majority rules in Locational Analysis. The relationship between Simpson decisions (those preferred by a majority of voters) and Pareto solutions is well-known among the researchers in voting theory. The application of these concepts to Locational Analysis was first given by [6, 46] for problems without locational constraints and later extended by [23, 24] to the constrained case. It is worth noting that this line of research offers interesting open problems, some of them already solved in [19].

3.2 Bicriteria Problems

For many practical situations it is sufficient to deal with two criteria. This allows to obtain a better knowledge of different solution sets and their properties. Most of the references dealing with this type of problems consider the median and the center or some modification of them as objective functions. Using the notation of Problem (19.2), this type of problems can be formulated as

$$\underset{X_p \subseteq S \subseteq X, |X_p|=p}{v-\min} \left(\sum_{a \in A} w_a d(a, X_p), \max_{a \in A} u_a d(a, X_p) \right), \tag{19.7}$$

where w_a and u_a are the weights associated with the demand facility a by the median and center criteria, respectively. Therefore, the first part of this subsection is devoted to analyze this kind of problems and the last part considers other bicriteria location problems.

In order to study Problem (19.7), we first notice that these two functions are convex, so by [65] the properly efficient solution set coincides with the set of minimizers of the convex combination of these two criteria, that is, the cent-dian problem, see (19.5). [21] proposes an axiomatic characterization of this criterion, which leads to an interpretation of the parameter $\lambda \in \mathbb{R}$ as a marginal rate of compensation.

The first reference we found that considers the bicriteria problem with median and center objectives is [81]. In this paper, the access cost of users is defined by a non-decreasing, continuous function of the distances which are measured by l_p-norms. For determining the set of efficient points, it provides a simple and practical approach based on the Big Square-Small Square method. Later, [3] shows that all the efficient solutions for Problem (19.7) can be obtained by solving constrained problems. These problems consist of minimizing the weighted sum of the distances so that the minimax function satisfies a varying upper bound. This result has the advantage that solving a constrained median problem is simpler than solving directly a cent-dian problem. In the plane, [115] studies the unweighted case with squared Euclidean distances and proposes a polynomial time algorithm to find the set of Pareto optimal locations based on the use of the farthest point Voronoi diagram.

The bicriteria problem with median and center objective functions in the presence of forbidden regions was considered in [129]. They use a direct search procedure based on Hooke and Jeeves algorithm to solve the rectangular norm planar location problem with forbidden regions, which is interesting because of its simplicity and versatility. A bicriteria location problem with a line barrier is considered in [88]. Their solution approach is based on [74].

The multifacility planar case (with interaction), using l_1-norm, is studied by [10]. They present a fuzzy goal programming model for locating new facilities in a region bounded by a convex polygon. Later, [11] proposes an interactive method to solve the problem above. In order to obtain a satisfactory solution for the Decision Maker (D-M), this procedure requires the D-M to know how much he/she can concede from the most satisfactory fuzzy goals at each current solution to improve the degree of satisfaction of other objectives.

Now that we have analyzed the bicriteria problems with median and center objective functions, we will start with the second part of this section where we study bicriteria problems where at least one the objective functions is none of them. [107] considers the bicriteria 2-Facility median problem using l_1-norm with interaction in $\mathbb{R}^d$ and gives a polynomial algorithm for determining all efficient locations. This algorithm is based on a discretization of the original continuous problem using geometrical and combinatorial arguments.

In a variety of practical settings the new facility to be located cannot be classified as being either purely desirable or obnoxious. These facilities falling somewhere between these two extremes are called semidesirable. As an illustrative example, consider the problem of locating a new chemical factory. For public safety, air pollution, and other reasons, such a facility should not be situated too close to populations centers. On the other hand, there are decreasing marginal benefits from locating the factory further away, because transportation cost for the users is steadily increasing. [25] presents a critical overview of the mathematical methods commonly used in semi-obnoxious facility location.

A common method to solve this kind of problems is to consider a bicriteria problem where each one of these objectives represents an attractive and a repulsive criterion respectively. [16] considers a bicriteria semidesirable location problem where the objective functions are the median criterion and the minimization of the weighted sum of Euclidean distances raised to a negative power. To solve the problem, they develop a heuristic method based on the computation of a trajectory determined by combining the first order necessary condition with the truncated Taylor series of the convex combination of these two criteria. Notice that this trajectory may not represent the complete set of efficient solutions. [30] considers a semi-obnoxious location problem where the objectives are the transportation and environmental costs. Since the usual solution set has, in general, infinite cardinality, they propose as solution a finite feasible set representing the best compromise solutions using the concept of α-**dominance.** Other applications of global optimization techniques to semi-obnoxious bicriteria location problems can be found in [31, 14]. On the other hand, [116] considers a semidesirable location problem using a bicriteria model with the center and anti-center (minimax) objective functions. He presents a geometrical characterization of the efficient set as well as the trade-off curve and develops a polynomial time algorithm for finding them. Finally, [134] considers planar bicriteria semi-obnoxious location problems where the importance of the obnoxious criterion with respect to the cost objective is not determined in advance.

3.3 Multicriteria Problems

As it was announced, the third subsection is devoted to the general case of multicriteria location problems where more than two objective functions are considered.

We start by mentioning the paper by [17]. It presents an axiomatic foundation of objective functions employed in multicriteria location theory that allows to characterize single objective reductions of these multicriteria problems. This procedure also simplifies the search of the Decision-Maker for suitable objective functions on the basis of desirable properties.

The first references that we consider in this section deal with the multicriteria median problem, that is, a multicriteria location problem where each of the involved objective functions is of the median type. This problem can be considered as the first actual multicriteria problem (more than two criteria not being distance functions) in continuous location. The formulation of this problem is as follows:

$$v - \min_{x\in X} \left(\sum_{a\in A} w_a^i \gamma(x-a) \right)_{i\in I},$$

where w_a^i is the weight associated to the demand facility a by the ith criterion. [33] studies this problem when the l_1-norm is used and develops a graphic-type algorithm that generates the set of all efficient solutions. [76] extends the analysis of this problem for the case of l_p-norms. In [106] the multicriteria median problem with polyhedral gauges is investigated. In addition, both papers also deal with the multicriteria center problem, that is, all the involved objective functions are of the center type. This problem can be formulated as

$$v - \min_{x\in X} \left(\max_{a\in A} u_a^i \gamma(x-a) \right)_{i\in I},$$

where u_a^i is the weight associated to the demand facility a by the ith criterion. For these two problems [76] analyzes the set of lexicographic locations, the set of Pareto locations and the set of max-ordering locations. A relationship between these three sets is established; and they develop efficient algorithms to compute the lexicographic location set for these two kinds of problems. Moreover, using the convex hull of their optimal solutions, they give a geometrical description of the set of efficient and properly efficient solutions for the case of median objectives with squared Euclidean norm. Finally, they develop an algorithm, based on a combinatorial approach, to compute efficient solution sets for the multicriteria median problem with l_1- and $l_\infty-$norm.

The multicriteria median problem with a general norm is studied in [125] which introduces the null vector condition for characterizing the set of properly efficient solutions. This condition is based on the computation of the cone generated by the subdifferentials of the functions considered in the multicriteria problem. They also analyze the relationship between the set of properly efficient solutions of this problem and the set of properly efficient solutions of the point-objective location problem defined by the demand points of the considered median objective functions. In the polyhedral case, they develop an algorithm to compute the set of efficient solutions with polynomial complexity. For the case of only one strict norm and assuming that the demand points are not collinear, [124] proves that the set of efficient solutions can be obtained as the limit of the set of weakly efficient solutions with a polyhedral gauge converging to the original strict norm.

A mixed version of the multicriteria median and center problem is analyzed by [52] which considers a multicriteria problem where all the objectives are either median or center ones. In particular, they characterize the set of max-ordering locations using the lexicographic and Pareto location sets. Three different strategies are proposed to find efficiently this set based on: 1) a direct approach, 2) the decision space approach, and 3) the objective space approach. Finally, they introduce the lexicographic max-ordering locations as a further specialization of max-ordering locations, which can be found efficiently.

One of the most general approaches to locate new facilities is the so called ordered median problem (see [108, 126]). Indeed, this criterion includes as particular instances the median, the center and the cent-dian problems among others. The multicriteria version of this problem with polyhedral gauges is studied in [109]. In this paper, the authors give geometrical characterizations of the set of efficient solutions and a polynomial time algorithm to compute it.

An alternative multicriteria location problem where neither center nor median objective functions are used, is proposed by [59]. In this paper, the authors consider the multicriteria minmax regret which combines the robustness approach using the minmax regret criterion together with Pareto-optimality. Its formulation is as follows:

$$\min_{x\in\mathbb{R}^2}\max_{w\in W}\sum_{a\in A} w_a\|x-a\|_2^2 - \sum_{a\in A} w_a\|x(w)-a\|_2^2,$$

where $W \subseteq \mathbb{R}^{|A|}$ and $x(w)$ is the optimal solution of $\min_{x\in\mathbb{R}^2}\sum_{a\in A} w_a \|x - a\|_2^2$. For the bicriteria case, the set of efficient locations is characterized as a particular set of line segments. Using this result the authors also give an algorithm for the general multicriteria case based on the solutions of bicriteria problems.

The important issue of equity measurement in Locational Analysis has also been modeled as a multiobjective problem. The interested reader can find a good review and a framework for this problem in [99]. A more recent approach as a multiobjective problem is given in [113]. For further details on this subject the reader is referred to [8, 9, 56, 98, 102, 103].

Another fruitful area of research in this field deals with the so called vectorial best approximation location problem. We are given two real linear spaces X, Y and a convex cone $C \subset Y$. We also consider a vectorial norm $|||\cdot|||$ being a mapping from X into C that satisfies for $x, \bar{x} \in X$ and $\lambda \in \mathbb{R}$:

i) $|||x||| = 0_Y$ (the null element in Y) if and only if $x = 0_X$,

ii) $|||\lambda x||| = |\lambda| |||x|||$,

iii) $|||x||| + |||\bar{x}||| - |||x+\bar{x}||| \in C$ ($|||x+\bar{x}||| \leq_C |||x||| + |||\bar{x}|||$).

The vectorial best approximation problem is

$$v - \min_{u \in V} |||x - u|||,$$

where $V \subset X$ is the feasible (constrained) set.

It is clear that most of the problems considered in the above sections fall into this very general formulation. In particular, the reader can check that considering, $X = (\mathbb{R}^n)^{|A|}$, $Y = \mathbb{R}^{|A|}$, $C = \mathbb{R}_+^{|A|}$, $|||z||| = (\|z^a\|_a)_{a \in A}$ being $z = (z^a)_{a \in A}$, $z^a \in \mathbb{R}^n$, $a \in A$; and $V = \{u \in X \;:\; u^a = u^b \quad \forall a, b \in A\}$, we get the so called point-objective problem.

Early references in the literature stating the relationships between multicriteria location problems and this general vectorial best approximation optimization problem can be found in [45, 142, 143, 144, 145].

The interested reader will find very important results scanning this line of research. However, they are scattered in journals that are hardly considered by location analysts (locators).

In [45] several topological properties as well as a geometrical description of the set of vectorial best approximants is given. On the other hand, [144, 145] emphasize more the conditions of the Kolmogorov type that characterize weakly and properly efficient solutions of the vectorial approximation problem.

Another topic pursued by the authors in this field is the use of general duality results characterizing the different notions of efficiency. In [142, 143, 148] the reader can find characterizations using duality under different hypotheses, with their corresponding applications to multicriteria location problems.

We also want to recall the concept of ε-optimality in vectorial approximation location problems. Without entering the details of this concept, we would like to mention at least that powerful results are known. The results are based on a generalization of Ekeland's variational principle (see [54]) for vector approximation problems (see [138]) that has been later applied to get results in approximating efficient solution sets [82, 147, 146]. The interested reader can find all the details in the references above and those cited therein.

We finish this section by mentioning a different multicriteria location problem. The goal is to find efficient designs (shapes) for a given area provided that disutilities for the users are known. [35, 36] study this problem and give necessary and sufficient conditions for a design to be efficient.

4. Multicriteria Network Location Problems

In a general network location problem, one or several facilities are to be placed in a graph optimizing a function of the distances between these facilities and the set of demand facilities located in the graph. Therefore the main difference with respect to the continuous problem is that the decision space is a network. This fact provides many intrinsic peculiarities both in the theoretical and practical point of view. In particular, this kind of models adapt better to some specific real world situations, as for instance road networks, power lines, etc. This justifies that many efforts have been devoted to improve the performance of facility systems to deal with network location problems. These problems can be classified depending on the graph structure (general graph, trees, etc.), the type of objective function (center, median, etc.) or the number of objective functions considered (single criterion or multicriteria problem).

We start introducing some basic notation to understand the formulation of this type of problems. Let $N = (G, l)$ denote a network with underlying graph $G = (V, E)$, where the node set is V (demand points) and the edge set is E. Therefore, we write the edge that joins the nodes v and v' as $[v, v']$.

The *length* of an edge $e \in E$ is denoted by $l(e) = l(v, v')$ and it represents the cost of going once through the edge to satisfy the demand of one user. By $\bar{d}(v, v')$, we denote the length of the shortest path between v and v' measured by l.

A point x on an edge $e = [v, v']$ is determined by a value t, $0 \leq t \leq l(e)$, which represents the length of the proportion of the edge between x and v, the point x is then denoted by $x = p(e, t) = p([v, v'], t)$. Hence, for instance in the case of an undirected graph, the distance from this point to another node v_k is:

$$\bar{d}(x, v'') := \bar{d}(v'', x) := \min\{\bar{d}(v'', v) + t \ , \ \bar{d}(v'', v') + l(e) - t\}.$$

Notice that the function $d(\cdot, v)$ for any $v \in V$ is concave over each edge of the graph, in fact, it is a concave piecewise linear function. Besides, if the graph is a tree, this function is convex over paths what implies that the sum of the distances from x to each node is a convex function over each path of the tree. This property allows to apply results of convex analysis to the resolution of location problems stated on a tree graph.

The set of all the points of a network (G, l) is denoted by $P(G)$. It should be noted that this set also contains the node set. Therefore, in order to locate p service facilities, we have to consider the distance from a node to a set of p points, $X_p \subseteq P(G)$, as

$$d(v, X_p) = \min_{x \in X_p} \bar{d}(v, x).$$

In order to present the references considering networks multicriteria location problems, analogously to the continuous case, we have divided this section in two subsections. In the first one, we consider the case of two criteria, i.e., bicriteria problems and in the second one, we deal with the general case where more than two objective functions are used.

4.1 Bicriteria Problems

In this subsection we analyze bicriteria location problems on networks. The most popular models are those with median and center objective functions. Similar to the continuous case this problem is

$$\underset{X_p \subseteq P(G),\, |X_p|=p}{v-\min} \left(\sum_{v \in V} w_v d(v, X_p), \max_{v \in V} u_v d(v, X_p) \right), \tag{19.8}$$

where w_v and u_v are the weights associated to the demand facility v by the median and center criteria, respectively. We start with the analysis of these problems.

A first method to handle this bicriteria problem is to transform it into a single objective problem via scalarization. This can be mathematically expressed by the minimization of different single objective functions as: cent-dian, generalized center or medi-center, among others. The cent-dian objective function, already defined in (19.5), was introduced by [69], who coined the term cent-dian for the point of a graph that minimizes the convex combination of the center and median functions. On the second hand, the generalized center objective, introduced and studied by [78], minimizes the difference between the center and the median functions,

$$\min_{X_p \subseteq P(G),\, |X_p|=p} \left\{ \max_{v \in V} u_v d(v, X_p) - \sum_{v \in V} w_v d(v, X_p) \right\}.$$

This criterion allows to deal with distributional justice considerations in the access to the facilities and corresponds to an aggregation procedure of semi-obnoxious location problems (center, anti-median).

Finally, the medi-center problem, considered in [71], minimizes one criterion subject to a restriction on the value of the other:

$$\begin{aligned} \min \quad & \max_{v \in V} u_v d(v, X_p) \\ \text{s.t.} \quad & \sum_{v \in V} w_v d(v, X_p) \le \mu \\ & X_p \subseteq P(G), \ |X_p| = p \end{aligned} \tag{19.9}$$

or

$$\begin{aligned} \min \quad & \sum_{v \in V} w_v d(v, X_p) \\ \text{s.t.} \quad & \max_{v \in V} u_v d(v, X_p) \le \mu', \\ & X_p \subseteq P(G), \ |X_p| = p \end{aligned} \tag{19.10}$$

where μ and μ' are upper bounds to the median and center objectives respectively.

The first approach that we consider in order to deal with a bicriteria location problem with center and median objective functions, is the parametric analysis of the cent-dian problem. This parametric analysis is very informative in a general network, however it does not provide a complete characterization of the efficient solution set. It is due to the non-convexity of these objective functions (recall that distances are concave). For the particular case oftree networks and due to the convexity properties of this case, this parametric analysis gives the whole set of efficient solutions, similarly to the continuous problems.

In any case, the solution set of the cent-dian problem for any parameter defining the convex combination of the center and median objectives is included in the set of efficient solutions of the bicriteria problem defined by these two criteria. Thus, its characterization continues to be interesting from multicriteria point of view. [69] shows that the cent-dian of a tree has the attractive property of being located either at the center of a tree or at a vertex on the path connecting the center and a median. Unfortunately, a cent-dian of a general graph does not satisfy this property in general. [70] proposes a procedure based on the computation of an upper bound to identify a cent-dian of an undirected graph; and traces its location as it moves from a graph median to its center as the weight of the latter objective is increased and of the former is decreased.

Since a one-to-one correspondence between cent-dian or generalized center solutions and efficient solutions does not exist, [110] analyzes a different solution concept for this bicriteria problem that provides some compromise between them. This is the Tchebychev cent-dian solution which is the set oflexicographic solutions of a bicriteria problem with the cent-dian and the weighted Tchebychev norm of the center and median criteria. This new solution concept allows to identify all Pareto locations on any network by means of a parametric analysis. Besides, he proposes an algorithm to generate the set of Tchebychev cent-dian solutions.

The models above only consider the case of locating one facility, however we can find situations where more than one facility is required. Hence, we analyze the **p-faciltiy** case, where the goal is to locate p points on the network so that the demand of the given facilities is covered by the closest new facility,

$$\min_{X_p \subseteq P(G),\, |X_p|=p} \left(\lambda \sum_{v \in V} w_v d(v, X_p) + (1-\lambda) \max_{v \in V} u_v d(v, X_p) \right).$$

[119] studies the unweighted p-facility cent-dian network location problem. They give a finite dominating set and also provide a solution method that solves this problem based on an exhaustive search in the set of all combinations of p points within this finite dominating set. For the case, $p = 2$, [120] provides a

different algorithm based on an exhaustive search that solves the problem with complexity $O(|V|^2)$.

[136] considers the weighted **p-cent-dian** problem on tree networks. The authors identify a set of points of polynomial size which is guaranteed to contain an optimal solution. Then, they exploit some convexity properties to develop a $O(|V|\log|V|)$ time algorithm that solves this problem.

The generalized center problem is introduced in [78], which includes an algorithm to solve this problem. Moreover, this paper provides a complete characterization of the cent-dian problem in the case of a tree. For the case of a general network, the authors present a new algorithm to find the set of cent-dian solutions which is conceptually much simpler than that developed by [70], although with the same computational complexity. This algorithm is based on the computation of the lower envelope of the bottleneck points and local minima of the median and maximum distance objective functions on the image space.

The third approach that we are looking at to study Problem (19.8) considers the medi-center problems, see (19.9) and (19.10). [71] analyzes this bicriteria location problem on general undirected graphs using the cent-dian problem and two medi-center problems. Between these two medi-center problems, a duality relation is stated, where solving one problem is equivalent to solving the other one when the upper bounds defining the constraints correspond to each other in a definite way. [71] presents a procedure for the identification of all efficient solutions based on solving only one of the two constrained problems. Finally, the author shows that the cent-dian problem is in some sense a special case of these medi-center problems since its solutions correspond to the extreme points of the solution set of a medi-center problem when the upper bounds in the constraint vary. [77] considers a medi-center problem, which minimizes the average travel time subject to the constraint that no individual response will be more than a determined number of time units long. Efficient algorithms are developed for locating a single facility on a tree. This efficiency is again due to the fundamental convexity characteristic for the distance measures on trees.

Most of the models dealing with network location problems use points to represent the facilities to be located. However, there are circumstances where these facilities cannot be modelled by points on a network, as for instance the problem of locating railroad lines, highways, transit routes, pipelines, etc. In order to solve these situations, some models have been developed where the goal is to locate an extensive facility (see [100] for a survey of this type of problems). In particular, one can find some papers in the literature considering multicriteria problems with path or tree shaped facilities.

Locating a path using the cent-dian criterion can be formulated as:

$$
\begin{array}{ll}
\min & \lambda \sum_{a\in A} w_a d(a,P) + (1-\lambda)\max_{a\in A} u_a d(a,P) \\
\text{s.t.} & d(v,P) = \min_{x\in P} d(v,x) \\
 & l(P) \leq L \\
 & P \subseteq P(G) \quad \text{being a path.}
\end{array}
$$

where $l(P)$ is the length of the path P and L its upper bound. The case of a tree shaped facility can be formulated in a similar way by considering this shape instead of a path. For the case of a path, [92] gives a complete characterization of the cent-dian function for tree graphs. To solve this problem, [4] proposes an efficient algorithm based on dynamic programming. For the case of a subtree, [137] presents an algorithm to find an optimal solution based on two facts: 1) that the point solution for the cent-dian problem belongs to an optimal subtree; and 2) the characterization of a finite set of breakpoints of the considered objective function. Its overall complexity is $O(|V|\log|V|)$.

After the analysis of the references considering bicriteria location problems with median and center objective functions we will study other bicriteria models in the second part of this section.

[139] considers a biobjective multifacilily minimax location problem on a tree network, which involves as objectives the maximum of the weighted distances between specified pairs of new and existing facilities, and the maximum of the weighted distances between specified pairs of new facilities. They develop an algorithm for constructing the efficient frontier and also provide a general result which gives necessary and sufficient conditions for a location vector to be efficient.

The problem of determining the absolute center of a network with two objective functions was studied by [128]. The authors consider a bicriteria problem where the objective functions are two center criteria using independent lengths on each edge. The problem is solved by a polynomial time algorithm based on [87].

The minimization of the superior section in a graph consists of finding the path, such that, the edge with the longest length is minimum. Applications can be found for instance in transportation of hazardous materials, where the weight associated to each edge is the risk of accident on that edge. [60] considers the bicriteria location problem of locating a path on a tree with respect to the minimization of the eccentricity or farthest distance and the superior section. They propose an algorithm that obtains all the efficient paths with complexity $O(|V|^3)$ based on two results: 1) on paths, the superior section function is a maximum function over the edge lengths and it may use a progressive reduction of the original tree and 2) there exist linear time algorithms to find path centers on trees. Moreover, they propose modifications of this algorithm that can be applied to a variety of bicriteria path problems on trees where one of the objective functions is the superior section.

The balance criterion is an equity objective function defined as the difference of the distance from the service facility to the farthest and to the nearest demand point. This model is induced by the situation when e.g. according to a designed schedule or by some equity reason, the distances to the facility are to be as balanced as possible. [83] studies a bicriteria location problem with the center and balance criterion. The set of efficient solutions for this problem is generated by minimizing a constrained problem, namely the center objective function subject to an upper bound on the balance objective.

At the beginning of this section we have considered the cent-dian objective function as an approach to deal with bicriteria location problems with center and median objectives. However, [38] considers two cent-dian objective functions in a bicriteria location problem on a network, where one function minimizes the distance and the other one minimizes the cost. The efficient solutions of this problem are derived by a polynomial algorithm based on computational geometry.

4.2 Multicriteria Problems

Considering more than two objective functions implies that several methods very useful in bicriteria problems, as for instance those based on projections onto the image space in order to find the efficient solution set, are useless. Hence, different techniques are needed to deal with these problems.

The point-objective location problem in networks is considered in [80]. The authors give a polynomial time algorithm for the set of efficient points on a general network and a linear time algorithm for the problem on trees.

In the case of tree networks, [94] considers a more general multicriteria location problem where each objective function is a continuous convex function constrained to a compact set. He characterizes the set of efficient solutions as a subtree delimited by the optimal solutions of each criteria. Besides, he provides a procedure for determining such a set. Finally, extensions to the case of non-convex feasible regions are analyzed.

The single facility multicriteria median problem on networks can be formulated as follows:

$$v-\min_{x\in P(G)}\left(\sum_{v\in V} w_v^i \bar{d}(v,x)\right)_{i\in I},$$

where w_v^i is the weight associated to the demand facility v by the ith criterion. Due to the non-convexity of this problem searching the efficient solutions is not restricted to a specific part of the network but rather it should be extended to all its edges, [127]. They develop a polynomial time algorithm to determine the efficient solution set. The procedure, first, determines the distance function for each objective with their corresponding breakpoints and then removes edges according to a simple rule.

Also for this problem, [74] develops a polynomial time algorithm to find the lexicographic and efficient solutions. The complexity of the algorithm is considerably improved for the case of tree graphs. The analysis in this case is based heavily on the partition of the objective function into subedges. For the case of lexicographical solutions, it reduces the search over a finite set of vectors. For the Pareto locations case, a procedure based on two stages is developed. In the first stage, the set of efficient points on an edge is obtained, while in the second, these points are tested for global domination.

In multicriteria network location analysis, we can also find models considering the location of a semi-obnoxious facility. [75] presents different models using criteria of median type with positive and negative weights. To solve these problems, they propose efficient algorithms based on the methodology used by [74]. These results are extended to models with maximin and minimax objectives. Recently, also ε-approximated solutions to the semi-obnoxious location problem have been discussed in [134].

5. Multicriteria Discrete Location Problems

Discrete location models consider the problem of determining where to locate one or several facilities within a finite set of given potential places to cover the demand of a region. Therefore, the mathematical formulations of these models mainly rely on (mixed) integer programming. Thus, one of the most classical models in this area, namely the $p-$median can be formulated as:

$$\begin{array}{ll} \min & \sum_{x\in PL\subseteq X}\sum_{a\in A} w_a d(a,x) y_{a,x} \\ \text{s.t.} & \sum_{x\in PL} y_{a,x} = 1 \quad \forall a\in A \\ & z_x - y_{a,x} \geq 0 \quad \forall x \in PL \\ & \sum_{x\in PL} z_x = p \\ & z_x, y_{a,x} \in \{0,1\} \quad \forall a\in A \text{ and } \forall x \in PL, \end{array}$$

where PL represents the finite set of potential locations for the service facilities. It should be mentioned the narrow relationship between discrete and networks location problems, especially when the latter is restricted to the vertex locations, [96, 97]. Multicriteria discrete location problems add to the above models the consideration of several criteria to be optimized simultaneously. The reader can find an introduction to these models in Chapter 8 of [40]. Depending on the different criteria used to locate the new facilities we can find a large variety of models in this field of location theory. This fact makes a difference between continuous or network multicriteria location problems and multicriteria discrete ones. In the former, most of the papers deal with specific problems (median, center, cent-dian,...) and focus on theoretical results. In the latter, the effort is put more on applications than on methodological results. As a consequence our presentation in this section does not classify the papers by areas. To review this material, we mainly follow a chronological scheme combined with a description of the most used techniques.

A possible approach to solve this type of problems are the interactive algorithms. These procedures help the D-M to explore and analyze his/her preferences in conjunction with an exploration of the set of feasible solutions. In other words, it combines what is desirable with a consideration of what is possible. [133] elaborates a specific interactive algorithm for a multicriteria location problem involving public facilities. Besides they give arguments showing that practical problems involving the location of public facilities are really multicriteria problems. [114] considers an interactive procedure which is an extension of the classical reference point approach to solve various multicriteria transshipment problems with facility location. In this new approach, the decision maker forms his/her requirements in terms of aspiration and reservation levels, i.e., he/she specifies acceptable and required values for the given objectives. [112] develops an interactive process that generates the solutions belonging to the symmetrically efficient set which is applied to discrete location problems. Notice that symmetric efficiency is a new solution concept based on the principle of impartiality, i.e., on the assumption that any permutation of the achievement vector is equally good as the original achievement vector.

A second approach to deal with multicriteria discrete location problems is goal programming. It is a very valuable tool, since it gives the D-M the opportunity to include many aspects of problems that usually are not included by other methodologies (e.g. quality of life, compliance with states laws, etc.). In what follows, we present four references that have used this procedure to solve multicriteria discrete location problems. [7] considers the location and size of day nurseries within a town by means of a multicriteria discrete model and its solution consists of finding a compromise among three conflicting objectives which represent educational needs, accessibility and budget considerations. [90] applies a branch and bound integer goal programming approach to a multicriteria location-allocation problem. [5] describes a model for evaluating and determining locations of fire stations. The model considers multiple objectives that incorporate both travel times and travel distances from stations to demand sites. [66] considers the problem of locating disposal or treatment centres and routing hazardous wastes through an underlying transportation network. The considered objectives are: minimization of total operating cost, minimization of total perceived risk, minimization of maximum individual risk and minimization of maximum individual disutility. In order to solve the problem, the author shows how monotonically increasing penalty functions can be used to obtain more satisfactory solutions. Location of waste disposals of several materials have also been addressed using other multicriteria techniques as in [1, 67, 89].

The third resolution method that we analyze are the enumeration procedures. [13] develops an implicit enumeration algorithm to determine the set of efficient points in zero-one multiple criteria problems. The algorithm is specialized on the solution of a particular class of facility location problems. The procedure is complemented with the use of the utility function of the decision maker to identify a subset of efficient candidates for the final selection. [57] applies this resolution method to a multicriteria model for locating one or more undesirable facilities to service a region. The objectives are to minimize the total cost of the facilities located, the total opposition to the facilities, and the maximun disutility imposed on any individual. Opposition and disutility are assumed to be nonlinearly decreasing functions of distances, and increasing functions of facilities size.

The point-objective location problem has also been considered in the discrete case. [32] studies the discrete version of this model with rectilinear distances and develops an enumerative algorithm that checks efficiency for each one of the candidate sites.

There also exist results that establish the relationship between the efficient solution set of a bicriteria (median-center) problem and the solution set of a single criterion problem resulting from the combination of both objective functions. In particular, the parametric analysis of the cent-dian problem only gives a subset of the efficient solutions of the considered problem. However, a modification of the cent-dian problem allows to obtain a criterion whose parametric analysis provides the whole set of efficient solutions, [18]. In addition, this paper suggests a solution procedure for the cent-dian problem.

[157, 158] employ five newly developed multiple attribute decision making methods for different versions of the manufacturing plant site selection problem. They consider the single plant strategy with qualitative and quantitative data and cover the multiplant strategy with budget constraints and relocation strategies.

An algorithm for generating an approximate representation of the efficient solutions in biobjective problems which are modeled as mixed integer linear programs is developed by [135]. A geometrical measure of the error is given to assure that the deviation of the approximation from the exact solution set is within a maximum allowable error. The author illustrates the algorithm with a biobjective model which seeks to locate p facilities in a set of potential facility sites to maximize the objectives of single coverage and multiple coverage over a set of demand points.

[91] proposes a facility site selection algorithm. Since in facility site selection it is common to find imprecise assessments of alternatives versus criteria as well as weighting factors, the conventional quantitative approaches may not be applicable. The paper suggests the application of the hierarchical structure analysis to aggregate the decision maker's linguistic assessments about weighting factors

and the suitability of facility sites. This procedure allows the decision-makers to obtain the final ranking of the alternatives automatically.

A general approach to consider multicriteria problems is to apply weights to the criteria to obtain overall scores for the purposes of simplifying the comparison. DEA (Data Envelopment Analysis) is an interesting and non-subjective method for obtaining weights. [141] presents an application of DEA called "profiling" in order to assist in the choice of a location for a particular facility when various criteria are considered. This application provides much greater discrimination than conventional DEA which greatly eases the site selection process.

The classical uncapacitated facility location has also been analyzed from a multicriteria point of view. In particular, [104] considers a biobjective model for this problem where one objective is to maximize the net profit and the other to maximize the profitability of the investment. To solve the problem, they develop a heuristic procedure to generate the efficient solutions which has computational advantages over existing methods. On the other hand, [58] presents the multicriteria version of this problem (where each objective represents a different scenario) and develops two approaches to obtain the set of efficient solutions based on the decomposition of the problem into two nested subproblems and the use of multicriteria dynamic programming.

[111] develops the concept of the lexicographic minimax solution (lexicographic center) being a refinement of the standard minimax approach to location problems. It is shown that the lexicographic minimax approach complies with both the Pareto-optimality (efficient) principle (crucial in multiple criteria optimization) and the principle of transfers (essential for equity measures) whereas the standard approach may violate both these principles. Computational algorithms are developed for the lexicographic minimax solution of discrete location problems.

An application of multicriteria discrete location analysis consists of locating regional service offices in the expanded operating territories of a large property and liability insurer. These offices serve as first line administrative centers for sales support and claims processing. For solving this real situation, [12] proposes a zero-one linear multicriteria programming formulation where the criteria and constraints of the model reflect investment and operating cost, budget considerations and a measure of the service level provided. The reader can also find another application of multiobjective integer programming to spatial decision for housing mobility planning in [85]. In addition, [53] gives an analysis of a part of the distribution system of the company BASF AG, which involves the construction of warehouses at various locations. The authors evaluate 14 different scenarios and each one of these scenarios is evaluated with the minimal cost solution obtained through linear programming and the resulting average delivery time at this particular solution. It is illustrated that a bicriteria analysis is certainly superior to a decision based on the cost or the service criterion alone.

6. Conclusions

We have shown in this chapter that location problems are multicriteria by their own nature. Location decisions are typically group decisions and different quality criteria have to be taken into account. The three main areas of location problems have been reviewed: continuous, network and discrete location problems. When looking at the references discussed, one can easily see that still many interesting open problems remain. In the continuous as well as the network cases multifacility problems are not adequately treated yet. Also the development of efficient algorithms is still in an early stage.

Moreover, the location of new facilities conditioned to the existence of other facilities that have already been located (conditional problems) have attracted the attention of researchers in Locational Analysis. Thus, this kind of problems opens a future avenue of research in the multicriteria case. Although some references have dealt with nonconvex problems, they only consider particular situations. The study of general models is another open line of research. Nonconvexities in the objective function may be modelled by the ordered median function that has been proven to be very useful in different problems of Locational Analysis.

For discrete problems a more systematic treatment of the different problem types is missing. For all three areas there is nearly no software available. Summing up, we can conclude that although an amazing number of publications dealing with multicriteria location problems is around, a lot of work is still waiting for the research community.

Acknowledgments

The research of the second and third authors is partially supported by Spanish research grants BMF2001-2378 and BMF2001-4028.

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VII

APPLICATIONS