Chapter 10

ON THE MATHEMATICAL FOUNDATIONS OF MACBETH

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Abstract MACBETH (Measuring Attractiveness by a Categorical Based Evaluation Technique) is a multicriteria decision analysis approach that requires only qualitative judgements about differences of value to help an individual or a group quantify the relative attractiveness of options. This chapter presents an up-to-date survey of the mathematical foundations of MACBETH. Reference is also made to real-world applications and an extensive bibliography, spanning back to the early 1990's, is provided.

Keywords: MACBETH, questioning procedure, qualitative judgements, judgmental inconsistency, cardinal value measurement, interaction.

1. Introduction

Let X (with $\#X = n \geq 2$) be a finite set of elements (alternatives, choice options, courses of action) that an individual or a group, *J*, wants to compare in terms of their relative attractiveness (desirability, value).

Ordinal value scales (defined on *X*) are quantitative representations of preferences that reflect, numerically, the order of attractiveness of the elements of *X* for *J*. The construction of an ordinal value scale is a straightforward process, provided that *J* is able to rank the elements of *X* by order of attractiveness – either directly or through pairwise comparisons of the elements to determine their relative attractiveness. Once the ranking is defined, one needs only to assign a real number $v(x)$ to each element x of *X*, in such a way that:

1 $v(x) = v(y)$ if and only if *J* judges the elements x and y to be equally attractive;

2 $v(x) > v(y)$ if and only if *J* judges x to be more attractive than y.

The problem, however, is that, in a multiple criteria decision analysis, conclusions based on a additive value model may be quantitatively meaningless, because "to be quantitatively meaningful a statement should be unaffected by admissible transformations of all the quantities involved." [53, p. 91]. A necessary condition is that each value scale should be unique up to a positive affine transformation (an interval scale), as it is with a value difference scale. A value difference scale (defined on *X*) is a quantitative representation of preferences that is used to reflect, not only the order of attractiveness of the elements of *X* for *J*, but also the differences of their relative attractiveness, or in other words, the strength of *J*'s preferences for one element over another. Unfortunately, the construction of an interval value scale is usually a difficult task.

Both numerical and non-numerical techniques have been proposed and used to build a value difference scale (hereafter, simply called a value scale) – see [51] for a survey. Examples of numerical techniques are direct rating and difference methods – see descriptions in [61, 62] and [41]. They require *J* to be able to produce, either directly or indirectly, numerical representations of his or her strengths of preferences, which is not a natural cognitive task. Non-numerical techniques, such as the bisection method (also described by the same authors), are based on indifference judgements, forcing *J* to compare his or her strengths of preferences between two pairs of elements of *X*, therefore involving at least three different elements in each judgement. This requires *J* to perform an intensive cognitive task and is prone to be substantively meaningless – "substantive meaningfulness (...) requires that the qualitative relations (...) being modelled should be unambiguously understood by the decision maker." [53, p. 91].

The aforementioned difficulties inspired the development of MACBETH "**M**easuring **A**ttractiveness by a **C**ategorical **B**ased **E**valuation **T**echnique".

The original research on the MACBETH approach was carried out in the early 1990's – see [2, 29] and [35] – as a response to the following question:

How can a value scale be built on X, both in a qualitatively and quantitatively meaningful way, without forcing J to produce direct numerical representations of preferences and involving only two elements of X for each judgement required from J?

Using MACBETH, J is asked to provide preferential information about two elements of X at a time, firstly by giving a judgement as to their relative attractiveness (ordinal judgement) and secondly, if the two elements are not deemed to be equally attractive, by expressing a qualitative judgement about the difference of attractiveness between the most attractive of the two elements and the other. Moreover, to ease the judgemental process, six semantic categories of difference of attractiveness, "very weak", "weak", "moderate", "strong", "very strong" or "extreme", or a succession of these (in case hesitation or disagreement arises) are offered to J as possible answers. This is somewhat in line with similar ideas previously proposed by Saaty [59] in a ratio measurement framework, or by Freeling [52] and Belton [40] in difference value measurement. By pairwise comparing the elements of X a matrix of qualitative judgements is filled in, with either only a few pairs of elements, or with all of them (in which case $n \cdot (n-1)/2$ comparisons would be made by J).

A brief review of the previous research on MACBETH is offered in Section 2, together with the evolution of its software's development. It shows that, on a technical level, MACBETH has evolved through the course of theoretical research and also through its extension to the multicriteria value measurement framework in numerous practical applications (see Section 10). Its essential characteristics, however, have never changed.

Sections 3 through 9 of this chapter present an up-to-date survey of the mathematical foundations of MACBETH. Section 3 describes the two MACBETH modes of questioning mentioned above (both involving only two elements at a time) used to acquire preferential information from J, as well as the types of information that can be deduced from each of them. The subsequent sections are devoted to an up-to-date rigorous survey of the mathematical foundations of MACBETH. Section 4 addresses the numerical representation of those different types of information. These numerical representations are only possible if J's responses satisfy certain rational working hypotheses. Section 5 deals with the "consistency / inconsistency" of the preferential information gathered from J and Section 6 explores the practical problem of testing the consistency of preferential information. How should an inconsistency be dealt with? The answer to this question is the subject of Section 7. Sections 8 and 9 present what MACBETH proposes to J once the preference information provided by J is consistent. Finally, Section 10 lists several real-world applications of multicriteria value analysis in which the MACBETH approach was used.

This chapter will use the following notation:

- *J* is an evaluator, either a individual or group.
- *X* (with $\#X = n \geq 2$) is a finite set of elements (alternatives, choice options, courses of action) that *J* wants to compare in terms of their relative attractiveness (desirability, value).
- $\Delta att(x, y)$ is the "difference of attractiveness between x and y for *J*", where x and y are elements of *X* such that x is more attractive than y for *J*.
- $\Delta att(x, y) \succ \Delta att(z, w)$ means that $\Delta att(x, y)$ is greater than Δatt (z, w).
- ϕ is an empty set.
- $\mathbb{R}$ is the set of real numbers.
- $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geq 0\}$.
- $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$.
- $\mathbb{R}^*_+ = \mathbb{R}_+ \setminus \{0\}$.
- $\mathbb{Z}$ is the set of integer numbers.
- $\mathbb{N}$ is the set of non-negative integer numbers.
- $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$.
- $\mathbb{N}_{s,t} = \{s, s+1, \ldots, t\} = \{x \in \mathbb{N} \mid s \leq x \leq t\}$ where $s, t \in \mathbb{N}$, and $s < t$.
- The transpose of a matrix *A* will be denoted by ${}^tA$.

2. Previous Research and Software Evolution

In order to build an interval (value) scale based on the qualitative judgements of difference of attractiveness formulated by *J*, it is necessary that the six MACBETH categories "very weak", "weak", "moderate", "strong", "very strong" or "extreme" be represented by non-overlapping (disjoint) intervals of real numbers. The basic idea underlying the initial development of MACBETH was that the limits of these intervals should not be arbitrarily fixed a priori, but determined simultaneously with numerical value scores for the elements of *X*. Research was then conducted on how to test for the existence of such intervals and how to propose numerical values for the elements of *X* and for the limits of the intervals – see [2, Chapter IV]. This gave rise to the formulation of a chain of four linear programs – see [31, 29, 30] and [32] – that, implemented in GAMS, were used in the first real-world applications of MACBETH as a decision aiding tool to derive value scores and criteria weights in the framework of an additive aggregation model – see [42, 43, 35] and [37]. Theoretical research conducted at the same time, and first presented in 1994 at the 11th International Conference on MCDM, demonstrated the equivalence of the approach by constant thresholds and the approach by measurement conditions – see [36].

The first MACBETH software was developed in 1994. In it, the objective function used in the GAMS implementation to determine a value scale was modified, on the basis of a simple principle – see [37] and [38] – that makes it possible, for simple cases, to determine the scale "by hand" [34]. However, complete procedures to address and manage all cases of inconsistency were not available at that time. Therefore, the software offered its users the possibility of obtaining a compromise scale in the case of inconsistency. This initial software was used in several real world applications – see, for example, [19, 21, 23, 24, 32, 39] and [48]. However, it had several important limitations:

1. The determination of suggestions was still heuristic and did not guarantee the minimal number of changes necessary to achieve consistency;

2. It was not possible for the evaluator to hesitate between several semantic categories when expressing judgements. It, therefore, did not enable one to facilitate the management of group judgemental disagreements;

3. It forced the evaluator to first provide all of the judgements before it could run any procedure. Consequently, judgemental inconsistency could only be detected for a full matrix of judgements. As a result, suggestions of changes to resolve inconsistency could only then be discussed, a restriction that did not lend itself to good interaction.

Subsequent theoretical research was therefore concentrated on resolving these problems. Results reported in [46] and [56], allowing inconsistencies to be dealt with in a mathematically sound manner, were the turning point in the search for a more interactive formulation. Indeed, it was then possible to implement a procedure that automatically detects "inconsistency", even for an incomplete matrix ofjudgements, in a new software called M-MACBETH – see `www.m-macbeth.com` and [16] – which has been used to produce some of the figures in this paper. The objective of abandoning the suggestion of a compromise scale could also finally be achieved, since the origin of the inconsistency could now be found (detection of elementary incompatible systems) and explained to *J*. M-MACBETH finds the minimal number of necessary changes and, for any number of changes not greater than five, suggests all of the possible ways in which the inconsistency can be resolved. Furthermore, it is able to provide suggestions of multiple category changes, where a "k categories change" is considered to be equivalent to k "1 category changes".

Real-world applications in the specific context of bid evaluation (see references in Section 10) inspired research regarding the concepts of "robustness" [46] and sensitivity [9], the results of which were then included in the software, together with the possibility of addressing potential imprecision (uncertainty) associated with impacts of options, incorporating reference levels for one criterion at any time, and graphically representing comparisons of options on any two groups of criteria. These issues are out of the scope of the present chapter and they are not also included in the version of the software, limited to scoring and weighting, embedded into the HIVIEW3 software in 2003 – see [45] and www.catalyze.co.uk.

3. Types of Preferential Information

3.1 Type 1 Information

Type 1 information refers to preferential information obtained from *J* by means of Questioning Procedure 1.

Let x and y be two different elements of *X*.

Questioning Procedure 1 *A first question (Q1) is asked of J:*
Q1: Is one of the two elements more attractive than the other?
J's response (R1) can be: "Yes", or "No", or "I don't know".
If R1 = "Yes", a second question (Q2) is asked:
Q2: Which of the two elements is the most attractive?

The responses to Questioning Procedure 1 for several pairs of elements of *X* enable the construction of three binary relations on *X:*

$$P = \{(x,y) \in X \times X : x \text{ is more attractive than } y\}$$

$$I = \{(x,y) \in X \times X : x \text{ is not more attractive than } y \text{ and } y \text{ is not more attractive than } x, \text{ or } x = y\}$$

$$? = \{(x,y) \in X \times X : x \text{ and } y \text{ are not comparable in terms of their attractiveness}\}.$$

P is asymmetric, *I* is reflexive and symmetric, and ? is irreflexive and symmetric. Note that $? = X \times X \setminus (I \cup P \cup P^{-1})$, with $P^{-1} = \{(x,y) \in X \times X \mid yPx\}$.

DEFINITION 37 *Type 1 information about X is a structure* $\{P, I, ?\}$ *where P, I and ? are disjoint relations on X, P is asymmetric, I is reflexive and symmetric, and* $? = X \times X \setminus (I \cup P \cup P^{-1})$.

3.2 Type 1+2 Information

Suppose that type 1 information $\{P, I, ?\}$ about X is available.

Questioning Procedure 2 *The following question (Q3) is asked, for all $(x, y) \in P$:*

Q3: How do you judge the difference of attractiveness between x and y?

J's response (R3) would be provided in the form "d_s" (where $d_1, d_2, \ldots, d_Q$ ($Q \in \mathbb{N} \setminus \{0, 1\}$) are semantic categories of difference of attractiveness defined so that, if $i < j$, the difference of attractiveness d_i is weaker than the difference of attractiveness d_j) or in the more general form (possibility of hesitation) "d_s to d_t", with $s \leq t$ (the response "I don't know" is assimilated to the response "d_1 to d_Q").

REMARK 34 *When $Q = 6$ and d_1 = very weak, d_2 = weak, d_3 = moderate, d_4 = strong, d_5 = very strong, d_6 = extreme, Questioning Procedure 2 is the mode of interaction used in the MACBETH approach and its M-MACBETH software.*

R3 responses give rise to relations C_{st} $(s, t \in \mathbb{N}, 1 \leq s \leq t \leq Q)$ where $C_{st} = \{(x, y) \in P \mid \Delta_{att}(x, y)$ is "d_s to d_t"$\}$. They enable the construction of an asymmetric relation on $P : \{((x, y), (z, w)) \in P \times P \mid \exists\, i, j, s, t \in \mathbb{N}$ with $1 \leq i \leq j < s \leq t \leq Q, (x, y) \in C_{st}, (z, w) \in C_{ij}\}$. Hereafter, C_{ss} will simply be referred to as C_s.

DEFINITION 38 *Type 1+2 information about X is a structure $\{P, I, ?, P^e\}$ where $\{P, I, ?\}$ is type 1 information about X and P^e is an asymmetric relation on P, the meaning of which is "$(x, y) P^e (z, w)$ when $\Delta_{att}(x, y) \succ \Delta_{att}(z, w)$".*

4. Numerical Representation of the Preferential Information

4.1 Type 1 Scale

Suppose that type 1 information $\{P, I, ?\}$ about X is available.

DEFINITION 39 *A type 1 scale on X relative to {P,I} is a function $\mu : X \rightarrow \mathbb{R}$ satisfying Condition 1.*

CONDITION 1 $\forall\, x, y \in X$, $[xPy \Rightarrow \mu(x) > \mu(y)]$ *and* $[xIy \Rightarrow \mu(x) = \mu(y)]$.

Let $Sc_1(X, P, I) = \{\mu : X \rightarrow \mathbb{R} \mid \mu$ is a type 1 scale on X relative to $\{P, I\}\}$. When X, P and I are well determined, $Sc_1(X, P, I)$ will be noted Sc_1.

When $? = \phi$ and $Sc_1(X, P, I) \neq \phi$, each element of $Sc_1(X, P, I)$ is an ordinal scale on X.

4.2 Type 1+2 Scale

Suppose that type 1+2 information $\{P, I, ?, P^e\}$ about X is available.

DEFINITION 40 *A type 1+2 scale on X relative to* $\{P, I, ?, P^e\}$ *is a function* $\mu : X \to \mathbb{R}$ *satisfying Condition 1 and Condition 2.*

CONDITION 2 $\forall\, x, y, z, w \in X,\ [(x, y)P^e(z, w) \Rightarrow \mu(x) - \mu(y) > \mu(z) - \mu(w)]$.

Let $Sc_{1+2}(X, P, I, P^e) = \{\mu : X \to \mathbb{R} \mid \mu$ is a type 1+2 scale on X relative to $\{P, I, P^e\}\}$. When X, P, I and P^e are well determined, $Sc_{1+2}(X, P, I, P^e)$ will be noted Sc_{1+2}.

5. Consistency – Inconsistency

DEFINITION 41 *Type 1 information* $\{P, I, ?\}$ *about X is*

- *consistent when* $Sc_1(X, P, I) \neq \phi$
- *inconsistent when* $Sc_1(X, P, I) = \phi$.

DEFINITION 42 *Type 1+2 information* $\{P, I, ?, P^e\}$ *about X is*

- *consistent when* $Sc_{1+2}(X, P, I, P^e) \neq \phi$
- *inconsistent when* $Sc_{1+2}(X, P, I, P^e) = \phi$.

When $Sc_{1+2}(X, P, I, P^e) = \phi$, one can have $Sc_1(X, P, I) = \phi$ or $Sc_1(X, P, I) \neq \phi$. In the first case, the message “no ranking” will appear in M-MACBETH; it occurs namely when J declares, in regards to elements x, y and z of X, that $[xIy, yIz$ and $xPz]$ or $[xPy, yPz$ and $zPx]$. In the second case, the message “inconsistent judgement” will appear in M-MACBETH.

Although this is the only difference between the types of inconsistency introduced in M-MACBETH, it is interesting to mention, from a theoretical perspective, that one could further distinguish two sub-types of inconsistency (sub-type a and sub-type b) when $Sc_{1+2}(X, P, I, P^e) = \phi$ and $Sc_1(X, P, I) \neq \phi$.

Sub-type a inconsistency arises when there is a conflict between type 1 information and P^e that makes the simultaneous satisfaction of conditions 1 and 2 impossible. These kinds of conflicts are found essentially in four types of situations; namely when $x, y, z \in X$ exist such that

$$\begin{aligned} & [\, xPy, yPz, xPz \text{ and } (y, z)P^e(x, z) \,] \\ \text{or} \quad & [\, xPy, yPz, xPz \text{ and } (x, y)P^e(x, z) \,] \\ \text{or} \quad & [\, xIy, yPz, xPz \text{ and } (x, z)P^e(y, z) \,] \\ \text{or} \quad & [\, xIy, zPy, zPx \text{ and } (z, x)P^e(z, y) \,]. \end{aligned}$$

Sub-type b inconsistency arises when there is no conflict between type 1 information and P^e but at least one conflict exists inside P^e that makes satisfying Condition 2 impossible. An example of this type of conflict is (see Figure 10.1):

$$xPy, xPw, yPz, wPz, xPz, yPw$$
$$(x, y) \in C_1, (y, z) \in C_2$$
$$(x, w) \in C_3, (w, z) \in C_2.$$

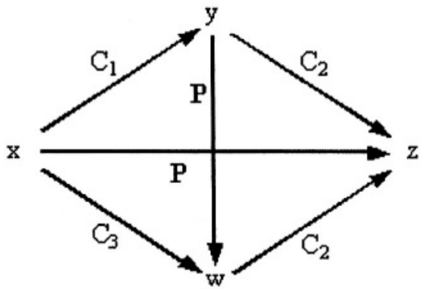

Figure 10.1. Example of sub-type b inconsistency.

In such a case, Condition 2 cannot be respected, because one should have

$$\begin{cases} \mu(x) - \mu(w) > \mu(y) - \mu(z) & (1) \\ \mu(w) - \mu(z) > \mu(x) - \mu(y) & (2) \end{cases}$$

which is impossible.

On the other hand, it is easily shown that the following two systems are compatible, that is, there is no conflict between type 1 information and P^e:

$$\begin{cases} \mu(x) - \mu(w) > \mu(y) - \mu(z) \\ \mu(x) - \mu(y) > 0 \\ \mu(x) - \mu(w) > 0 \\ \mu(x) - \mu(z) > 0 \\ \mu(y) - \mu(z) > 0 \\ \mu(w) - \mu(z) > 0 \\ \mu(y) - \mu(w) > 0 \end{cases} \qquad \begin{cases} \mu(w) - \mu(z) > \mu(x) - \mu(y) \\ \mu(x) - \mu(y) > 0 \\ \mu(x) - \mu(w) > 0 \\ \mu(x) - \mu(z) > 0 \\ \mu(y) - \mu(z) > 0 \\ \mu(w) - \mu(z) > 0 \\ \mu(y) - \mu(w) > 0 \end{cases}$$

For a detailed study of inconsistency, see [46].

6. Consistency Test for Preferential Information

6.1 Testing Procedures

Suppose that $X = \{a_1, a_2, \ldots, a_n\}$.

During the interactive questioning process conducted with J, each time that a new judgement is obtained, the consistency of all the responses already provided is tested. This consistency test begins with a pre-test aimed at detecting the (potential) presence of cycles within the relation P and, if no such cycle exists, making a permutation of the elements of X in such a way that, in the matrix of judgements, all of the cells P or C_{ij} will be located above the main diagonal.

When there is no cycle in P, the consistency of type 1 information $\{P, I, ?\}$ is tested as follows:

- If $? \neq \phi$, a linear program named **LP-test$_1$** is used;
- if $? = \phi$, rather than linear programming, a method named **DIR-test$_1$** is used, which has the advantage of being easily associated with a very simple visualization of an eventual ranking within the matrix of judgements.

When $\{P, I, ?\}$ is consistent, the consistency of type 1+2 information $\{P, I, ?, P^e\}$ is tested with the help of a linear program named **LPσ-test$_{1+2}$**.

6.2 Pre-test of the Preferential Information

The pre-test of the preferential information is based on Property 1. (Evident because $\#X$ is finite).

PROPERTY 1 *Let* $X^* \subset X$; *if* $\forall\, x \in X^*, \exists\, y \in X^*$ *such that* xPy, *then* $\exists\, x_1, x_2, \ldots, x_p \in X^*$ *such that* $x_1Px_2P\ldots Px_pPx_1$ *(cycle).*

The pre-test consists of seeking a permutation $\varphi : \mathbb{N}_{1,n} \to \mathbb{N}_{1,n}$ such that

$$\forall\, i, j \in \mathbb{N}_{1,n},\ [\, i > j \Rightarrow a_{\varphi(i)}(notP)a_{\varphi(j)}\,].$$

The permutation of the elements of X is made by the algorithm PRETEST, that detects cycles within P and sorts the elements(s) of X.

PRETEST:

1 $s \leftarrow n$;

2 among $a_1, a_2, \ldots, a_s$ find a_i which is not preferred over any other:
if a_i exists, go to 3.;
if not, return FALSE ($Sc_1 = \phi$, according to Property 1); finish.

3 permute a_i and a_s;

4 $s \leftarrow s - 1$:
if $s = 1$, return TRUE; finish.
If not, go to 2.

6.3 Consistency Test for Type 1 Information

Suppose that PRETEST detected no cycle within P and that the elements of X were renumbered as follows (to avoid the introduction of a permutation in the notation):

$$\forall\, i, j \in \mathbb{N}_{1,n},\ [\, i > j \Rightarrow a_i (notP) a_j \,].$$

6.3.1 Consistency Test for Incomplete ($? \neq \phi$) Type 1 Information.

Consider the linear program **LP-test₁** with variables $x_1, x_2, \ldots, x_n$:

$$\begin{array}{ll} \min x_1 & \\ \text{subject to} & \\ x_i - x_j \geq d_{\min} & \forall\, (a_i, a_j) \in P \\ x_i - x_j = 0 & \forall\, (a_i, a_j) \in I \text{ with } i \neq j \\ x_i \geq 0 & \forall\, i \in \mathbb{N}_{1,n} \end{array}$$

where $d_{\min}$ is a positive constant, and the variables $x_1, x_2, \ldots, x_n$ represent the numbers $\mu(a_1), \mu(a_2), \ldots, \mu(a_n)$ that should satisfy Condition 1 so that μ is a type 1 scale.

The objective function min x_1 of **LP-test**$_1$ is obviously arbitrary. It is trivial that $Sc_1 \neq \phi \Leftrightarrow$ **LP-test**$_1$ is feasible.

6.3.2 Consistency Test for Complete ($? = \phi$) Type 1 Information.

When $? = \phi$ and the elements of X have been renumbered (after the application of PRETEST), another simple test (**DIR-test**$_1$) allows one to verify if $P \cup I$ is a complete preorder on X. **DIR-test**$_1$ is based on Proposition 1 (Proved in [46]).

PROPOSITION 6 *If* $[\, \forall\, i, j \in \mathbb{N}_{1,n}$ *with* $i < j,\ (a_i, a_j) \in P \cup I\,]$ *then* $P \cup I$ *is a complete preorder on X if and only if* $\forall\, i, j \in \mathbb{N}_{1,n}$ *with* $i < j : \left[a_i P a_j \Rightarrow \begin{cases} \forall\, s \leq i, \forall\, t \geq j, a_s P a_t \\ \exists\, s : i \leq s \leq j-1 \text{ and } a_s P a_{s+1} \end{cases} \right].$

Proposition 1 means that when the "P cases" of the matrix of judgements forms a "staircase", a ranking exists such that each step of the "staircase" rests, at least partly, on the principal diagonal of the matrix.

6.4 Consistency Test for Type 1+2 Information

It would be possible to test the consistency of type 1+2 information with a linear program based on Conditions 1 and 2. However, the more efficient linear program **LP-test**$_{1+2}$, which includes "thresholds conditions" equivalent to Conditions 1 and 2, is used instead. **LP-test**$_{1+2}$ is based on Lemma 1 (Proved in [46]).

LEMMA 1 *Let* $\mu : X \rightarrow \mathbb{R}$. μ *satisfies Conditions 1 and 2 if and only if there exist Q "thresholds"* $0 < \sigma_1 < \sigma_2 < \ldots < \sigma_Q$ *that satisfy Conditions 3, 4 and 5.*

CONDITION 3 $\forall\,(x, y) \in I,\ \mu(x) = \mu(y)$.

CONDITION 4 $\forall\, i, j \in \mathbb{N}_{1,Q}$ *with* $i \leq j$, $\forall\,(x, y) \in C_{ij},\ \sigma_i < \mu(x) - \mu(y)$.

CONDITION 5 $\forall\, i, j \in \mathbb{N}_{1,Q-1}$ *with* $i \leq j$, $\forall\,(x, y) \in C_{ij},\ \mu(x) - \mu(y) < \sigma_{j+1}$.

Program **LP-test**$_{1+2}$ has variables $x_1 (= \mu(a_1)), \ldots, x_n (= \mu(a_n)), \sigma_1, \ldots, \sigma_Q$:

$$
\begin{array}{ll}
\min x_1 & \\
\text{subject to} & \\
x_p - x_r = 0 & \forall\,(a_p, a_r) \in I \text{ with } p < r \\
\sigma_j + d_{\min} \leq x_p - x_r & \forall\, i, j \in \mathbb{N}_{1,Q} \text{ with } i \leq j,\ \forall\,(a_p, a_r) \in C_{ij} \\
x_p - x_r \leq \sigma_{j+1} - d_{\min} & \forall\, i, j \in \mathbb{N}_{1,Q-1} \text{ with } i \leq j,\ \forall\,(a_p, a_r) \in C_{ij} \\
d_{\min} \leq \sigma_1 & \\
\sigma_{i-1} + d_{\min} \leq \sigma_i & \forall\, i \in \mathbb{N}_{2,Q} \\
x_i \geq 0 & \forall\, i \in \mathbb{N}_{1,n} \\
\sigma_i \geq 0 & \forall\, i \in \mathbb{N}_{1,Q}
\end{array}
$$

Taking into account Lemma 1, it is trivial that $Sc_{1+2} \neq \phi$ if and only if the linear program **LP-test**$_{1+2}$, which is based on Conditions 3, 4 and 5, is feasible.

7. Dealing with Inconsistency

When a type 1+2 information $\{P, I, ?, P^e\}$ about X is inconsistent, it is convenient to be able to show J systems of constraints that render his or her judgements inconsistent and modifications of these judgements that would render **LP**σ**-test**$_{1+2}$ feasible.

7.1 Systems of Incompatible Constraints

Suppose that **LP-test**$_{1+2}$ is not feasible or, in other words, that the following system is incompatible (variables $x_1 (= \mu(a_1)), \ldots, x_n (= \mu(a_n)), \sigma_1, \ldots, \sigma_Q$ nonnegative):

$$
\left\{
\begin{array}{lll}
x_p - x_r = 0 & \forall\,(a_p, a_r) \in I \text{ with } p < r & \text{(t1)} \\
\sigma_i < x_p - x_r & \forall\, i, j \in \mathbb{N}_{1,Q} \text{ with } i \leq j,\ \forall\,(a_p, a_r) \in C_{ij} & \text{(t2)} \\
x_p - x_r < \sigma_{j+1} & \forall\, i, j \in \mathbb{N}_{1,Q-1} \text{ with } i \leq j,\ \forall\,(a_p, a_r) \in C_{ij} & \text{(t3)} \\
0 < \sigma_1 & & \text{(t4)} \\
\sigma_{i-1} < \sigma_i & \forall\, i \in \mathbb{N}_{2,Q} & \text{(t5)}
\end{array}
\right.
$$

Conventions:

- $\mathbb{R}^{m \times n}$ is the set of the real matrices with m lines and n columns.
- Matrix $M \in \mathbb{R}^{m \times n}$ is "non-zero" $(M \neq 0)$ if at least one of its elements is not null.
- Matrix $M \in \mathbb{R}^{m \times n}$ is positive or null $(M \geq 0)$ if all of its elements are positive or null.

The system of incompatible constraints can be written in the matrix format as follows:

$$\begin{cases} C \cdot Z \gtrdot 0 & \text{(by grouping constraints (t2))} \\ D \cdot Z > 0 & \text{(by grouping constraints (t3))} \\ E \cdot Z > 0 & \text{(by grouping constraints (t4) and (t5))} \\ B \cdot Z = 0 & \text{(by grouping constraints (t1))} \end{cases}$$

where

$$Z = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ \sigma_1 \\ \sigma_2 \\ \vdots \\ \sigma_Q \end{pmatrix}$$

$$\begin{aligned} C &\in \mathbb{R}^{p_1 \times (n+Q)} \text{(where } p_1 \text{ is the number of constraints (t2))} \\ D &\in \mathbb{R}^{p_2 \times (n+Q)} \text{(where } p_2 \text{ is the number of constraints (t3))} \\ E &\in \mathbb{R}^{p_3 \times (n+Q)} \text{(where } p_3 \text{ is the number of constraints (t4) and (t5))} \\ B &\in \mathbb{R}^{r \times (n+Q)} \text{(where } r \text{ is the number of constraints (t1))} \end{aligned}$$

Note: if $r = 0$, one could consider that $B = 0 \in \mathbb{R}^{1 \times (n+Q)}$ without losing generality.

Let A be the matrix $\begin{bmatrix} C \\ D \\ E \end{bmatrix} \in \mathbb{R}^{p \times (n+Q)}$ $(p = p_1 + p_2 + p_3)$. The system of incompatible constraints can be written more simply as

$$S \begin{cases} A \cdot Z > 0 & \text{(by grouping constraints (t2), (t3), (t4) and (t5))} \\ B \cdot Z = 0 & \text{(by grouping constraints (t1)).} \end{cases}$$

In order to detect incompatibilities between the constraints (t1), (t2), (t3), (t4) and (t5) and propose eventual corrections, we apply Proposition 2 (Proved in [46]), which is a corollary of Mangasarian's [55] version of the *Theorem of the Alternative.*

PROPOSITION 7 *The system* S $\{A \cdot Z > 0;\ B \cdot Z = 0\}$ *admits a solution* $Z \in \mathbb{R}^{(n+Q)\times 1}$ *or there exists* $Y \in \mathbb{R}^{p\times 1}, V, W \in \mathbb{R}^{r\times 1}$ *with* $Y \neq 0, Y \geq 0, V \geq 0, W \geq 0$ *such that* ${}^tA \cdot Y + {}^tB \cdot (V - W) = 0$ *and* $\forall i \in \mathbb{N}_{1,r},\ V_i \cdot W_i = 0$ *but never both.*

The interest of Proposition 2 is that vectors *Y, V* and *W* have positive or null components, thus making it compatible with linear programming (see Sections 7.3 and 7.4)

7.2 Example 1

Suppose that $X = \{a_1, a_2, a_3, a_4\}$ and that *J* has formulated the following judgements:

- $P = \{(a_1, a_2), (a_1, a_3), (a_2, a_3), (a_3, a_4)\}$
- $(a_1, a_2) \in C_1,\ (a_1, a_3) \in C_4,\ (a_2, a_3) \in C_2,\ (a_3, a_4) \in C_2.$

Suppose that *J* also judges that $a_2 P a_4$ and that $(a_2, a_4) \in C_3$. **LP-test$_1$** is feasible: the judgements are compatible with a ranking. **LP-test$_{1+2}$** is not feasible: the software informs *J* that his or her judgements are "inconsistent".

Suppose now that *J* confirms his or her judgements. One must then have:

$$
\begin{array}{llllll}
\sigma_1 < x_1 - x_2 & (1) & x_1 - x_2 < \sigma_2 & (2) & 0 < \sigma_1 & (11) \\
\sigma_2 < x_2 - x_3 & (3) & x_2 - x_3 < \sigma_3 & (4) & \sigma_1 < \sigma_2 & (12) \\
\sigma_2 < x_3 - x_4 & (5) & x_3 - x_4 < \sigma_3 & (6) & \sigma_2 < \sigma_3 & (13) \\
\sigma_3 < x_2 - x_4 & (7) & x_2 - x_4 < \sigma_4 & (8) & \sigma_3 < \sigma_4 & (14) \\
\sigma_4 < x_1 - x_3 & (9) & x_1 - x_3 < \sigma_5 & (10) & \sigma_4 < \sigma_5 & (15) \\
 & & & & \sigma_5 < \sigma_6 & (16)
\end{array}
$$

or, in matrix format (which one can denote as $A \cdot Z > 0$):

$$\begin{pmatrix} 1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{pmatrix} > 0$$

Since it is known, according to Proposition 2, that the system has no solution, there necessarily exists $Y \in \mathbb{R}^{16\times 1}(Y \neq 0, Y \geq 0)$ such that ${}^tA \cdot Y = 0$. Thus, positive or null (but not all null) real numbers $y_1, y_2, \ldots, y_{16}$ exist such that $\sum_{i=1}^{16} y_i \cdot Col_i = 0$ (where Col_i is the column i of the matrix ${}^tA$).

In this simple example, one can see that it is enough to make $y_2 = y_5 = y_8 = y_9 = 1$ and $y_1 = y_3 = y_4 = y_6 = y_7 = y_{10} = y_{11} = y_{12} = y_{13} = y_{14} = y_{15} = y_{16} = 0$:

$$1 \cdot \underbrace{\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{Col_2} +1 \cdot \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{Col_5} +1 \cdot \underbrace{\begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}}_{Col_8} +1 \cdot \underbrace{\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}}_{Col_9} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

These four vectors correspond to the four constraints (2), (5), (8) and (9) above:

$$\left.\begin{array}{ll} \sigma_4 > x_2 - x_4 & (8) \\ x_1 - x_3 > \sigma_4 & (9) \end{array}\right\} \Rightarrow x_1 - x_3 > x_2 - x_4 \quad (*)$$

$$\left.\begin{array}{ll}\sigma_2 > x_1 - x_2 & (2)\\ x_3 - x_4 > \sigma_2 & (5)\end{array}\right\} \Rightarrow x_3 - x_4 > x_1 - x_2 \quad (**)$$

$(*)$ and $(**)$ bring to the contradiction $x_1 - x_4 > x_1 - x_4$. The incompatibility between $(*)$ and $(**)$ is presented in M-MACBETH as shown in Figure 10.2.

Diff.	Couples		Couples	Diff.
strong	a1 - a3	>	a2 - a4	moderate
weak	a3 - a4	>	a1 - a2	very weak

Figure 10.2. Example of incompatibility between $(*)$ and $(**)$.

Note that the problem disappears if

$$\begin{array}{ll} & (a_1, a_3) \in C_3 \text{ instead of } C_4 \quad ((*) \text{ disappears})\\ \text{or} & (a_2, a_4) \in C_4 \text{ instead of } C_3 \quad ((*) \text{ disappears})\\ \text{or} & (a_3, a_4) \in C_1 \text{ instead of } C_2 \quad ((**) \text{ disappears})\\ \text{or} & (a_1, a_2) \in C_2 \text{ instead of } C_1 \quad ((**) \text{ disappears}).\end{array}$$

Note also that the inconsistency would not be eliminated for any modification of the judgement "$(a_2, a_3) \in C_2$".

If *J* confirms the judgement "$(a_2, a_4) \in C_3$", M-MACBETH calculates the different possibilities (four in example 1) that *J* can follow to make his or her judgements consistent with a "minimal" number of changes of category (one in Example 1). (We will specify in Section 7.4 the meaning of this notion).

In M-MACBETH, the "suggestions" of changes are presented (graphically) in the matrix of judgements. They are:

- to replace the judgement $(a_1, a_3) \in C_4$ with the judgement $(a_1, a_3) \in C_3$
- or to replace the judgement $(a_2, a_4) \in C_3$ with the judgement $(a_2, a_4) \in C_4$
- or to replace the judgement $(a_3, a_4) \in C_2$ with the judgement $(a_3, a_4) \in C_1$
- or to replace the judgement $(a_1, a_2) \in C_1$ with the judgement $(a_1, a_2) \in C_2$.

7.3 Identifying Constraints which Cause Inconsistency

Let us detail the various stages of our search for "suggestions". The first step consists of determining the constraints (t1), (t2) and (t3) which are "the origin of the incompatibilities" present in the system

$$S\left\{\begin{array}{l} A \cdot Z > 0\\ B \cdot Z = 0\end{array}\right. \quad \text{(see Section 7.1)}$$

We consider that a constraint is "at the origin of an incompatibility" when it is part of a system S' that

- is a "sub-system" of S,
- is incompatible,
- does not contain any incompatible "sub-system".

Mathematically, this idea can be represented by Definition 7.

DEFINITION 43 *An incompatible elementary system (SEI) is a system*

$$S' \begin{cases} A' \cdot Z > 0 \\ B' \cdot Z = 0 \end{cases}$$

such that

1 $A' \in \mathbb{R}^{p' \times (n+Q)}$ *is a sub-matrix of A, and* $B' \in \mathbb{R}^{r' \times (n+Q)}$ *is a sub-matrix of B;*

2 S' *is incompatible;*

3 If $\begin{cases} A'' \in \mathbb{R}^{p'' \times (n+Q)} \text{ is a sub-matrix of } A', \\ B'' \in \mathbb{R}^{r'' \times (n+Q)} \text{ is a sub-matrix of } B', \\ p'' + r'' < p' + r' \end{cases}$ *then* $\begin{cases} A'' \cdot Z > 0 \\ B'' \cdot Z = 0 \end{cases}$ *is compatible.*

However, our goal is not to determine all the SEI that could be extracted from the constraints using **LPσ-test$_{1+2}$**. We just want to find all of the judgements of the type $(a_s, a_t) \in C_{ij}$ that "generate" an incompatibility. In Section 7.4.3, we will explain how we use these judgements.

We know that an inconsistency occurs when the system

$$S \begin{cases} A \cdot Z > 0 \\ B \cdot Z = 0 \end{cases}$$

is incompatible; that is, $\exists Y \in \mathbb{R}^p$ and V, $W \in \mathbb{R}^r$ such that

$$\begin{cases} {}^tA \cdot Y + {}^tB \cdot (V - W) = 0 \\ Y \geq 0,\ V \geq 0,\ W \geq 0 \\ \forall\, i \in \mathbb{N}_{1,r},\ V_i \cdot W_i = 0 \\ \exists\, i_0 \in \mathbb{N}_{1,p} \text{ such that } Y_{i_0} \neq 0 \end{cases}$$

In such a case, if $i_0 \leq p_1 + p_2$, where p_1 is the number of constraints (t2) and p_2 is the number of constraints (t3) (see Section 7.1), a constraint of the type $x_s - x_t < \sigma_j$ or $x_s - x_t > \sigma_j$ will correspond to S.

Consider, then, the system (with $i \le p_1 + p_2$):

$$\text{Syst-}Y_i \begin{cases} {}^tA \cdot Y + {}^tB \cdot (V - W) = 0 \\ Y_i = 1 \end{cases}$$

If Syst-Y_i is compatible, for one of its solutions it corresponds to a system of incompatible constraints (t1), (t2), (t3), (t4) and (t5) where at least one constraint (that which corresponds to $Y_i = 1$) is of the type $x_s - x_t < \sigma_j$ or $x_s - x_t > \sigma_j$ and is part of a SEI. If Syst-Y_i is incompatible, the constraint that corresponds to Y_i is not part of any SEI.

To find all of the constraints (t2) and (t3) which are part of a SEI, it is sufficient to study the compatibility of all of the systems Syst-Y_i, for $i = 1, 2, \dots, p_1 + p_2$.

We will proceed in a similar way, using the systems Syst-V_i and Syst-W_i, to find all of the constraints (t1) which are part of a SEI:

$$\text{Syst-}V_i \begin{cases} {}^tA \cdot Y + {}^tB \cdot (V - W) = 0 \\ W_i = 0 \\ V_i = 1 \end{cases}$$

and

$$\text{Syst-}W_i \begin{cases} {}^tA \cdot Y + {}^tB \cdot (V - W) = 0 \\ V_i = 0 \\ W_i = 1 \end{cases}$$

It is not necessary to examine all of the systems Syst-Y_i, Syst-V_i and Syst-W_i:

- If Syst-Y_i is compatible and has the solution *Y, V, W,* then
  - $\forall\, j > i$ such that $Y_j \neq 0$, Syst-Y_i is compatible;
  - $\forall\, j \in \mathbb{N}_{1,r}$ such that $V_j \neq 0$, Syst-V_i is compatible;
  - $\forall\, j \in \mathbb{N}_{1,r}$ such that $W_j \neq 0$, Syst-W_i is compatible.
- If Syst-V_i is compatible and has the solution *Y, V, W,* then
  - $\forall\, j > i$ such that $V_j \neq 0$, Syst-V_i is compatible;
  - $\forall\, j \in \mathbb{N}_{1,r}$ such that $W_j \neq 0$, Syst-W_i is compatible.
- If Syst-W_i is compatible and has the solution *Y, V, W,* then
  - $\forall\, j > i$ such that $W_j \neq 0$, Syst-W_i is compatible.

It is for this reason that a “witness-vector” $T \in \mathbb{N}^{p_1+p_2+2\cdot r}$ must be used, initially null, updated as follows:

- For any solution *Y, V, W* of a system Syst-Y_i, Syst-V_i or Syst-W_i do
  - $\forall\, j \in \mathbb{N}_{1,p_1+p_2}$, $[\, Y_j \neq 0 \Rightarrow T_j = 1\,]$

- $\forall\, j \in \mathbb{N}_{1,r}$, $[\, V_j \neq 0 \Rightarrow T_{p_1+p_2+j} = 1\,]$
- and $[\, W_j \neq 0 \Rightarrow T_{p_1+p_2+r+j} = 1\,]$.

To find the interesting pairs, the compatibility of at most p_1+p_2+2r systems should be studied. The general algorithm to seek equations (t1) and inequalities (t2) and (t3) that are part of a SEI is the following:

- $T = (0, 0, \ldots, 0)$
- for $i = 1, 2, \ldots, p_1 + p_2$ do:
  - $T_i = 0$,
  - then if Syst-Y_i compatible and *Y, V, W* solution of Syst-Y_i

    then update *T*
- for $i = 1, 2, \ldots, r$ do:
  - if $T_{p_1+p_2+i} = 0$,
  - then if Syst-V_i compatible and *Y, V, W* solution of Syst-V_i

    then update *T*
- for $i = 1, 2, \ldots, r$ do:
  - if $T_{p_1+p_2+r+i} = 0$,
  - then if Syst-W_i compatible and *Y, V, W* solution of Syst-W_i

    then update *T*.

In this way one obtains the set of all of the equations and inequalities that make up the SEI.

7.4 Augmentation – Reduction in a Judgement with p Categories

7.4.1 Preliminaries. Notation:

- Judgement $(x, y) \in C_{ij}$ will be represented by element (x, y, i, j) of $X \times X \times \mathbb{N}_{1,Q} \times \mathbb{N}_{1,Q}$.
- Judgement $(x, y) \in I$ will be represented by element $(x, y, 0, 0)$ of $X \times X \times \mathbb{N} \times \mathbb{N}$.

DEFINITION 44 *A reduction in judgement* (s, t, i, j) *with* p *categories* $(1 \leq p \leq Q + i)$ *is the replacement of this judgement*

- *by the judgement* $(s, t, i - p, i - p)$ *if* $i \geq p$

- *by the judgement* $(t, s, p-i, p-i)$ *if* $i < p$.

DEFINITION 45 *An augmentation of the judgement* (s, t, i, j) *with* p *categories* $(1 \leq p \leq Q - j)$ *is the replacement of this judgement by the judgement* $(s, t, j+p, j+p)$.

DEFINITION 46 *A change of judgement* (s, t, i, j) *with* p *categories is an augmentation or a reduction of the judgement with* p *categories.*

Comment: It is evident that one obtains the same final judgement as a result of "1 reduction of a judgement with p categories" or the "p successive reductions of a category of 1 judgement".

Convention: A "change in judgement (s, t, i, j) with p categories" will be represented by $(s, t, i, j, p) \in X \times X \times \mathbb{N}_{1,Q} \times \mathbb{N}_{1,Q} \times \mathbb{Z}$ (augmentation if $p > 0$, reduction if $p < 0$).

7.4.2 Exploitation of the Constraints of SEI. Let us recall from 7.3 that

- if $T_i > 0$, it has a corresponding constraint (t2) or (t3) or (t1) that is part of an SEI;
- if $T_i = 0$, it has no corresponding constraint that is part of an SEI.

These variables, then, provide us with an indication as to the future "modification" to be made to the judgements associated with these constraints. Indeed, suppose that $T_i > 0$:

a) if $1 \leq i \leq p_1$, a constraint $\sigma_u < x_s - x_t$ which is part of an SEI corresponds to variable T_i; if a change in its judgement $(s, t, \ldots, \ldots)$ can help to eliminate the SEI, it ensures that it will be a "reduction" (evident).

b) if $p_1 + 1 \leq i \leq p_1 + p_2$, a constraint $x_s - x_t < \sigma_u$ which is part of an SEI corresponds to variable T_i; if a change in its judgement $(s, t, \ldots, \ldots)$ can help to eliminate the SEI, it ensures that it will be an "augmentation" (evident).

c) if $p_1 + p_2 + 1 \leq i \leq p_1 + p_2 + r$, a constraint $x_s - x_t = 0$ which is part of an SEI corresponds to variable T_i; if a change in its judgement $(s, t, 0, 0)$ can help to eliminate the SEI, it ensures that it will be a "reduction".

d) if $p_1 + p_2 + r + 1 \leq i \leq p_1 + p_2 + 2r$, a constraint $x_s - x_t = 0$ which is part of an SEI corresponds to variable T_i; if a change in its judgement $(s, t, 0, 0)$ can help to eliminate the SEI, it ensures that it will be an "augmentation" (proof similar to that of c).

Proof of c):

Being $h = i - (p_1 + p_2)$, one knows (by the definition of T_i) that $\exists\, Y \in \mathbb{R}^p$, $\exists\, V, W \in \mathbb{R}^r$ with $Y \geq 0,\ V \geq 0,\ W \geq 0,\ Y \neq 0,\ V_h \neq 0$ and $W_h = 0$ such that ${}^t(A') \cdot Y + {}^t(B') \cdot (V - W) = 0$ or, if one notes $LineB_j$ the jth line of B',

$$ {}^t(A') \cdot Y + {}^tLineB_h \cdot V_h + \sum_{\substack{j=1 \\ j\neq h}}^{r} {}^tLineB_j \cdot V_j - \sum_{\substack{j=1 \\ j\neq h}}^{r} {}^tLineB_j \cdot W_j = 0 $$

(because $W_h = 0$).

The corresponding SEI $\begin{cases} A' \cdot Z > 0 \\ B' \cdot Z = 0 \end{cases}$ can be written $\begin{cases} A' \cdot Z > 0 \\ x_s - x_t = 0 \\ B'' \cdot Z = 0, \end{cases}$ where

$$ B'' = \begin{bmatrix} LineB_1 \\ \vdots \\ LineB_{h-1} \\ LineB_{h+1} \\ \vdots \\ LineB_r \end{bmatrix} \text{ (the matrix } B' \text{ without line } LineB_h). $$

If one considers an "augmentation" of judgement $(s, t, 0, 0)$, the constraint $x_s - x_t = 0$ would be replaced by the constraint $x_s - x_t > 0$. The new system $\begin{cases} A' \cdot Z > 0 \\ x_s - x_t > 0 \\ B'' \cdot Z = 0 \end{cases}$ can be written $\begin{cases} A'' \cdot Z > 0 \\ B'' \cdot Z = 0, \end{cases}$ where $A'' = \begin{bmatrix} A' \\ LineB_h \end{bmatrix}$ (the matrix A' "augmented" with line $LineB_h$).

The system is still incompatible; indeed, if one poses

- $Y' = (Y_1, Y_2, \ldots, Y_p, V_h) \in \mathbb{N}^{p+1}$
- $V' = (V_1, \ldots, V_{h-1}, V_{h+1}, \ldots, V_r) \in \mathbb{N}^{r-1}$
- $W' = (W_1, \ldots, W_{h-1}, W_{h+1}, \ldots, W_r) \in \mathbb{N}^{r-1}$.

$$ {}^t(A') \cdot Y + {}^tLineB_h \cdot V_h + \sum_{\substack{j=1 \\ j\neq h}}^{r} {}^tLineB_j \cdot V_j - \sum_{\substack{j=1 \\ j\neq h}}^{r} {}^tLineB_j \cdot W_j = 0 $$

can be written: ${}^t(A'') \cdot Y' + {}^t(B'') \cdot (V' - W') = 0$, where $Y' \neq 0$ (since $Y \neq 0$), which proves the incompatibility of the system.

Each "suggestion" of a potential change $(T_i = 1)$ of a judgement $(s, t, \ldots, \ldots)$ can thus be stored in a vector S of $\mathbb{N}^4$ where

$$
\begin{aligned}
S_1 &= s \\
S_2 &= t \\
S_3 &= \begin{cases} 1 & \text{if } \exists\, i \in \mathbb{N}_{1,p_1} \cup \mathbb{N}_{p_1+p_2+1,p_1+p_2+r} \text{ such that } T_i = 1 \\ & \quad \text{(reduction)} \\ 0 & \text{otherwise} \end{cases} \\
S_4 &= \begin{cases} 1 & \text{if } \exists\, i \in \mathbb{N}_{p_1+1,p_1+p_2} \cup \mathbb{N}_{p_1+p_2+r+1,p_1+p_2+2r} \text{ such that} \\ & T_i = 1 \quad \text{(augmentation)} \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

We will denote by *PreSugg* the set of these "pre-suggestions". In the case of example 1 (see Section 7.3) one has

$$PreSugg = \{(a_1, a_3, 1, 0), (a_3, a_4, 1, 0), (a_1, a_2, 0, 1), (a_2, a_4, 0, 1)\}.$$

7.4.3 Search for Suggestions.

DEFINITION 47 *Changing judgements by m categories is any set Modi f_m of the form $Modif_m = \{(s_1, t_1, i_1, j_1, p_1), (s_2, t_2, i_2, j_2, p_2), \ldots, (s_u, t_u, i_u, j_u, p_u)\ |\forall v \in \mathbb{N}_{1,u}, (s_v, t_v, i_v, j_v, p_v)$ is a change of judgement (s_v, t_v, i_v, j_v) with p_v categories$\}$ such that $\sum_{v=1}^{u} |p_v| = m$*

Within Example 1, $\{(a_1, a_2, 1, 1, 2), (a_3, a_4, 2, 2, -1)\}$ is a "change of judgements with 3 categories", which consists of

- to replace the judgement $(a_1, a_2) \in C_1$ with the judgement $(a_1, a_2) \in C_3$ (augmentation of 2 categories)
- to replace the judgement $(a_3, a_4) \in C_2$ with the judgement $(a_3, a_4) \in C_1$ (reduction of 1 category)

Notation: the set of "judgement changes with m categories" which renders the judgements consistent will be denoted by $Sugg_m$.

Within Example 1,

- $\{(a_1, a_2, 1, 1, 2), (a_3, a_4, 2, 2, -1)\} \in Sugg_3$
- $\{(a_1, a_3, 4, 4, -1)\}$, $\{(a_3, a_4, 2, 2, -1)\}$, $\{(a_1, a_2, 1, 1, 1)\}$ and $\{(a_2, a_4, 3, 3, 1)\} \in Sugg_1$,

these are the 4 changes suggested in Section 7.3.

Once the PreSugg group is determined, the third step is to:

- determine the "minimum number of changes" (some possibly successive) necessary to render the judgements consistent;
- determine all of the combinations of such "minimal" changes.

More rigorously, this means

- find $m_0 = \min\{m \in \mathbb{N}^* | Sugg_m \neq \emptyset\}$
- clarify $Sugg_m$

In Example 1, we have already seen that $m_0 = 1$ (since $Sugg_1 \neq \emptyset$).
We will proceed as follows for all cases of inconsistency (see Figure 10.3).

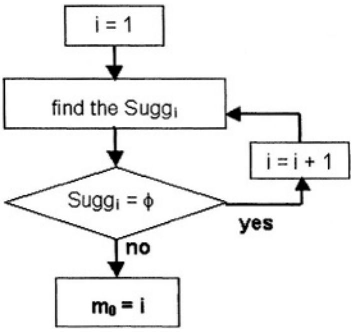

Figure 10.3. Procedure for all cases of inconsistency.

At each step i,

- the set of all "judgement changes of i categories", built on the basis of element PreSugg are considered;
- for each of the elements in this group:
  - carry out the modifications included in the selected item;
  - test the consistency of the new matrix of judgements; if it is consistent, store the element in $Sugg_i$;
  - restore the matrix to the initial judgements.

It is worth mentioning that we consider the possibility of changing a judgement by several categories.

This algorithm is always convergent since one can always give consistent judgements in a finite number of changes.

We emphasize that in practice, the cases of inconsistency that require more than 2 "changes of 1 category" are almost non-existent. The main reason being that any change in judgement that generates an inconsistency is immediately announced to *J,* who must then confirm or cancel his or her judgement.

This procedure allows one to avoid

- coarse errors of distraction (by cancelling the judgement);
- the "accumulation" of inconsistencies since, if *J* confirms his or her judgement, suggestions of changes that will eliminate the inconsistency are made.

7.5 Example 2

Suppose that $X = \{a_1, a_2, a_3, a_4\}$ and that *J* has formulated the following consistent judgements:

- $P = \{(a_1, a_2), (a_1, a_3), (a_2, a_3), (a_3, a_4)\}$
- $(a_1, a_2) \in C_1, (a_1, a_3) \in C_4, (a_2, a_3) \in C_2, (a_3, a_4) \in C_3$

Suppose that *J* adds that $a_2 P a_4$ and that $(a_2, a_4) \in C_3$:M-MACBETH informs *J* that his or her judgements are "inconsistent".

If *J* confirms the judgement $(a_2, a_4) \in C_3$, M-MACBETH will display the message: "Inconsistent judgements: MACBETH has found 6 ways to render the judgements matrix consistent with 2 category changes."

This time, it will be necessary to make at least 2 "changes of 1 category" to render the judgements consistent; there are 6 distinct combinations of such changes. Each of these 6 suggestions is presented graphically (see Figure 10.4) within the table of judgements, accompanied by SEI which, moreover, shows why the suggestions made eliminate this incompatibility: Figure 10.4 presents the first of six suggestions.

8. The MACBETH Scale

8.1 Definition of the MACBETH Scale

Suppose that $Sc_{1+2} \neq \phi$ and $a_1(P \cup I)a_2 \ldots a_{n-1}(P \cup I)a_n$. The linear program LP-MACBETH with variables $x_1, \ldots, x_n, \sigma_1, \ldots, \sigma_Q$ is therefore feasible:

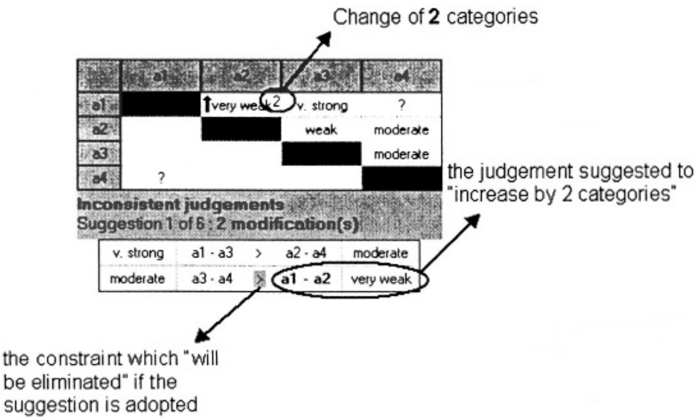

Figure 10.4. Suggestion of change to resolve inconsistency.

$$
\begin{array}{lll}
\min x_1 & & \\
\text{subject to} & & \\
x_p - x_r = 0 & \forall\, (a_p, a_r) \in I \text{ with } p < r & \text{(t1)} \\
\sigma_i + \frac{1}{2} \leq x_p - x_r & \forall\, i, j \in N_{1,Q} \text{ with } i \leq j, \forall (a_p, a_r) \in C_{ij} & \text{(t2')} \\
x_p - x_r \leq \sigma_{j+1} - \frac{1}{2} & \forall\, i, j \in N_{1,Q-1} \text{ with } i \leq j, \forall (a_p, a_r) \in C_{ij} & \text{(t3')} \\
\sigma_1 = \frac{1}{2} & & \text{(t4')} \\
\sigma_{i-1} + 1 \leq \sigma_i & \forall\, i \in N_{2,Q} & \text{(t5')} \\
x_i \geq 0 & \forall\, i \in N_{1,n} & \\
\sigma_i \geq 0 & \forall\, i \in N_{1,Q} &
\end{array}
$$

DEFINITION 48 *Any function EchMac* $: X \to \mathbb{R}$ *such that* $\forall\, i \in \mathbb{N}_{1,n}$, $EchMac(a_i) = x_i^*$ *– where* $(x_1^*, \ldots, x_n^*)$ *is an optimal solution of LP-MACBETH – is called a basic MACBETH scale.*

DEFINITION 49 $\forall\, a \in \mathbb{R}_+^*, \forall\, b \in \mathbb{R}$ *with* $(a, b) \neq (1, 0)$, $a \cdot EchMac + b$ *is a transformed MACBETH scale.*

8.2 Discussing the Uniqueness of the Basic MACBETH Scale

Nothing guarantees that a LP-MACBETH optimal solution is unique. For example, consider the matrix of judgements and the basic MACBETH scale shown is Figure 10.5.

One can verify that, $\forall\, x \in [6, 7], (8, x, 5, 2, 1, 0)$ is still an optimal solution of LP-MACBETH. Thus, a basic MACBETH scale is not necessarily unique.

	a1	a2	a3	a4	a5	a6
a1	no	very weak	weak	moderate	moderate	strong
a2		no	very weak	moderate	moderate	moderate
a3			no	weak	moderate	moderate
a4				no	very weak	very weak
a5					no	very weak
a6						no

	Macbeth basic
a1	**8.00**
a2	6.50
a3	5.00
a4	2.00
a5	1.00
a6	**0.00**

Figure 10.5. Matrix of judgements and basic MACBETH scale.

As long as the MACBETH scale is interpreted as a technical aid whose purpose is to provide the foundation for a discussion with *J*, this does not constitute a true problem. However, we have observed that in practice decision makers often adopt the MACBETH scale as the final scale. It is, therefore, convenient to guarantee the uniqueness of the MACBETH scale. This is obtained technically, as follows (where S_{mac} is the group of the constraints of LP-MACBETH):

Step 1) solution of LP-MACBETH
$\rightarrow$ optimal solution $x_1, x_2, \ldots, x_n$
$\rightarrow \mu(a_1) = x_1, \mu(a_n) = x_n = 0$ (remark: $\mu(a_1)$ is unique)

Step 2) for $i = 2$ to $n-1$
to solve $\max x_i$ under $\begin{cases} S_{mac} \\ x_1 = \mu(a_1), \ldots, x_{i-1} = \mu(a_{i-1}) \end{cases}$
$\rightarrow$ optimal solution $x_1, x_2, \ldots, x_n$
$\rightarrow xmax = x_i$
to solve $\min x_i$ under $\begin{cases} S_{mac} \\ x_1 = \mu(a_1), \ldots, x_{i-1} = \mu(a_{i-1}) \end{cases}$
$\rightarrow$ optimal solution $x_1, x_2, \ldots, x_n$
$\rightarrow xmin = x_i$

$$\mu(a_i) = \frac{xmin + xmax}{2}$$

Thus,

- to calculate $\mu(a_2)$, the variable x_1 is "fixed" to the value $\mu(a_1)$, the minimum and maximum values of x_2 are calculated and the average of the two results is taken as the value of $\mu(a_2)$;
- to calculate $\mu(a_3)$, the variable x_1 is "fixed" to the value of $\mu(a_1)$, the variable x_2 is "fixed" to the value of $\mu(a_2)$, the minimum and maximum values of x_3 are calculated and the average of the two values is taken as the value of $\mu(a_3)$;

- etc.

This method guarantees that $\mu(a_1), \mu(a_2), \dots \mu(a_n)$ are unique for a given preferential information $\{P, I, ? = \phi, P^e\}$. It permits us to speak of "the" basic MACBETH scale, instead of "one" MACBETH scale.

8.3 Presentation of the MACBETH Scale

The MACBETH scale that corresponds to $\{P, I, ? = \phi, P^e\}$ consistent information is represented in two ways in M-MACBETH: a table and a "thermometer". In the example in Figure 10.6, the transformed MACBETH scale represented in the thermometer was obtained by imposing the values of the elements d and c as 100 and 0 respectively.

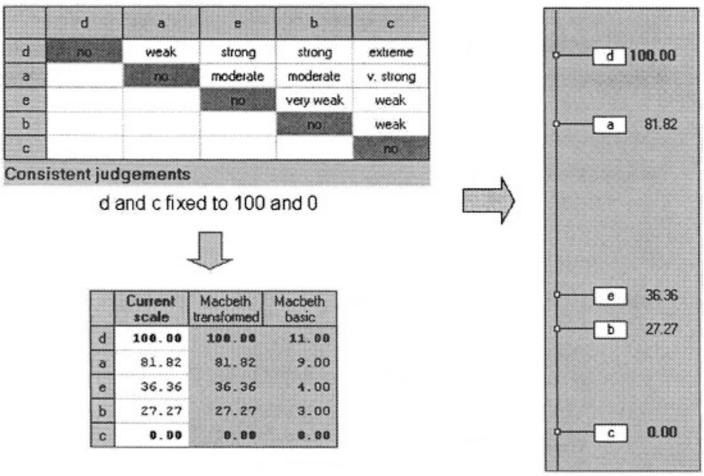

Figure 10.6. Representations of the MACBETH scale.

Even though the values attributed to c and d are fixed, in general an infinite number of scales that satisfy Conditions 1 and 2 exist. It is, thus, necessary to allow J to, should he or she want to, modify the values suggested. This is the subject of the next section.

9. Discussion About a Scale

Suppose that, in the example in Figure 10.6, J considers that the element a is badly positioned when compared to elements c and d and therefore J wants to redefine the value of a. It is then interesting to show J the limits within which the value of a can vary without violating the preferential information provided by *J*. Let us suppose in this section that we have a type 1+2 information about X which is consistent and that ? $= \phi$.

Let μ_0 be a particular scale of Sc_{1+2}, L and H be two fixed elements of X with *HPL (H* more attractive than *L)* and a be an element of X (not indifferent to L and not indifferent to H) that *J* would like to have repositioned.

Let

- $Sc_{(\mu_0,H,L)} = \{\mu \in Sc_{1+2} \mid \mu(H) = \mu_0(H)$ and $\mu(L) = \mu_0(L)\}$ (scales for which values associated with *H* and *L* have been fixed)
- $Sc_{(\mu_0,\hat{a})} = \{\mu \in Sc_{1+2} \mid \forall\, y \in X$ with y not indifferent to a: $\mu(y) = \mu_0(y)\}$ (scales for which the values of all of the elements of X except a and its eventual equals have been fixed).

We call *free interval* associated to interval a :

$$\left]\inf_{\mu\in Sc_{(\mu_0,H,L)}} \mu(a), \sup_{\mu\in Sc_{(\mu_0,H,L)}} \mu(a)\right[$$

We call *dependent interval* associated to interval a :

$$\left]\inf_{\mu\in Sc_{(\mu_0,\hat{a})}} \mu(a), \sup_{\mu\in Sc_{(\mu_0,\hat{a})}} \mu(a)\right[$$

In the example in Figure 10.6, if one selects a, two intervals are presented to *J* (see Figure 10.7) which should be interpreted as follows:

$$\forall\, \mu \in Sc_{1+2}\ ,\ [\,\mu(c) = 0, \mu(d) = 100\,] \Rightarrow 66.69 \le \mu(a) \le 99.98.$$

$$\forall\, \mu \in Sc_{1+2}\ ,\ [\,\mu(c) = 0, \mu(d) = 100, \mu(e) = 36.36, \mu(b) = 27.27\,]$$
$$\Rightarrow 72.74 \le \mu(a) \le 90.9.$$

The closed intervals (in the example [66.69, 99.98] and [72.74, 90.9]) that have been chosen to present to *J* are not the precise free and dependent intervals associated to a (which, by definition, are open); however, by taking a precision of 0.01 into account, they can be regarded as the "greatest" closed intervals included in the free and dependent intervals.

M-MACBETH permits the movement of element a with the mouse but, obviously, only inside of the dependent interval associated to a.

If *J* wants to give element a a value that is outside of the dependent interval (but still inside the free interval), the software points out that the values of the other elements must be modified. If *J* confirms the new value of a, a new MACBETH scale is calculated, taking into account the additional constraint that fix the new value of a.

The ("closed") free interval is calculated by integer linear programming. The ("closed") dependent interval could be also calculated in the same manner. However, M-MACBETH computes it by "direct" calculation formulas which make the determination of these intervals extremely fast – for details, see [46].

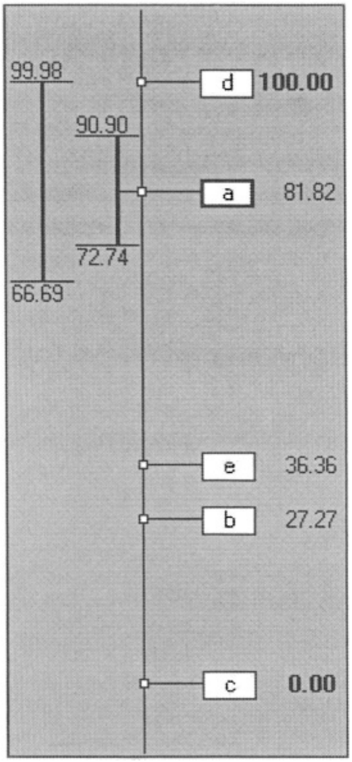

Figure 10.7. "Greatest" closed intervals included in the free and dependent intervals.

10. MACBETH and MCDA

The MACBETH approach and the M-MACBETH software have been used to derive preference scales or value functions and scaling constants in many public and private applications of multicriteria additive value analysis, some of them reported in the literature:

- Evaluation ofbids in international public calls for tenders and contractors' choice – see [3, 5, 10, 11, 12, 19, 20, 21, 33] and [58].
- Management of European structural programs – see [37, 42] and [43].

- Public policy analysis, prioritization of projects, resources allocation and conflict management – see [4, 22, 23, 24, 25, 26, 33] and [60].
- Suppliers performance evaluation – see [7] and [57].
- Credit scoring – see [8].
- Strategic town planning – see [14] and [15].
- Environmental management and evaluation of flood control measures – see [1, 6] and [17].
- Portfolio management – see [27].
- Airport management – see [39].
- Human resources evaluation and management – see [50, 47, 48] and [54].
- Total Quality Management – see [11].
- Firms' competitiveness, resource allocation and risk management – see [13] and [49].
- Location of military facilities – see [28].
- Applications in the telecommunications sector – see [18] and [44].

It is worth noting that in all these applications MACBETH was applied in a constructive framework of multicriteria additive aggregation, whose theoretical foundations are reviewed in James Dyer's chapter in this book (Chapter 7).

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V

NON-CLASSICAL MCDA APPROACHES