Chapter 7

Formulation of Equations

7.1 Remarks on Formulating Problems

Procedure for Formulating Problems via Lagrange's Equations. Although there is a wide variety of dynamics problems, the following procedure is generally applicable:

1. Identify and classify all constraints and given forces.
2. Choose suitable coordinates – any set that completely specifies the configuration of the system.
3. Write the kinetic energy, the potential energy, the nonconservative given forces, and the constraints in terms of the choosen coordinates.
4. Substitute the results into Lagrange's equations.

The result is a set of ordinary differential equations, the solution of which gives the motion, that is, the path through the configuration space.

Choice of Coordinates. The goal is to choose the coordinates that make formulating the problem easiest; although this is frequently obvious, especially in simple problems, general guides may be helpful:

1. It is almost always desirable to choose the minimal number of coordinates, i.e. generalized coordinates. These coordinates eliminate the holonomic constraints directly and give the fewest number of differential equations.

2. The holonomic constraints frequently indicate the most desirable coordinates; for example, if the motion is confined to be on the surface of a sphere, spherical coordinates are indicated.

3. Because of the simplicity in T for rectangular coordinates, they should always be considered.

7.2 Unconstrained Particle

Rectangular (Cartesian) Coordinates. Since $n = N = 3$ for a particle moving without constraints in 3-D, the rectangular components (x, y, z) serve as generalized coordinates:

$$q_1 = x\,, \qquad q_2 = y\,, \qquad q_3 = z$$

In these coordinates, from Eqn. (4.2),

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \tag{7.1}$$

Apply Lagrange's equations, Eqns. (6.29),

$$\frac{d}{d}\left(\frac{\partial T}{\partial \dot{q}_s}\right) - \frac{\partial T}{\partial q_s} - Q_s = 0\,; \qquad s = 1, 2, 3 \tag{7.2}$$

The three equations are

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x} - F_x = 0\,, \\ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}}\right) - \frac{\partial T}{\partial y} - F_y = 0\,, \\ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{z}}\right) - \frac{\partial T}{\partial z} - F_z = 0 \end{aligned} \tag{7.3}$$

From Eqn. (7.1) the terms in these equations are:

$$\frac{\partial T}{\partial \dot{x}} = m\dot{x}\,, \qquad \frac{\partial T}{\partial \dot{y}} = m\dot{y}\,, \qquad \frac{\partial T}{\partial \dot{z}} = m\dot{z}$$

$$\frac{\partial T}{\partial x} = 0\,, \qquad \frac{\partial T}{\partial y} = 0\,, \qquad \frac{\partial T}{\partial z} = 0$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) = m\ddot{x}\ , \qquad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}}\right) = m\ddot{y}\ , \qquad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{z}}\right) = m\ddot{z}$$

$$Q_1 = F_x\ , \qquad Q_2 = F_y\ , \qquad Q_3 = F_z$$

Thus the equations of motion are

$$m\ddot{x} - F_x = 0\ , \qquad m\ddot{y} - F_y = 0\ , \qquad m\ddot{z} - F_y = 0 \qquad (7.4)$$

Cylindrical Coordinates. Choosing $(q_1, q_2, q_3) = (r, \phi, z)$, the transformation equations giving the rectangular coordinates in terms of the generalized coordinates are Eqns. (1.17) (Fig. 7-1):

$$\begin{aligned} x &= r\cos\phi \\ y &= r\sin\phi \\ z &= z \end{aligned} \qquad (7.5)$$

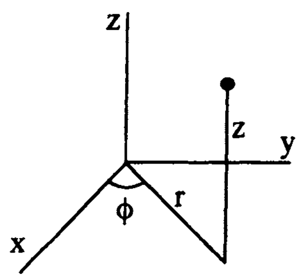


Fig. 7-1

We need to express the kinetic energy in cylindrical coordinates. Differentiating,

$$\dot{x} = \dot{r}\cos\phi - r\dot{\phi}\sin\phi\ ; \qquad \dot{y} = \dot{r}\sin\phi + r\dot{\phi}\cos\phi\ ; \qquad \dot{z} = \dot{z} \quad (7.6)$$

Substituting into Eqn. (7.1):

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2\right) \qquad (7.7)$$

Also

$$Q_1 = F_r\ , \qquad Q_2 = F_\phi\ , \qquad Q_3 = F_z$$

Lagrange's equations then give the equations of motion; the terms are

$$\frac{\partial T}{\partial \dot{r}} = m\dot{r}\,, \qquad \frac{\partial T}{\partial \dot{\phi}} = mr^2\dot{\phi}\,, \qquad \frac{\partial T}{\partial \dot{z}} = m\dot{z}$$

$$\frac{\partial T}{\partial r} = mr\dot{\phi}^2\,, \qquad \frac{\partial T}{\partial \phi} = 0\,, \qquad \frac{\partial T}{\partial z} = 0$$

Substituting into Eqns. (6.29):

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{r}}\right) - \frac{\partial T}{\partial r} - F_r = 0$$
$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}}\right) - \frac{\partial T}{\partial \phi} - F_\phi = 0$$
$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{z}}\right) - \frac{\partial T}{\partial z} - F_z = 0$$

$$\begin{aligned} &m\ddot{r} - mr\dot{\phi}^2 - F_r = 0 \\ &2mr\dot{r}\dot{\phi} + mr^2\ddot{\phi} - F_\phi = 0 \\ &m\ddot{z} - F_z = 0 \end{aligned} \tag{7.8}$$

If the force is given in rectangular components, Eqns. (6.15) may be used to get the generalized force components:

$$F_s = \sum_{i=1}^{3} F_i \frac{\partial u_i}{\partial q_s}\,; \qquad s = 1, 2, 3$$

$$F_r = F_x \frac{\partial x}{\partial r} + F_y \frac{\partial y}{\partial r} + F_z \frac{\partial z}{\partial r}$$

$$F_\phi = F_x \frac{\partial x}{\partial \phi} + F_y \frac{\partial y}{\partial \phi} + F_z \frac{\partial z}{\partial \phi}$$

$$F_z = F_x \frac{\partial x}{\partial z} + F_y \frac{\partial y}{\partial z} + F_z \frac{\partial z}{\partial z}$$

$$\begin{aligned} F_r &= F_x \cos\phi + F_y \sin\phi \\ F_\phi &= -F_x r \cos\phi + F_y r \cos\phi \\ F_z &= F_z \end{aligned} \tag{7.9}$$

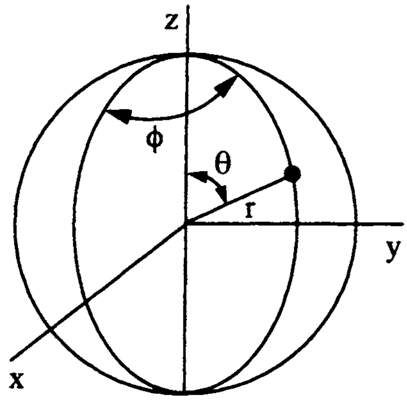


Fig. 7-2

Spherical Coordinates. Choosing $(q_1, q_2, q_3) = (r, \theta, \phi)$, the transformation equations are Eqns. (1.19) (see Fig. 7-2):

$$\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned} \tag{7.10}$$

We now need to express the kinetic energy in spherical coordinates. Differentiating,

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{1}{2}m(\dot{r}^2 + r^2\sin^2\theta\dot{\phi}^2 + r^2\dot{\theta}^2) \tag{7.11}$$

The generalized force components are

$$Q_1 = F_r\ , \qquad Q_2 = F_\theta\ , \qquad Q_3 = F_\phi$$

The partials are

$$\frac{\partial T}{\partial \dot{r}} = m\dot{r}\ ; \qquad \frac{\partial T}{\partial \dot{\theta}} = mr^2\dot{\theta}\ ; \qquad \frac{\partial T}{\partial \dot{\phi}} = mr^2\sin^2\theta\dot{\phi}$$

$$\frac{\partial T}{\partial r} = mr\sin^2\theta\dot{\phi}^2 + mr\dot{\theta}^2$$

$$\frac{\partial T}{\partial \theta} = mr^2\sin\theta\cos\theta\dot{\phi}^2$$

$$\frac{\partial T}{\partial \phi} = 0$$

Thus Lagrange's equations give

$$\begin{aligned}
&\frac{d}{dt}(m\dot{r}) - mr\sin^2\theta\dot{\phi}^2 - mr\dot{\theta}^2 - F_r = 0 \\
&\frac{d}{dt}(mr^2\dot{\theta}) - mr^2\sin\theta\cos\theta\dot{\phi}^2 - F_\theta = 0 \\
&\frac{d}{dt}(mr^2\sin^2\theta\dot{\phi}) - F_\phi = 0
\end{aligned} \tag{7.12}$$

If the force is given in rectangular components,

$$\begin{aligned}
F_s &= \sum_{i=1}^{3} F_i \frac{\partial u_i}{\partial q_s}\,; \qquad s = 1,2,3 \\
F_r &= F_x \sin\theta\cos\phi + F_y \sin\theta\sin\phi + F_z\cos\theta \\
F_\theta &= F_x r\cos\theta\cos\phi + F_y r\cos\theta\sin\phi - F_z r\sin\theta \\
F_\phi &= -F_x r\sin\theta\sin\phi + F_y r\sin\theta\cos\phi
\end{aligned} \tag{7.13}$$

Note that F_r is a force whereas F_θ and F_ϕ are moments. Equations (7.12) will be our starting point for a future topic – central force motion (Chapter 10).

7.3 Constrained Particle

One Holonomic Constraint in Planar Motion. Assume the constraint is rheonomic (scleronomic will be a special case), as shown on Fig. 7-3:

$$y = f(x,t)$$

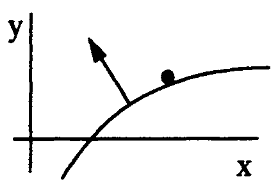


Fig. 7-3

Since the constraint is holonomic, it can be eliminated directly by embedding as follows

$$\dot{y} = f_x \dot{x} + f_t$$
$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}m(\dot{x}^2 + f_x^2\dot{x}^2 + 2f_x\dot{x}f_t + f_t^2)$$

Lagrange's equation gives the differential equation of motion; the terms are

$$\frac{\partial T}{\partial \dot{x}} = \frac{m}{2}(2\dot{x} + 2f_x^2\dot{x} + 2f_xf_t)$$

$$\underbrace{\frac{d}{dt}}_{\substack{\text{total}\\ \text{derivative}}} \underbrace{\left(\frac{\partial T}{\partial \dot{x}}\right)}_{\substack{\text{partial}\\ \text{derivative}\\ \text{holding}\\ x\ \&\ t\ \text{fixed}}} = m\Big[\ddot{x} + 2f_x(f_{xx}\dot{x} + f_{xt})\dot{x} + f_x^2\ddot{x} + (f_{xx}\dot{x} + f_{xt})f_t + f_x(f_{tx}\dot{x} + f_{tt})\Big]$$

$$\underbrace{\frac{\partial T}{\partial x}}_{\substack{\text{partial derivative}\\ \text{holding } \dot{x}\ \&\ t\ \text{fixed}}} = \frac{1}{2}m(2f_xf_{xx}\dot{x}^2 + 2f_{xx}\dot{x}f_t + 2f_x\dot{x}f_{tx} + 2f_tf_{tx})$$

Substitution into the following equation then gives the equation of motion:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x} - F_x = 0 \tag{7.14}$$

where F_x is the x-component of the *given* force. The constraint does not appear (either geometrically or as a force) because it has been explicitly removed.

As a special case, suppose the constraint is scleronomic, $y = f(x)$; then Eqn. (7.14) reduces to:

$$m\left[(1 + f_x^2)\ddot{x} + f_x\, f_{xx}\, \dot{x}^2\right] - F_x = 0 \tag{7.15}$$

Nonholonomic Constraints. Here, $\ell > 0$, $DOF = N - L$, $L = L' + \ell$, $n = N - L' > DOF$. This is because a nonholonomic constraint does not affect the accessibility of the configuration space; one particle in 3-space subject to a nonholonomic constraint has 3 generalized coordinates.

Most nonholonomic constraints arise as constraints on velocities, for example in problems involving the rolling of one body on another.

As an example, suppose a particle is subject to a single catastatic nonholonomic constraint

$$a\delta x + b\delta y + c\delta z = 0$$

Comparison with Eqn. (5.15) with $n = 3$ and $\ell = 1$ gives

$$B_{11} = a\,, \qquad B_{12} = b\,, \qquad B_{13} = c$$

Equations (6.29) then give, in rectangular coordinates,

$$\begin{aligned} m\ddot{x} - F_x + \lambda a = 0 \\ m\ddot{y} - F_y + \lambda b = 0 \\ m\ddot{z} - F_z + \lambda c = 0 \end{aligned} \tag{7.16}$$

These three equations, along with

$$a\dot{x} + b\dot{y} + c\dot{z} = 0$$

provide four equations in the four unknowns x, y, z, and λ.

7.4 Example – Two Link Robot Arm

Problem Definition. Consider two rigid bodies connected together and moving in a plane as shown (Fig. 7-4). This may be considered a typical robot arm. $\underline{M}_1$ and $\underline{M}_2$ are motor torques.

All the constraints are holonomic and there are two generalized coordinates. We choose

$$q_1 = \theta_1\,; \qquad q_2 = \theta_2$$

We could choose ϕ instead of θ_2 but θ_2 is somewhat easier since we need the velocities and angular velocities relative to the inertial frame (x, y) for use in determining the kinetic energy.

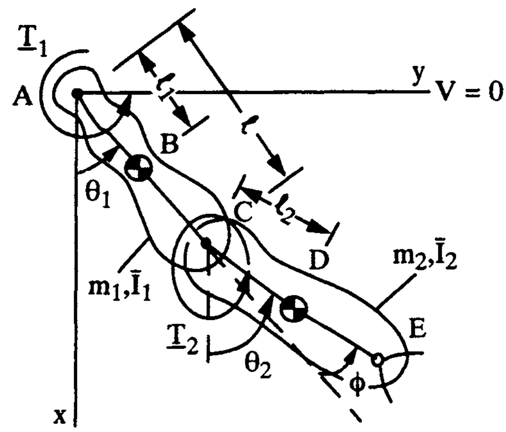


Fig. 7-4

Kinetic Energy. For two rigid bodies, Eqn. (1.56) gives

$$T = \sum_{i=1}^{2} \left(\frac{1}{2} m_i \, \bar{v}_i^2 + \frac{1}{2} \bar{I}_i \, \omega_i^2 \right) \tag{7.17}$$

First consider body # 1 (Fig. 7-5).

$$\begin{aligned} v_B &= \ell_1 \dot{\theta}_1 \\ T_1 &= \frac{1}{2} m_1 v_B^2 + \frac{1}{2} \bar{I}_1 \omega_1^2 = \frac{1}{2} m_1 \ell_1^2 \dot{\theta}_1^2 + \frac{1}{2} \bar{I}_1 \dot{\theta}_1^2 = \frac{1}{2} I_1 \dot{\theta}_1^2 \end{aligned} \tag{7.18}$$

where

$$I_1 = \bar{I}_1 + m_1 \ell_1^2$$

is the moment of inertia about A by the parallel axis theorem. This agrees with the alternative expression given by Eqn. (1.57).

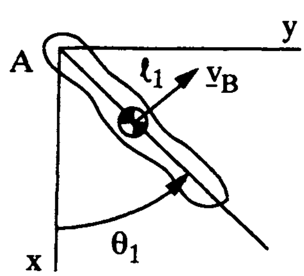


Fig. 7-5

Now consider body # 2 (Fig. 7-6). Let $\{\hat{\imath}_1, \hat{\jmath}_1\}$, $\{\hat{\imath}_2, \hat{\jmath}_2\}$, and $\{\hat{\imath}_3, \hat{\jmath}_3\}$ be reference frames fixed in the ground, in link AC, and in link CD, respectively. Use Eqn. (1.25) to relate the velocity of D relative to $\{\hat{\imath}_3, \hat{\jmath}_3\}$ to its velocity relative to $\{\hat{\imath}_1, \hat{\jmath}_1\}$ and let $\underline{\omega}$ be the angular velocity of $\{\hat{\imath}_3, \hat{\jmath}_3\}$ relative to $\{\hat{\imath}_1, \hat{\jmath}_1\}$:

$$\underline{v}_D = \underline{v}_C + \underline{v}_{\text{rel}} + \underline{\omega} \times \underline{r}$$

where

$$\underline{v}_D = \frac{d\underline{r}_D}{dt}, \qquad \underline{v}_C = \frac{d\underline{r}_C}{dt}$$

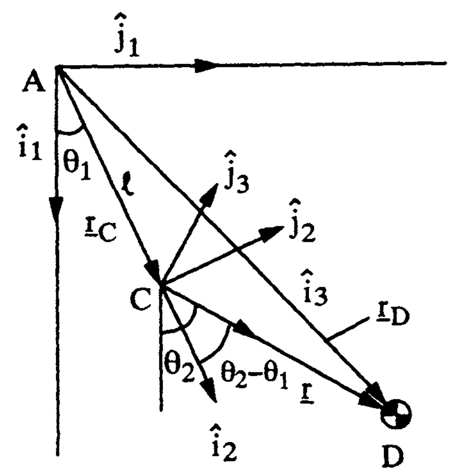


Fig. 7-6

The terms are

$$\underline{v}_C = \ell\dot{\theta}_1\hat{\jmath}_2\,, \qquad \underline{v}_{\text{rel}} = \underline{0}\,, \qquad \underline{\omega} = \dot{\theta}_2\hat{k}_3\,, \qquad \underline{r} = \ell_2\hat{\imath}_3$$

The required unit vector transformation is

$$\hat{\jmath}_2 = \sin(\theta_2 - \theta_1)\hat{\imath}_3 + \cos(\theta_2 - \theta_1)\hat{\jmath}_3$$

Substituting,

$$\begin{aligned} \underline{v}_D &= \ell\dot{\theta}_1\hat{\jmath}_2 + \underline{0} + \dot{\theta}_2\hat{k}_3 \times \ell_2\hat{\imath}_3 \\ &= \ell\dot{\theta}_1\sin(\theta_2 - \theta_1)\hat{\imath}_3 + \left[\ell_2\dot{\theta}_2 + \ell\dot{\theta}_1\cos(\theta_2 - \theta_1)\right]\hat{\jmath}_3 \qquad (7.19) \end{aligned}$$

(This also could be obtained geometrically from the law of cosines.) T_2 is then

$$\begin{aligned} T_2 &= \frac{1}{2}m_2 v_D^2 + \frac{1}{2}\bar{I}_2 \omega_2^2 \\ &= \frac{1}{2}m_2\left[\ell^2\dot{\theta}_1^2 + \ell_2^2\dot{\theta}_2^2 + 2\ell\ell_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_2-\theta_1)\right] + \frac{1}{2}\bar{I}_2\dot{\theta}_2^2 \quad (7.20) \end{aligned}$$

Note that is is quadratic in the velocity components with displacement dependent coefficients; the constant and linear terms are missing.

Consequently,

$$T = T_1 + T_2 = \frac{1}{2}I'\dot{\theta}_1^2 + \frac{1}{2}I_2\dot{\theta}_2^2 + m_2\ell\ell_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_2-\theta_1) \qquad (7.21)$$

where

$$I' = I_1 + m_2\ell^2\ ; \qquad I_2 = \bar{I}_2 + m_2\ell_2^2$$

Potential Energy and Lagrangian. Since the only given force is gravity,

$$V = -m_1 g\ell_1\cos\theta_1 - m_2 g(\ell\cos\theta_1 + \ell_2\cos\theta_2) \qquad (7.22)$$

$$\begin{aligned} L &= T - V = \frac{1}{2}I'\dot{\theta}_1^2 + \frac{1}{2}I_2\dot{\theta}_2^2 + m_2\ell\ell_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_2-\theta_1) \\ &\quad + m_1 g\ell_1\cos\theta_1 + m_2 g(\ell\cos\theta_1 + \ell_2\cos\theta_2) \qquad (7.23) \end{aligned}$$

Lagrange's Equations. Equations (6.34) are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}_1}\right) - \frac{\partial L}{\partial\theta_1} = Q_1\ ; \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}_2}\right) - \frac{\partial L}{\partial\theta_2} = Q_2 \qquad (7.24)$$

Computing some of the terms

$$\frac{\partial L}{\partial\dot{\theta}_1} = I'\dot{\theta}_1 + m_2\ell\ell_2\dot{\theta}_2\cos(\theta_2-\theta_1)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}_1}\right) = I'\ddot{\theta}_1 + m_2\ell\ell_2\left[\ddot{\theta}_2\cos(\theta_2-\theta_1) - \dot{\theta}_2\sin(\theta_2-\theta_1)(\dot{\theta}_2-\dot{\theta}_1)\right]$$

$$\frac{\partial L}{\partial\theta_1} = m_2\ell\ell_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_2-\theta_1) - m_1 g\ell_1\sin\theta_1 - m_2 g\ell\sin\theta_1$$

Finally, substitution gives the equations of motion

$$\begin{aligned} I'\ddot{\theta}_1 + A\cos(\theta_2 - \theta_1)\ddot{\theta}_2 - A\sin(\theta_2 - \theta_1)\dot{\theta}_2^2 + B\sin\theta_1 &= M_1 \\ I_2\ddot{\theta}_2 + A\cos(\theta_2 - \theta_1)\ddot{\theta}_1 + A\sin(\theta_2 - \theta_1)\dot{\theta}_1^2 + C\sin\theta_2 &= M_2 \end{aligned} \tag{7.25}$$

where

$$Q_1 = M_1, \, , \qquad Q_2 = M_2$$

and

$$A = m_2\ell\ell_2\,; \qquad B = m_1g\ell_1 + m_2g\ell\,; \qquad C = m_2g\ell_2$$

Equations (7.25) are dynamically coupled and highly nonlinear, making their solution difficult.

Linearizing for small angles and small angular rates, and setting $M_1 = M_2 = 0$ gives the special case of the double physical pendulum.

$$\begin{aligned} I'\ddot{\theta}_1 + A\ddot{\theta}_2 + B\theta_1 &= 0 \\ I_2\ddot{\theta}_2 + A\ddot{\theta}_1 + C\theta_2 &= 0 \end{aligned} \tag{7.26}$$

These equations are linear but still dynamically coupled. They may be easily dynamically uncoupled by a change of variables. Modal analysis of these equations for typical cases reveals two modes, one rapid and one relatively slow.

7.5 Example – Rolling Disk

Problem Definition. A thin homogeneous disk of radius r rolls without slipping on a horizontal plane (Fig. 7-7). Recall that a rigid body in motion in 3-space without contraints has six degrees of freedom and every point of the body is specified by 6 independent parameters, say (x, y, z), the location of some body-fixed point with respect to an inertial frame, and (ϕ, ψ, θ), three angles of body-fixed lines. If the body rolls on a plane, there is one holonomic constraint ($y = 0$ in our case) and $n = N - L' = 6 - 1 = 5$. For the disk we choose $(x, z, \phi, \psi, \theta)$ as generialized coordinates as shown.

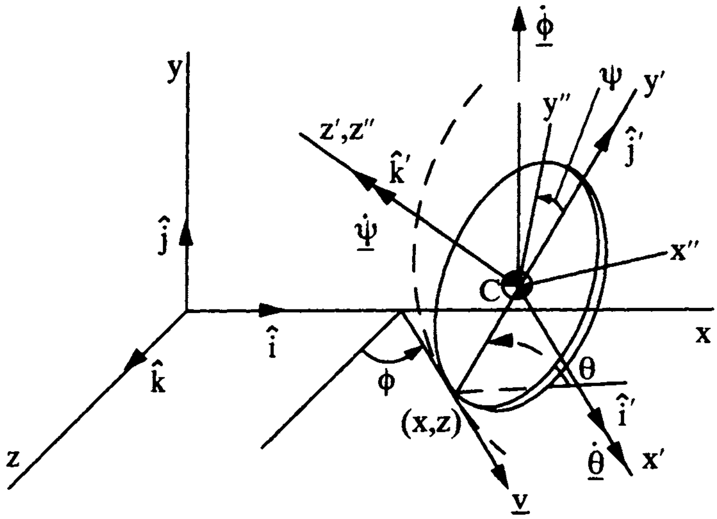


Fig. 7-7

Introduce the following reference frames as shown on Fig. 7-7:

1. $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ ground fixed (inertial).
2. $\{\hat{\imath}', \hat{\jmath}', \hat{k}'\}$ neither ground nor body fixed with $\hat{k}'$ along the axis of symmetry of the disk and $\hat{\jmath}'$ along a diameter of the disk passing through the contact point and the disk center.
3. $\{\hat{\imath}'', \hat{\jmath}'', \hat{k}''\}$ body fixed with $\hat{k}''$ along the axis of symmetry of the disk.

By the definition of pure rolling, the velocity of the disk at the contact point is zero and the contact point itself moves with velocity in the (x, y) plane in the $\hat{\imath}'$ direction; therefore the rolling constraint is

$$\underline{v} = -r\dot{\psi}\hat{\imath}' = -r\dot{\psi}(\cos\phi\hat{k} + \sin\phi\hat{\imath})$$

Since $\underline{v} = \dot{z}\hat{k} + \dot{x}\hat{\imath}$, we have in component form

$$\dot{x} = -r\dot{\psi}\sin\phi$$
$$dx + r\sin\phi \, d\,\psi = 0$$
$$\delta x + r\sin\phi \, \delta\,\psi = 0 \qquad (7.27)$$

$$\dot{z} = -r\dot{\psi}\cos\phi$$
$$dz + r\cos\phi \, d\,\psi = 0$$
$$\delta z + r\cos\phi \, \delta\,\psi = 0 \qquad (7.28)$$

These are two nonholonomic constraints. (A special case is rolling along a straight line; now, $\theta = \pi/2$, $\phi =$ constant and these constraints are integrable. This shows that the issue of nonholonomic constraints doesn't usually come up in two-dimensional problems.)

Kinetic Energy. For a rigid body,

$$T = \frac{1}{2}m\overline{v}^2 + \frac{1}{2}\overline{I}\omega^2 \tag{1.56}$$

First consider the translational term:

$$\overline{v}^2 = \dot{\overline{x}}^2 + \dot{\overline{y}}^2 + \dot{\overline{z}}^2 \tag{7.29}$$

From Fig. 7-8:

$$\begin{aligned}
\overline{x} &= x + r\cos\theta\cos\phi \\
\overline{y} &= r\sin\theta \\
\overline{z} &= z - r\cos\theta\sin\phi
\end{aligned}$$

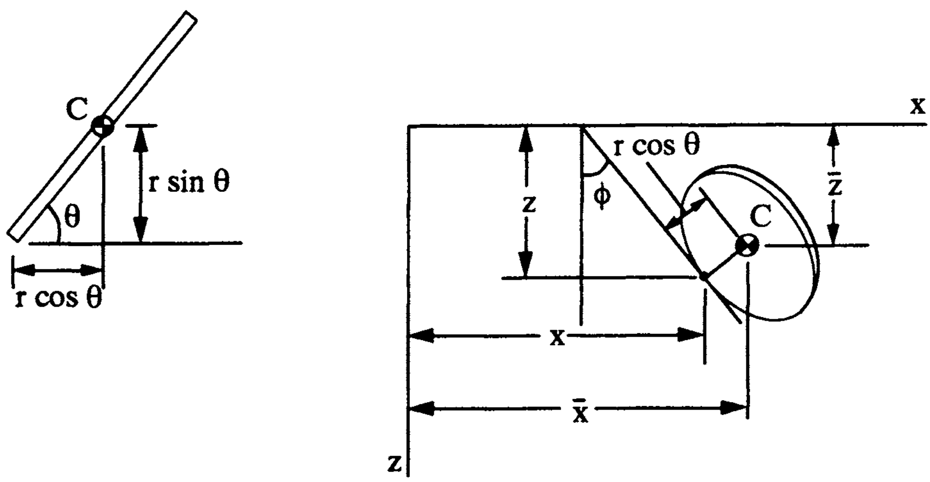


Fig. 7-8

Differentiating

$$\begin{aligned}
\dot{\overline{x}} &= \dot{x} - r\dot{\theta}\sin\theta\cos\phi - r\dot{\phi}\cos\theta\sin\phi \\
\dot{\overline{y}} &= r\dot{\theta}\cos\theta \\
\dot{\overline{z}} &= \dot{z} + r\dot{\theta}\sin\theta\sin\phi - r\dot{\phi}\cos\theta\cos\phi
\end{aligned}$$

Substitution of these in Eqn. (7.29) gives

$$\begin{aligned}\bar{v}^2 &= \dot{x}^2 + \dot{z}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\cos^2\theta + 2r(-\dot{x}\dot{\theta}\sin\theta\cos\phi \\ &\quad -\dot{x}\dot{\phi}\cos\theta\sin\phi + \dot{z}\dot{\theta}\sin\theta\sin\phi - \dot{z}\dot{\phi}\cos\theta\cos\phi) \quad (7.30)\end{aligned}$$

Next consider the rotational term. Figure 7-7 shows the directions of the angular velocity components:

$$\begin{aligned}\underline{\omega} = \dot{\theta}\hat{i}' + \dot{\psi}\hat{k}' + \dot{\phi}\hat{j} &= \dot{\theta}\hat{i}' + \dot{\psi}\hat{k}' + \dot{\phi}(\cos\theta\hat{k}' + \sin\theta\hat{j}') \\ &= \dot{\theta}\hat{i}' + \dot{\phi}\sin\theta\hat{j}' + (\dot{\psi} + \dot{\phi}\cos\theta)\hat{k}' \quad (7.31)\end{aligned}$$

Because of the rotational mass symmetry, the (x', y', z') axes are principle axes of inertia; let

$$I = I_{x'x'} = I_{y'y'}\ ; \qquad J = I_{z'z'} \qquad (7.32)$$

(The products of inertia are of course all zero.) The angles (θ, ψ, ϕ) are Euler's angles. The rotational term is then

$$\begin{aligned}\frac{1}{2}\bar{I}\omega^2 &= \frac{1}{2}\left(I_{x'x'}\omega_{x'}^2 + I_{y'y'}\omega_{y'}^2 + I_{z'z'}\omega_{z'}^2\right) \\ &= \frac{1}{2}I\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right) + \frac{1}{2}J\left(\dot{\psi} + \dot{\phi}\cos\theta\right)^2 \qquad (7.33)\end{aligned}$$

Also

$$V = mgr\sin\theta \qquad (7.34)$$

and $L = T - V$.

Lagrange's Equations. The nonholonomic constraints are of the form

$$\sum_{s=1}^{5} B_{rs}\,\delta q_s = 0\ ; \qquad r = 1, 2$$

Letting $(q_1, \cdot\cdot, q_5) = (x, z, \theta, \phi, \psi)$ and comparing with the constraint equations, Eqns. (7.27) and (7.28),

$$\begin{matrix} B_{11} = 1, & B_{15} = r\sin\phi, & B_{12} = B_{13} = B_{14} = 0 \\ B_{22} = 1, & B_{25} = r\cos\phi, & B_{21} = B_{23} = B_{24} = 0 \end{matrix} \qquad (7.35)$$

The appropriate form of Lagrange's equation is:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) - \frac{\partial L}{\partial q_s} + \sum_{r=1}^{2}\lambda_r B_{rs} = 0\ ; \qquad s = 1, \cdot\cdot, 5 \qquad (7.36)$$

Combining Eqns. (1.56), (7.30), and (7.33) - (7.35), the resulting equations of motion are:

$$
\begin{aligned}
&\frac{d}{dt}[m\dot{x} + mr(-\dot{\theta}\sin\theta\cos\phi - \dot{\phi}\cos\theta\sin\phi] + \lambda_1 = 0\,,\\
&\frac{d}{dt}[m\dot{z} + mr(\dot{\theta}\sin\theta\sin\phi - \dot{\phi}\cos\theta\cos\phi)] + \lambda_2 = 0\,,\\
&\frac{d}{dt}[mr^2\dot{\theta} + mr(-\dot{x}\sin\theta\cos\phi + \dot{z}\sin\theta\sin\phi) + I\dot{\theta}]\\
&\quad +mr^2\dot{\phi}^2\cos\theta\sin\theta + mr(\dot{x}\dot{\theta}\cos\theta\cos\phi - \dot{x}\dot{\phi}\sin\theta\sin\phi\\
&\quad -\dot{z}\dot{\theta}\cos\theta\sin\phi - \dot{z}\dot{\phi}\sin\theta\cos\phi) - I\dot{\phi}^2\sin\theta\cos\theta\\
&\quad +J(\dot{\psi} + \dot{\phi}\cos\theta)\dot{\phi}\sin\theta + mgr\cos\theta = 0\,,\\
&\frac{d}{dt}[mr^2\dot{\phi}\cos^2\theta + mr(-\dot{x}\cos\theta\sin\phi - \dot{z}\cos\theta\cos\phi)\\
&\quad +I\dot{\phi}\sin^2\theta + J(\dot{\psi} + \dot{\phi}\cos\theta)\cos\theta] + mr(-\dot{x}\dot{\theta}\sin\theta\sin\phi\\
&\quad +\dot{x}\dot{\phi}\cos\theta\cos\phi + \dot{z}\dot{\theta}\sin\theta\cos\phi - \dot{z}\dot{\phi}\cos\theta\sin\phi) = 0\,,\\
&\frac{d}{dt}[J(\dot{\psi} + \dot{\phi}\cos\theta)] + \lambda_1 r\sin\phi + \lambda_2 r\cos\phi = 0\,.
\end{aligned}
\tag{7.37}
$$

The two nonholonimc constraint equations in velocity form are:

$$
\begin{aligned}
\dot{x} + r\dot{\psi}\sin\phi = 0\\
\dot{z} + r\dot{\psi}\cos\phi = 0
\end{aligned}
\tag{7.38}
$$

Equations (7.37) plus (7.38) are seven equations in the seven unknowns $x, z, \theta, \phi, \psi, \lambda_1$, and λ_2. They are highly coupled and highly nonlinear, and thus difficult to solve.

PROBLEMS

7/1. An unconstrained particle of mass m moves in 3-space under a force

$$F = X_0\hat{i} + Y_0\hat{j} + Z_0\hat{k}\,,$$

where X_0, Y_0, Z_0 are constants. Write the Lagrangian equations of motion in the generalized coordinates ξ, η ζ, which are connected to the Cartesian coordinates by

$$x = l(\xi^2 - \eta^2)\,, \qquad y = 2l\xi\eta\,, \qquad z = \zeta\,.$$

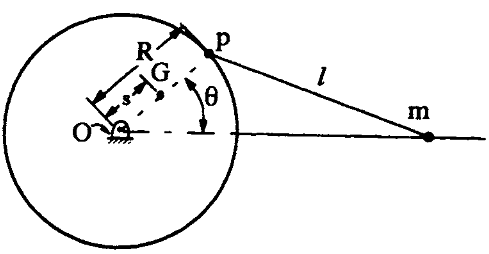
and state why ξ, η, and ζ are called parabolic, cyclindrical coordinates. Calculate an arc length ds in terms of ξ, η, and ζ. Denote the generalized force components by E, H, and Z, respectively.

7/2. Let the Cartesian coordinates of a 4-space be w, x, y, z. A particle of unit mass moves on the surface of a four-dimensional sphere of radius R under a potential force, and the potential energy is constant on the cylindrical surface $w^2 + x^2$ = constant. If θ, φ, and ψ are connected to w x y, and z by

$$w = R\cos\theta\cos\varphi\,, \qquad x = R\cos\theta\sin\varphi\,,$$
$$y = R\sin\theta\cos\psi\,, \qquad z = R\sin\theta\sin\psi\,,$$

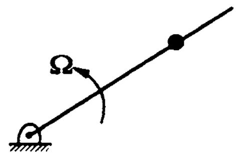
show that θ, φ, and ψ are suitable generalized coordinates, and construct Lagrange's equations in θ, φ, and ψ.

7/3. A heavy eccentric disk can rotate about a fixed, smooth, horizontal axis at O. Let its mass moment of inertia about the axis of rotation be I, and let its mass center G be a distance s from the axis of rotation. A massless connecting rod of length l is smoothly hinged to the disk at a point P a distance R from the axis of rotation, and connected to a particle of mass m, which is constrained to move on a smooth horizontal surface as shown. O, G, and P lie on a straight line. If gravity is the only force acting on the system, define suitable coordinates and construct Lagrange's equations of motion for this system.



Problem 7/3

7/4. A heavy bead of mass m slides on a smooth rod that rotates with constant angular velocity Ω about a fixed point lying on the rod centerline, as shown. What are Lagrange's equations of motion of the bead in suitably chosen generalized coordinates?



Problem 7/4

7/5. Use Lagrange's equations to find the equations of motion of the system described in the first example of Section 4.3.