Chapter 5

Generalized Coordinates

5.1 Theory of Generalized Coordinates

Remarks. As revealed by the simple pendulum (Fig. 5-1), for example, use of Newton's Second Law in rectangular coordinates has the shortcoming that both the constraint force T and the gravitational force W appear explicitly. In the energy method, however, the first of these doesn't appear at all and the second appears via a potential energy function (see Fig. 3-9).

A second important shortcoming is that coordinates x and y are awkward and, more fundamentally, one is redundant; only one coordinate is needed and θ is the obvious choice. We now take up this second point and introduce "generalized coordinates", of which an example is θ for the pendulum.

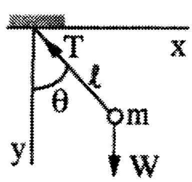


Fig. 5-1

Generalized Coordinates. Suppose a system of $N/3$ particles is constrained by L independent constraints, Eqns. (3.2):

$$\sum_{s=1}^{N} A_{rs} du_s + A_r dt = 0 \; ; \qquad r = 1, \cdot\cdot, L$$

Suppose further that $L'(< L)$ of these are holonomic and $L - L' = \ell$ are nonholonomic, i.e. if the integrated form of the holonomic constraints is

$$f_i(u_1, \cdot\cdot, u_N, t) = \alpha_i \; ; \qquad i = 1, \cdot\cdot, L' \tag{5.1}$$

then the Pfaffian form of the constraints is

$$\sum_{s=1}^{N} \frac{\partial f_i}{\partial u_s} du_s + \frac{\partial f_i}{\partial t} dt = 0 \; ; \qquad i = 1, \cdot\cdot, L' \tag{5.2}$$

$$\sum_{s=1}^{N} A_{js} \, du_s + A_j \, dt = 0 \; ; \qquad j = L' + 1, \cdot\cdot, L \tag{5.3}$$

Recall from Sections 2.3 and 2.7 that DOF $= N - L$ and DSAC $= N - L'$. From now on we will use the symbol n to denote the DSAC; that is $n = N - L'$ (note that, therefore, n will *not* denote the number of particles as before).

Now make the following definitions:

1. Any finite set of numbers $\{q_1, q_2, \cdot\cdot, q_{\overline{n}}\}$, $\overline{n} \geq n$, that completely defines the configuration of a system at a given instant is a set of *coordinates*.

2. Any set of numbers $\{q_1, \cdot\cdot, q_n\}$ is called a set of *generalized coordinates* where n is defined as above. Thus n is the least possible number of coordinates and excedes the DOF by the number of nonholonomic constraints.[1]

Transformation of Coordinates. We wish to transform from rectangular coordinates to generalized coordinates. Introduce transformation functions

$$q_s = \rho_s(u_1, \cdot\cdot, u_N, t) \; ; \qquad s = 1, \cdot\cdot, N \tag{5.4}$$

such that the first L' satisfy the holonomic constraints, i.e.

$$\rho_s(u_1, \cdot\cdot, u_N, t) = \alpha_s \; ; \qquad s = 1, \cdot\cdot, L' \tag{5.5}$$

where the α_s are constants, and the remaining $\rho_s(\cdot)$, $s = L' + 1, \cdot\cdot, N$ are

1. Single-valued.

2. Continuous with continuous derivatives.

3. Such that the Jacobian is not zero; that is,

$$J = \begin{vmatrix} \dfrac{\partial \rho_1}{\partial u_1} & \cdots & \dfrac{\partial \rho_1}{\partial u_N} \\ \vdots & & \vdots \\ \dfrac{\partial \rho_N}{\partial u_1} & \cdots & \dfrac{\partial \rho_N}{\partial u_N} \end{vmatrix} \neq 0 \,. \tag{5.6}$$

Under these restrictions, the transformation is one-to-one and onto (Fig. 5-2); therefore by the implicit function theorem the transformation can be inverted to give

$$\begin{aligned} u_s &= u_s(q_1, \cdot\cdot, q_N, t) \\ &= u_s(\alpha_1, \cdot\cdot, \alpha_{L'}, q_{L'+1}, \cdot\cdot, q_N, t) \,; \qquad s = 1, \cdot\cdot, N \end{aligned} \tag{5.7}$$

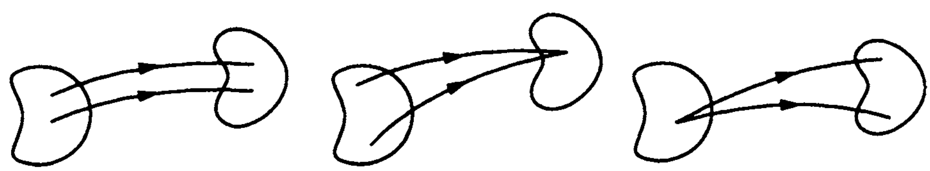


Fig. 5-2

Now re-label:

$$q_{L'+1} = q_1, \; \cdots, \; q_N = q_n$$

These then are the generalized coordinates; we have now

$$q_s = \rho_s(u_1, \cdot\cdot, u_N, t) \,; \qquad s = 1, \cdot\cdot, n$$

$$u_s = u_s(q_1, \cdot\cdot, q_n, t) \,; \qquad s = 1, \cdot\cdot, N \tag{5.8}$$

From the last of these, the differential displacements are related by

$$du_s = \sum_{k=1}^{n} \frac{\partial u_s}{\partial q_k} dq_k + \frac{\partial u_s}{\partial t} dt \,; \qquad s = 1, \cdot\cdot, N \tag{5.9}$$

and therefore the virtual displacements are related by

$$\delta u_s = \sum_{k=1}^{n} \frac{\partial u_s}{\partial q_k} \delta q_k \; ; \qquad s = 1, \cdot\cdot, N \tag{5.10}$$

Possible and Virtual Displacements in Generalized Coordinates. The nonholonomic constraints are

$$\sum_{s=1}^{N} A_{rs} du_s + A_r dt = 0 \; ; \qquad r = 1, \cdot\cdot, \ell \tag{5.11}$$

where $\ell = L - L'$. The generalized coordinates are not subject to the holonomic constraints; the discarded L' coordinates have accomplished this. Substituting Eqn. (5.9) into (5.11):

$$\sum_{s=1}^{N} A_{rs} \left(\sum_{k=1}^{n} \frac{\partial u_s}{\partial q_k} dq_k + \frac{\partial u_s}{\partial t} dt \right) + A_r dt = 0 \; ; \qquad r = 1, \cdot\cdot, \ell$$

$$\sum_{k=1}^{n} \left(\sum_{s=1}^{N} A_{rs} \frac{\partial u_s}{\partial q_k} \right) dq_k + \left(\sum_{s=1}^{N} A_{rs} \frac{\partial u_s}{dt} + A_r \right) dt = 0$$

$$\sum_{k=1}^{n} B_{rk} dq_k + B_r dt = 0 \; ; \qquad r = 1, \cdot\cdot, \ell \tag{5.12}$$

where

$$\begin{aligned} B_{rk} &= \sum_{s=1}^{N} A_{rs} \frac{\partial u_s}{\partial q_k} \\ B_r &= \sum_{s=1}^{N} A_{rs} \frac{\partial u_s}{\partial t} + A_r \end{aligned} \tag{5.13}$$

In terms of velocity components, these are

$$\sum_{k=1}^{n} B_{rk} \dot{q}_k + B_r = 0 \; ; \qquad r = 1, \cdot\cdot, \ell \tag{5.14}$$

As before, these equations define *possible* displacements and velocities. *Virtual* displacements satisfy

$$\sum_{k=1}^{n} B_{rk} \delta q_k = 0 \; ; \qquad r = 1, \cdot\cdot, \ell \tag{5.15}$$

It is important to realize that generalized coordinates are *general*; they can be distances, angles, etc. and have any dimensions.

Variation Operator. Recall Eqn. (3.28):

$$\frac{d}{dt}(\delta u) = \delta\left(\frac{du}{dt}\right) = \delta\dot{u}$$

or,

$$d(\delta u) = \delta(du) \tag{5.16}$$

It may be shown that also

$$d(\delta q) = \delta(dq) \tag{5.17}$$

In words, the d and δ operators are communicative in generalized coordinates.

5.2 Examples

Simple Pendulum. Let the rectangular components of the bob be (x, y); then the constraint is (see Fig. 5-1)

$$\sqrt{x^2 + y^2} = \ell$$

which is of the form $f(x, y) = \alpha$ and which is holonomic, scleronomic, and catastatic. We have:

N = number of rectangular components = 2
L' = number of holonomic constraints = 1
ℓ = number of nonholonomic constraints = 0
L = number of constraints = $L' + \ell = 1$
DOF = degrees of freedom = $N - L = 1$
n = number of generalized coordinates = $N - L' = 1$

According to our approach, we transform to new variables such that first is equal to α and second is convenient:

$$\begin{aligned} q_1 &= \alpha = \ell = \rho_1(x, y, t) \\ q_2 &= \tan^{-1}\frac{y}{x} = \rho_2(x, y, t) \end{aligned}$$

The inverse transformation is

$$\begin{aligned} x &= q_1 \cos q_2 = \alpha_1 \cos q_2 = \ell \cos q_2 \\ y &= q_1 \sin q_2 = \alpha_1 \sin q_2 = \ell \sin q_2 \end{aligned}$$

The Jacobian of this transformation is

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial q_1} & \dfrac{\partial x}{\partial q_2} \\ \dfrac{\partial y}{\partial q_1} & \dfrac{\partial y}{\partial q_2} \end{vmatrix} = \begin{vmatrix} \cos q_2 & -q_1 \sin q_2 \\ \sin q_2 & -q_1 \cos q_2 \end{vmatrix}$$

$$= q_1 \cos^2 q_2 + q_1 \sin^2 q_2 = q_1 = \ell \neq 0$$

Therefore the transformation satisfies all three requirements. Now relabel to get the generalized coordinate:

$$q_2 = q_1 = \theta$$

and the transformations are now

$$\theta = \tan^{-1} \frac{y}{x}$$

$$\begin{aligned} x &= \ell \cos \theta \\ y &= \ell \sin \theta \end{aligned}$$

The q_1 and q_2 in this problem are of course just the polar coordinates (Fig. 5-1) and the choice of θ could have been made by inspection.

In Section 2.3 it was shown that a rigid body in unconstrained motion has DOF = 6; thus it has 6 generalized coordinates. These are usually taken to be the coordinates of some body-fixed point, say the center of mass, and three angles defining the location of body fixed axes. A common choice of angles are the Euler angles; these are defined and used in Chapter 11.

Example. Three bars are hinged and lie in a plane such that one end is attached at 0 and the other end carries particle p (Fig. 5-2). Then either (x, y) or $(\theta_1, \theta_2, \theta_3)$ determine the location of p. This seems to imply that there is a relationship $f(\theta_1, \theta_2, \theta_3) =$ constant because it only takes two parameters to give the location of p. This, however, is not true because the transformations have the properties (see Fig. 5-3):

$$\begin{aligned} (\theta_1, \theta_2, \theta_3) &\longrightarrow (x, y) \quad \text{is one-to-one, but not onto} \\ (x, y) &\longrightarrow (\theta_1, \theta_2, \theta_3) \quad \text{is not one-to-one, but onto} \end{aligned}$$

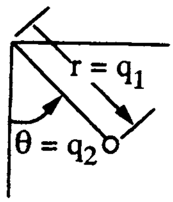


Fig. 5-3

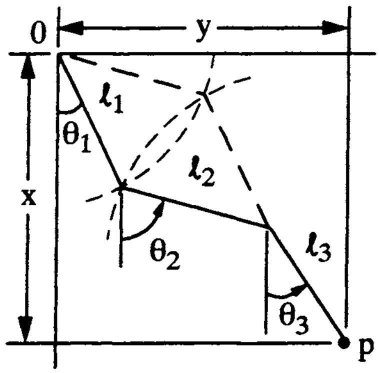


Fig. 5-4

A basic assumption is violated and the transformation theory does not apply.

Remarks

1. For specific problems, generalized coordinates usually suggest themselves from the geometry of the problem; the choice of θ for the simple pendulum is an example of this. The general theory of transformation is needed, however, because we desire to put the key equations of dynamics in terms of generalized coordinates.

2. In some cases there are isolated points in configuration space for which one or more of the three conditions on the coordinate transformation are not satisfied. In this case, the transformation is restricted to regions of the configuration space not containing these points.

Notes

1 In some texts, "generalized coordinates" are not necessarily minimal. Also, they are sometimes called Lagrangian coordinates.

PROBLEMS

5/1. A particle moves on the surface of a three-dimensional sphere.

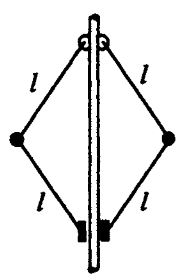
(a) Choose suitable generalized coordinates for the motion.

(b) What are the Eqns. (5.7) for this case?

(c) Examine the Jacobian.

5/2. A particle moves on the surface of a right circular cylinder whose radius expands according to the law $r = f(t)$ while its axis remains stationary. Answer the same questions as in Problem 5/1.

5/3. A centrifugal governor has the configuration shown. If unconstrained, six coordinates would be required to define the configurations of the flyballs. How many constraints must the Cartesian coordinates satisfy? What are they? Choose suitable generalized coordinates to describe the position of the flyballs. Construct Eqns. (5.7) for this problem.



Problem 5/3