Chapter 4

Variational Principles

4.1 Energy Relations

Kinetic Energy. Suppose a particle p has mass m, position $\underline{x}$, and velocity $\dot{\underline{x}}$ relative to an inertial frame of reference. Then the *kinetic energy* of the particle is defined as

$$T = \frac{1}{2}m\dot{\underline{x}}\cdot\dot{\underline{x}} = \frac{1}{2}m|\dot{\underline{x}}|^2 = \frac{1}{2}m\dot{x}^2 \tag{4.1}$$

For a system of n particles, with $\underline{x}^r$ and $\dot{\underline{x}}^r$ the position and velocity of particle r with mass m_r, $r = 1, \cdot\cdot, n$,

$$T = \sum_{r=1}^{n} T^r = \frac{1}{2}\sum_{r=1}^{n} m_r\dot{\underline{x}}^r\cdot\dot{\underline{x}}^r = \frac{1}{2}\sum_{r=1}^{n} m_r(\dot{x}^r)^2 \tag{4.2}$$

where

$$\dot{\underline{x}}^r = \dot{x}_1^r\hat{e}_1 + \dot{x}_2^r\hat{e}_2 + \dot{x}_3^r\hat{e}_3 \; ; \qquad r = 1, \cdot\cdot, n$$
$$(\dot{x}^r)^2 = (\dot{x}_1^r)^2 + (\dot{x}_2^r)^2 + (\dot{x}_3^r)^2$$

Now change to component form:

$$\dot{u}_1 = \dot{x}_1^1\,, \qquad \dot{u}_2 = \dot{x}_2^1\,, \qquad \dot{u}_3 = \dot{x}_3^1\,, \qquad \dot{u}_4 = \dot{x}_1^2\,, \cdots\,,$$

Thus Eqn. (4.2) gives[1]

$$T = \frac{1}{2}\sum_{s=1}^{N} m_s\dot{u}_s^2 \; ; \qquad N = 3n \tag{4.3}$$

Kinetic Energy in Catastatic System. Recall the fundamental equation, Eqn. (3.7):

$$\sum_{s=1}^{N}(m_s\ddot{u}_s - F_s)\delta u_s = 0$$

In a catastatic system, the constraints are (see Eqn. (3.2)):

$$\sum_{s=1}^{N} A_{rs}du_s = 0 \ ; \qquad r = 1, \cdot\cdot, L \tag{4.4}$$

The virtual displacements always satisfy Eqn. (3.3):

$$\sum_{s=1}^{N} A_{rs}\delta u_s = 0 \ ; \qquad r = 1, \cdot\cdot, L$$

Since the possible and the virtual displacements now satisfy the same equations, the fundamental equation may be written

$$\sum_{s=1}^{N}(m_s\ddot{u}_s - F_s)du_s = 0 \tag{4.5}$$

or as

$$\sum_{s=1}^{N} m_s\ddot{u}_s\dot{u}_s = \sum_{s=1}^{N} F_s\dot{u}_s \tag{4.6}$$

where the $\dot{u}_s$ satisfy

$$\sum_{s=1}^{N} A_{rs}\dot{u}_s = 0 \ ; \qquad r = 1, \cdot\cdot, L \tag{4.7}$$

Now differentiate T

$$\frac{dT}{dt} = \sum_{s=1}^{N} m_s\dot{u}_s\ddot{u}_s \tag{4.8}$$

Comparing Eqns. (4.6) and (4.8) gives:

$$\frac{dT}{dt} = \sum_{s=1}^{N} F_s\dot{u}_s \tag{4.9}$$

This states that in a catastatic system, the time rate of change of the kinetic energy equals the power of the given forces under possible velocities.

Energy Relations in Catastatic Systems. If the $\dot{u}_s$ are all continuous (as we are assuming because all forces are bounded) and if the number of particles is constant (as we are also assuming), we can integrate Eqn. (4.9) to get

$$T = \int \sum_{s=1}^{N} F_s \dot{u}_s dt + h = \int \sum_{s=1}^{N} F_s du_s + h \tag{4.10}$$

where h is a constant of integration.

Suppose that some of the given forces are conservative and some are not and let

F_s^c = s component of resultant of all conservative forces = $-\partial V/\partial u_s$
F_s^{nc} = s component of resultant of all nonconservative forces.

Then Eqn. (4.10) gives

$$\begin{aligned} T &= \int \sum_{s=1}^{N} F_s^c du_s + \int \sum_{s=1}^{N} F_s^{nc} du_s + h \\ &= -\int \sum_{s=1}^{N} \frac{\partial V}{\partial u_s} du_s + \int \sum_{s=1}^{N} F_s^{nc} du_s + h \end{aligned}$$

$$T + V = \int \sum_{s=1}^{N} F_s^{nc} du_s + h \tag{4.11}$$

If all forces are conservative and included in V, the total mechanical energy of the system is constant over time for actual motions:

$$T + V = h = \text{constant} \tag{4.12}$$

That is, in a closed system (catastatic and conservative), and only in a closed system, the total mechanical energy is a constant (is conserved). Note that these relations are true for all catastatic systems; they hold for holonomic or nonholonomic systems.

4.2 Central Principle

Central Principle. Now consider a general system. The fundamental equation, Eqn. (3.7), in vector form is

$$\sum_{r=1}^{n}(m_r\ddot{\underline{x}}^r - \underline{F}^r)\cdot\delta\underline{x}^r = 0 \tag{4.13}$$

where $\underline{F}^r$ are the resultants of the given forces. Consider

$$\frac{d}{dt}\left(\sum_{r=1}^{n} m_r\dot{\underline{x}}^r\cdot\delta\underline{x}^r\right) = \sum_{r=1}^{n} m_r\ddot{\underline{x}}^r\cdot\delta\underline{x}^r + \sum_{r=1}^{n} m_r\dot{\underline{x}}^r\cdot\frac{d}{dt}(\delta\underline{x}^r)$$

$$\sum_{r=1}^{n} m_r\ddot{\underline{x}}^r\cdot\delta\underline{x}^r = \frac{d}{dt}\left(\sum_{r=1}^{n} m_r\dot{\underline{x}}^r\cdot\delta\underline{x}^r\right) - \sum_{r=1}^{n} m_r\dot{\underline{x}}^r\cdot\delta\dot{\underline{x}}^r \tag{4.14}$$

From Eqn. (4.2) the variation of T is

$$\delta T = \underbrace{\sum_{r=1}^{n}\frac{\partial T}{\partial\underline{x}^r}\cdot\delta\underline{x}_r}_{0} + \sum_{r=1}^{n}\frac{\partial T}{\partial\dot{\underline{x}}^r}\cdot\delta\dot{\underline{x}}^r = \sum_{r=1}^{n} m_r\dot{\underline{x}}^r\cdot\delta\dot{\underline{x}}^r \tag{4.15}$$

Combining Eqns. (4.14) and (4.15) gives

$$\sum_{r=1}^{n} m_r\ddot{\underline{x}}^r\cdot\delta\underline{x}^r = \frac{d}{dt}\left(\sum_{r=1}^{n} m_r\dot{\underline{x}}^r\cdot\delta\underline{x}^r\right) - \delta T \tag{4.16}$$

Finally, using the fundamental equation, Eqn. (4.13),

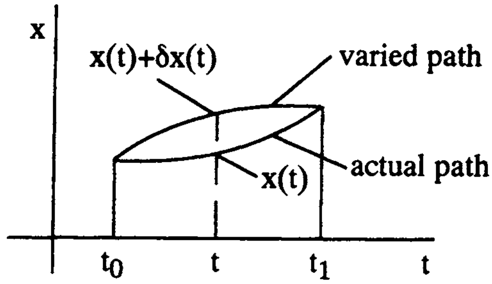
$$\frac{d}{dt}\left(\sum_{r=1}^{n} m_r\dot{\underline{x}}^r\cdot\delta\underline{x}^r\right) - \delta T = \sum_{r=1}^{n}\underline{F}^r\cdot\delta\underline{x}^r = \delta W \tag{4.17}$$

This is called the *central principle* by Hamel.

4.3 Hamilton's Principle

First Form. We will derive several forms of Hamilton's principle, each more specialized. Integrating Eqn. (4.17) between times t_0 and t_1,

$$\left[\sum_{r=1}^{n} m_r\dot{\underline{x}}^r\cdot\delta\underline{x}^r\right]_{t_0}^{t_1} = \int_{t_0}^{t_1}(\delta T + \delta W)dt \tag{4.18}$$



$E \subset \mathbb{E}^{N+1}$

Fig. 4-1

Now consider virtual displacements from the actual motion satisfying

$$\delta \underline{x}^r(t_0) = \delta \underline{x}^r(t_1) = \underline{0}$$

Figure 4-1 shows the situation in the event space. Note that the variations take place with time fixed; thus they are called *contemporaneous.* Equation (4.18) therefore reduces to:

$$\int_{t_0}^{t_1} (\delta T + \delta W) dt = 0 \tag{4.19}$$

This is the first form of *Hamilton's principle*, known as the *extended* or *unrestricted* form, which states that "The time integral of the sum of the virtual work and the variation of the kinetic energy vanishes when virtual displacements are made from the actual motion with endpoints held fixed".

Second Form. If all given forces are conservative,[2] Eqn. (3.42) applies:

$$\delta W = -\delta V$$

Recall that δW is the virtual work, not the variation of W, but that δV is the variation of V. Therefore, in this case,

$$\delta T + \delta W = \delta T - \delta V = \delta(T - V) \tag{4.20}$$

which is the variation in $T - V$. Define the Lagrangian function

$$L = T - V \tag{4.21}$$

Then Hamilton's principle for a conservative system is:

$$\int_{t_0}^{t_1} \delta L \, dt = 0 \tag{4.22}$$

or "The time integral of the variation of the Lagrangian function vanishes for the actual motion".

Since, in general, virtual changes from a possible state do not lead to another possible state, in Hamilton's principle constraints are generally violated and this is not a problem of the calculus of variations. The exception, stated earlier, is a holonomic system.[3]

Third Form. Now suppose the system is conservative and holonomic. Then the variations satisfy the constraints and Hamilton's principle is

$$\delta \int_{t_0}^{t_1} L \, dt = 0 \tag{4.23}$$

or "The time integral of the Lagrangian is stationary along the actual path relative to other possible paths having the same endpoints and differing by virtual displacements". This equation is usually referred to as simply *Hamilton's Principle.*

Remark. The derivation of the various forms of Hamilton's principle given here are completely reversible; that is, starting from them we may derive the corresponding fundamental equations. Thus Hamilton's principle is necessary and sufficient for a motion to be an actual motion. It is precisely an integrated form of the fundamental equation.

Example. A particle moves on a smooth surface with gravity the only given force (Fig. 4-2). We have:

$$\begin{aligned} T &= \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \\ V &= mgz \\ \dot{z} &= f_x\dot{x} + f_y\dot{y} \\ L &= T - V = \frac{1}{2}m\left[\dot{x}^2 + \dot{y}^2 + (f_x\dot{x} + f_y\dot{y})^2\right] - mgf \end{aligned}$$

Since the only given force is conservative and the only constraint is holonomic, the third form of Hamilton's principle applies:

$$\delta \int_{t_0}^{t_1} L \, dt = \delta \int_{t_0}^{t_1} \frac{m}{2}\left\{\dot{x}^2 + \dot{y}^2 + (f_x\dot{x} + f_y\dot{y})^2 - 2gf\right\} dt = 0$$

We could carry out these variations to get the equation of motion; we will not do this for this problem, but will do it for the following one.

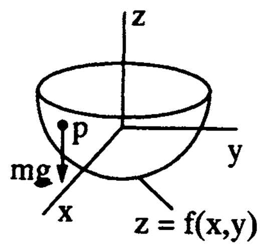


Fig. 4-2

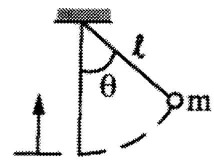


Fig. 4-3

Example. Consider again the simple pendulum (Fig. 4-3). We have:

$$T = \frac{1}{2}mv^2 = \frac{1}{2}(\ell\dot{\theta})^2$$

$$V = mg\ell(1 - \cos\theta)$$

$$L = T - V = \frac{1}{2}m\ell^2\dot{\theta}^2 - mg\ell(1 - \cos\theta)$$

The third form of Hamilton's principle applies:

$$\delta \int_{t_0}^{t_1} L\, dt = \delta \int_{t_0}^{t_1} \left[\frac{1}{2}m\ell^2\dot{\theta}^2 - mg\ell(1 - \cos\theta)\right] dt = 0$$

We will now carry out the variation to get the equation of motion:

$$\int_{t_0}^{t_1} \left[m\ell^2\dot{\theta}\,\delta\dot{\theta} - mg\ell\sin\theta\,\delta\theta\right] dt = 0$$

Integration by parts gives[4]

$$\int_{t_0}^{t_1} \dot{\theta}\,\delta\dot{\theta}\, dt = \int_{t_0}^{t_1} \dot{\theta}\frac{d}{dt}(\delta\theta)dt = \underbrace{\dot{\theta}\delta\theta\Big|_{t_0}^{t_1}}_{=0} - \int_{t_0}^{t_1} \delta\theta\,\ddot{\theta}\, dt$$

Consequently

$$\int_{t_0}^{t_1} \left(-\ell\ddot{\theta}\,\delta\theta - g\sin\theta\,\delta\theta\right) dt = 0$$

$$\int_{t_0}^{t_1} \left(\ell\ddot{\theta} + g\sin\theta\right) \delta\theta\, dt = 0$$

But $\delta\theta$ is an arbitrary virtual displacement; therefore by the Fundamental Lemma of the calculus of variations (see next section) we must have

$$\ddot{\theta} + \frac{g}{\ell}\sin\theta = 0$$

We see that getting equations of motion from Hamilton's principle is somewhat cumbersome. Lagrange's Equations, to be derived shortly, essentially carry out this variation in general for all problems and are much easier to use.

4.4 Calculus of Variations

Statement of the Problem. Because of the close connection between the variational principles of dynamics and the calculus of variations, the latter will be briefly reviewed. Attention will be restricted to the "simplest problem" of the calculus of variations, stated as follows. We seek the function $x = x(t)$, $t \in [t_0, t_1]$, that renders the integral

$$J = \int_{t_0}^{t_1} f(x, \dot{x}, t)dt \tag{4.24}$$

a minimum subject to fixed endpoints $x(t_0) = x_0$ and $x(t_1) = x_1$.

Euler – Lagrange Equation. In ordinary calculus, necessary conditions for the minimum of a function are obtained by considering the first and second derivatives. Analogously, in the calculus of variations necessary conditions are obtained by considering the first and second *variations* of J. The most important result, obtained from setting the first variation, δJ, to zero, is that if $x(t)$ minimizes J then it must satisfy the Euler-Lagrange equation:

$$f_x - \frac{d}{dt} f_{\dot{x}} = 0 \tag{4.25}$$

where subscripts indicate partial derivatives. A function satisfying this equation is called an *extremal*; it is a candidate for the minimizing function. Carrying out the differentiation gives the long form of the Euler – Lagrange equation:

$$f_x - f_{\dot{x}t} - f_{\dot{x}x}\dot{x} - f_{\dot{x}\dot{x}}\ddot{x} = 0 \tag{4.26}$$

Two key lemmas are needed to establish this result. The Fundamental Lemma states that if $M(t)$ is a continuous function on $[t_0, t_1]$ and

$$\int_{t_0}^{t_1} M(t)\eta(t)dt = 0$$

for all $\eta(t) \in C^1$ with $\eta(t_0) = \eta(t_1) = 0$ then $M(t) = 0$ for all $t \in [t_0, t_1]$.

The Du Bois - Reymond Lemma states that if $N(t)$ is continuous on $[t_0, t_1]$ and if

$$\int_{t_0}^{t_1} \dot{\eta} N \, dt = 0$$

for all $\eta(t) \in C'$ with $\eta(t_0) = \eta(t_1) = 0$ then $N(t) =$ constant for all $t \in [t_0, t_1]$.

The other necessary conditions, arising from consideration of the second variation, will not be discussed here.

Application to Dynamics. Consider a holononomic, conservative system with a single coordinate, x. Hamilton's principle for such a system is Eqn. (4.23):

$$\delta \int_{t_0}^{t_1} L(x, \dot{x}, t) dt = 0 \tag{4.27}$$

Applying Eqn. (4.25),

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \tag{4.28}$$

This is in fact Lagrange's equation for the system. (Lagrange's equations for general systems will derived in Chapter 6.)

Inverse Problem. In the inverse problem of the calculus of variations, we are given a two parameter family of curves

$$x = g(t, \alpha, \beta) \tag{4.29}$$

and we want to find a function $f(x, \dot{x}, t)$ such that the family members are the extremals of

$$J = \int_{t_0}^{t_1} f(x, \dot{x}, t) dt \tag{4.30}$$

Differentiating Eqn. (4.29) twice,

$$\dot{x} = g_t(t, \alpha, \beta) \tag{4.31}$$

$$\ddot{x} = g_{tt}(t, \alpha, \beta) \tag{4.32}$$

Under general conditions, Eqns. (4.29) and (4.31) may in principle be solved for α and β,

$$\begin{aligned} \alpha &= \rho(x, \dot{x}, t) \\ \beta &= \psi(x, \dot{x}, t) \end{aligned}$$

and then substituted into Eqn. (4.32) to obtain an equation of the form

$$\ddot{x} = G(x, \dot{x}, t) \tag{4.33}$$

This equation must be the Euler-Lagrange equation; that is, it must be identical to Eqn. (4.26). Substitute Eqn. (4.33) into Eqn. (4.26) and differentiate the result with respect to $\dot{x}$ to get

$$f_{\dot{x}t\dot{x}} + \dot{x} f_{\dot{x}x\dot{x}} + G f_{\dot{x}\dot{x}\dot{x}} + G_{\dot{x}} f_{\dot{x}\dot{x}} = 0$$

Letting $M = f_{\dot{x}\dot{x}}$, this becomes

$$\frac{\partial M}{\partial t} + \dot{x}\frac{\partial M}{\partial x} + G\frac{\partial M}{\partial \dot{x}} + G_{\dot{x}} M = 0 \tag{4.34}$$

Now define the function

$$\theta(t, \alpha, \beta) = \exp\left[\int G_{\dot{x}}\left(t, g(t, \alpha, \beta), g_t(t, \alpha, \beta)\right) dt\right] \tag{4.35}$$

The solution of Eqn. (4.34) may be shown to be of the form

$$M = \frac{\Phi(\varphi(x, \dot{x}, t),\ \psi(x, \dot{x}, t))}{\theta(t,\ \varphi(x, \dot{x}, t),\ \psi(x, \dot{x}, t))} = f_{\dot{x}\dot{x}} \tag{4.36}$$

where Φ is an arbitrary but nonzero function of φ and ψ. Integrating Eqn. (4.36) twice gives an expression for f:

$$f = \int\int M\ d\dot{x} d\dot{x} + \dot{x}\lambda(x, t) + \mu(x, t) \tag{4.37}$$

where λ and μ must be chosen so that f satisfies Eqn. (4.26). Since Φ is arbitrary there are an infinity of such functions f and thus the solution to the inverse problem is not unique.

Example. Consider again a system with a single generalized coordinate subject to a conservative force. From Newton's Second Law we know that the equation of motion is

$$\ddot{x} = F = -\frac{dV(x)}{dx}$$

where $V(x)$ is the potential energy function of F. We want to find a function $f(x, \dot{x}, t)$ such that this differential equation is the Euler-Lagrange equation for

$$J = \int_{t_0}^{t_1} f(x, \dot{x}, t)dt$$

That is,

$$\delta \int_{t_0}^{t_1} f(x, \dot{x}, t)dt = 0$$

From Eqn. (4.33) we see that

$$G = \ddot{x} = F = -\frac{dV(x)}{dx}$$

so that $G_{\dot{x}} = 0$ and Eqn. (4.35) gives

$$\theta(t, \alpha, \beta) = \exp\left[\int G_{\dot{x}} dt\right] = 1$$

Therefore, from Eqns. (4.36) and (4.37),

$$\begin{aligned} M &= \Phi = f_{\dot{x}\dot{x}} \\ f &= \int\int \Phi(x, \dot{x}, t) d\dot{x} d\dot{x} + \dot{x}\lambda(x, t) + \mu(x, t) \end{aligned}$$

To get the "simplest case", select $\Phi = 1$ to obtain

$$f = \frac{1}{2}\dot{x}^2 + \dot{x}\lambda + \mu$$

Substituting into Eqn. (4.26),

$$f_x - f_{\dot{x}t} - \dot{x} f_{\dot{x}x} - \ddot{x} f_{\dot{x}\dot{x}} = 0$$

$$\frac{\partial \mu(x, t)}{\partial x} - \frac{\partial \lambda(x, t)}{\partial t} = -\frac{dV(x)}{dx}$$

Thus we must have $\lambda = 0$ and $\mu = -V(x)$, and f becomes

$$f = \frac{1}{2}\dot{x}^2 - V = T - V = L$$

Hence the "simplest" variational problem that leads to the correct equation of motion for this case is

$$\delta \int_{t_0}^{t_1} L \, dt = 0$$

which is, of course, Hamilton's principle. Choosing other functions Φ gives other variational principles.

4.5 Principle of Least Action

Historical Remarks. Although Newton's Second Law gives highly accurate results in most situations, it doesn't seem to emanate from any deeper philosophic or scientific principle, a matter of great concern to eighteenth century scientists. Variational principles arose initially to meet this perceived need. The idea was that of all the possible motions of a dynamic system, the one that is actually followed is the one that minimizes some fundamental quantity; in other words, nature acts in the way that is most efficient.

The first successful variational principle was Fermat's principle of minimum time in optics. He starts "from the principle that Nature always acts in the shortest ways". With this principle, Fermat was able to derive the laws of refraction.

Maurpertius stated the Principle of Least Action (PLA) in dynamics from analogy to Fermat's principle. Maurepertius' viewpoint was that "nature in the production of her works always acts in the most simple ways". He stated the principle in metaphysical terms and never proved the PLA in the sense of showing that it was equivalent to the established laws of dynamics. Immediately after statement of the principle, a controversy started. On the one hand, some claimed, most notably Koenig, that the principle was not valid or that Leibniz had discovered it previously, or both! Even the great Voltaire, who knew little of mathematics and science, got into the act, satirizing the PLA in some of his books.

Euler sided with Maupertuis and managed to prove (in the sense just stated) the PLA for a particle, thus putting the principle an a sound basis. Many years later, Mach remarked that "Euler, a truly great man, lent his reputation to the PLA and the glory of his invention to Maupertuis; but he made a new thing of the principle, practical and useful". (Euler was also the first to consider the inverse problem.)

We now have the PLA in two forms, associated with the names of Lagrange and Jacobi. The latter's version has path length as the independent variable and may be viewed as a geometrical statement. In this view, the principle becomes a problem of finding geodesics in a Riemann space.

The previous Section shows that it is possible to generate an infinite number of variational principles. The only requirement is that they be "valid", that is, that they lead to the same equations of motion as does Newton's Second Law. It is surprising that the principle that is perhaps the most straight-forward and useful, that of Hamilton, did not emerge until much later than the PLA.

Noncontemporaneous Variations. In Hamilton's principle, the variations from the actual path take place with time fixed and the variations are zero at the endpoints. Now we relax this restriction and consider noncontemporaneous variations, as shown on Fig. 4-4.

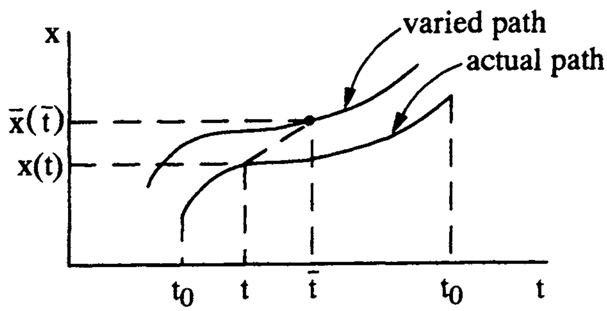


Fig. 4-4

Lagrange's Principle of Least Action. In the principle of least action we consider variations from the actual path with energy held fixed. We consider closed systems only, so that energy is conserved, and denote the noncontemporaneous variation operator by δ_t. Thus from Eqn. (4.12),

$$\delta_t\, T + \delta_t\, V = 0 \tag{4.38}$$

The relation between the operators δ and δ_t for a function $F(x, t)$ is given by

$$\delta_t\, F = \delta F + \frac{\partial F}{\partial t} \delta_t\, t \tag{4.39}$$

and is illustrated on Fig. 4-5.

Because the principle of least action is largely of historical interest only, the details of the derivation will be omitted and the results will be summarized.[5] The *action* is defined by

$$A = \int_{t_0}^{t_1} 2T\, dt \tag{4.40}$$

The Lagrange form of the principle of least action is

$$\delta_t\, A = 0 \tag{4.41}$$

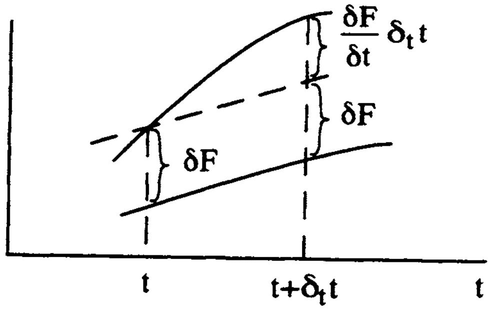


Fig. 4-5

In words, "the action is stationary for the actual path in comparison with neighboring paths having the same energy". The principle is both necessary and sufficient and thus it may be used to derive equations of motion. We note that the varied motion does not in general take the same time as the actual motion and in fact the varied motion is not in general a possible motion. Clearly, the factor 2 in Eqn. (4.40) may be omitted.

Jacobi's Principle of Least Action. Since energy is conserved, Eqn. (4.40) may be written

$$A' = \int_{t_0}^{t_1} 2\sqrt{T(h-V)}\,dt \tag{4.42}$$

However, from Eqn. (4.3) T is a quadratic function of the $\dot{u}_s$ and thus the integral in Eqn. (4.42) is homogeneous of degree one in the $\dot{u}_s$. This means that A depends only on the path in the configuration space and not in the event space.[6] Writing Eqn. (4.42) in terms of s, the path length in configuration space, gives

$$A' = \int \sqrt{2(h-V)}\,ds \tag{4.43}$$

Consequently, the Jacobi form of the principle is

$$\delta_t A' = 0 \tag{4.44}$$

This is a problem in the calculus of variations.

Notes

1 Strictly speaking, $T = T(\dot{x}_1^1, \dot{x}_2^1, \dot{x}_3^1, \dot{x}_1^2, \cdot\cdot, \dot{x}_3^n)$ and $T'(\dot{u}_1, \cdot\cdot, \dot{u}_N)$ are two different functions but we will use the same symbol, T, for both.

2 Hamilton's principle for a class of non-conservative systems may be found in "Some Remarks on Hamilton's Principle", G. Leitmann, *J. Appl. Mech.*, Dec. 1963.

3 This was proved in Section 3.3; an alternate proof will be given in Section 6.5.

4 Recall that $\int u\,dv = uv - \int v\,du$; here we take $u = \dot{\theta}$ and $dv = \frac{d}{dt}(\delta\theta)dt$.

5 See Rosenberg or Pars for the details.

6 See Pars

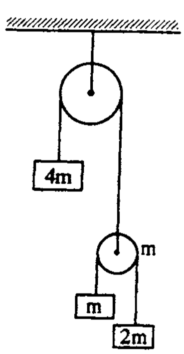
PROBLEMS

4/1. A weight of mass $4m$ is attached to a massless, inextensible string which passes over a frictionless, massless pulley, as shown on Fig. 4/1. The other end of this string is attached to the center of a frictionless, homogeneous pulley of mass m. A second massless inextensible string having masses m and $2m$ attached to its extremities passes over the pulley of mass m. Gravity is the only force acting on this system.

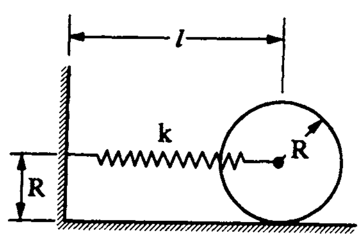
(a) Give the kinetic energy for this system;

(b) Give the energy integral, if one exists;

(c) Write down Hamilton's principle.



Problem 4/1

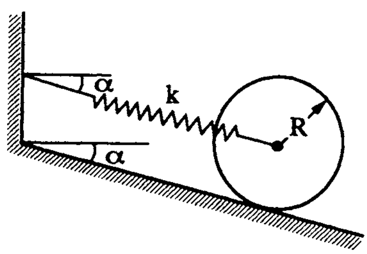


Problem 4/2

4/2. A homogeneous disk of mass M, constrained to remain in a vertical plane, rolls without sliding on a horizontal line as shown. A massless horizontal, linear spring of rate k is attached to the center of the disk and to a fixed point. If the free length of the spring is l, and the disk radius is R,

(a) Give the kinetic energy for this system;
(b) Give the energy integral, if one exists;
(c) Write down Hamilton's principle.

4/3. Give the same answers as in Problem 4/2 when the configuration is changed so that the line is inclined by the angle α to the horizontal, as shown.



Problem 4/3

4/4. Three particles of mass m_1, m_2, and m_3, respectively, are constrained to move so that they lie for all time on a straight line passing through a fixed point. For the force-free problem in Cartesian coordinates:

(a) Give the kinetic energy;
(b) Give the energy integral, if one exists;
(c) Write down Hamilton's principle.

4/5. A heavy, homogeneous inextensible string of given length remains for all time in a vertical plane. It lies in part on a smooth, horizontal table, and in part, it hangs vertically down over the table edge. What is Hamilton's principle?

4/6. A particle of mass m moves in the x, y plane under a force which is derivable from a potential energy function. The particle velocity is directed for all time toward a point P which moves along the x axis so that its distance from the origin is given by the prescribed function $\xi(t)$.

(a) How many degrees of freedom does the particle have?

(b) What is Hamilton's principle?

(c) Give the energy integral, if one exists.

4/7. One point of a rigid body is constrained to move on a smooth space curve defined by $f(x_0, y_0, z_0, t) = 0$. If the forces and moments acting on the body are conservative, give Hamilton's principle. Does an energy integral exist? If so, write it down. If none exists, explain why.