Chapter 2

Motion and Constraints

2.1 Newton's Second Law

Vector Form. Consider a system of n mass particles of masses m_1, $m_2, \ldots, m_n$. (Occasionally we will call a system of particles a *dynamic system.*) Let the position of particle r in an inertial reference frame be denoted $\underline{x}^r(t)$, as shown on Fig. 2-1. Let the resultant forces on the particles be bounded functions of the particles' positions, velocities, and time. Then Eqn. (1.1) gives

$$\begin{aligned} m_1\ddot{\underline{x}}^1(t) &= \sum \underline{F}^1(\underline{x}^1, \cdot\cdot, \underline{x}^n, \dot{\underline{x}}^1, \cdot\cdot\ \dot{\underline{x}}^n, t) \\ m_2\ddot{\underline{x}}^2(t) &= \sum \underline{F}^2(\underline{x}^1, \cdot\cdot, \underline{x}^n, \dot{\underline{x}}^1, \cdot\cdot\ \dot{\underline{x}}^n, t) \\ &\vdots \\ m_n\ddot{\underline{x}}^n(t) &= \sum \underline{F}^n(\underline{x}^1, \cdot\cdot, \underline{x}^n, \dot{\underline{x}}^1, \cdot\cdot\ \dot{\underline{x}}^n, t) \end{aligned} \tag{2.1}$$

or

$$m_r\ddot{\underline{x}}^r(t) = \sum \underline{F}^r(\underline{x}^1, \cdot\cdot, \underline{x}^n, \dot{\underline{x}}^1, \cdot\cdot, \dot{\underline{x}}^n, t)\,; \qquad r = 1, 2, \cdot\cdot, n \tag{2.2}$$

If none of the forces depend explicitly on time, we say the system is *autonomous.* Note that forces are not allowed to be functions of the particles' accelerations.[1]

In the "Newtonian" problem, unbounded forces are allowed provided they are measurable, that is if $\int \underline{F}^r dt$; $r = 1, \cdot\cdot, n$ are always bounded. A force which is unbounded but measurable is called an impulsive force. In the "strictly Newtonian" problem, all forces are bounded. For most

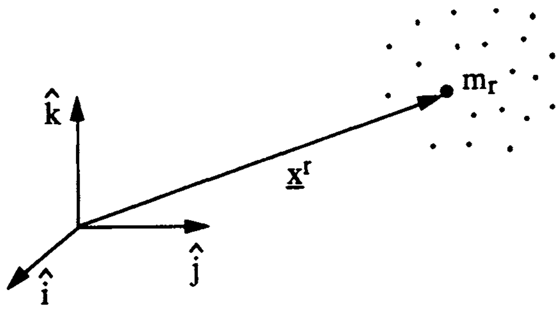


Fig. 2-1

of this book we restrict our attention to the strictly Newtonian problem. The exception is Chapter 13 where impulsive forces will be considered.

Recall that there are two ways Newton's Second Law can be used. One way (problem of the second kind) is to determine the forces acting on a system when the motion of the system is given. This is typically the situation at the design stage. For example, when designing a space launch system the motion is known (transition from earth surface to orbit location and speed) and Newton's Second Law can be used to predict the propulsive forces required, and hence the size of the vehicle. The second way is to determine the motion when the forces are given (problem of the first kind). This situation typically arises in the performance estimation of an existing system. For example, it may be of interest to determine the range of orbits accessible by an existing launch vehicle. In this book we approach dynamics as a problem of the first kind, although all the results obtained apply equally to either type of problem. Thus it is characteristic that the equations of motion of a particle system give the accelerations of the particles in terms of their positions, velocities, and time.

Component Form. Now introduce linearly independent unit vectors $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$. Then, if $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ are fixed in the inertial frame,

$$\begin{aligned} \underline{x}^r(t) &= x_1^r(t)\hat{e}_1 + x_2^r(t)\hat{e}_2 + x_3^r(t)\hat{e}_3 \\ \ddot{\underline{x}}^r(t) &= \ddot{x}_1^r(t)\hat{e}_1 + \ddot{x}_2^r(t)\hat{e}_2 + \ddot{x}_3^r(t)\hat{e}_3 \end{aligned} \tag{2.3}$$

Label the components of $\underline{x}^1$, $\underline{x}^2$, $\cdot\cdot$, $\underline{x}^n$ as follows

$$u_1 = x_1^1\ , \quad u_2 = x_2^1\ , \quad u_3 = x_3^1\ , \quad u_4 = x_1^2\ ,$$

$$u_5 = x_2^2 \,, \;\; \cdot\cdot, \;\; u_N = x_3^n \,; \quad N = 3n \tag{2.4}$$

Then we can write the Second Law as

$$\begin{aligned}
m_1\ddot{u}_1 &= \sum F_1^1(u_1, u_2, \cdot\cdot, u_N, \dot{u}_1, \dot{u}_2, \cdot\cdot, \dot{u}_N, t) \\
m_1\ddot{u}_2 &= \sum F_2^1(u_1, u_2, \cdot\cdot, u_N, \dot{u}_1, \dot{u}_2, \cdot\cdot, \dot{u}_N, t) \\
m_1\ddot{u}_3 &= \sum F_3^1(u_1, u_2, \cdot\cdot, u_N, \dot{u}_1, \dot{u}_2, \cdot\cdot, \dot{u}_N, t) \\
m_2\ddot{u}_4 &= \sum F_1^2(u_1, u_2, \cdot\cdot, u_N, \dot{u}_1, \dot{u}_2, \cdot\cdot, \dot{u}_N, t) \\
&\vdots \\
m_n\ddot{u}_N &= \sum F_3^n(u_1, u_2, \cdot\cdot, u_N, \dot{u}_1, \dot{u}_2, \cdot\cdot, \dot{u}_N, t)
\end{aligned} \tag{2.5}$$

or

$$m_s\ddot{u}_s = \sum F_s(u_1, \cdot\cdot, u_N, \dot{u}_1, \cdot\cdot, \dot{u}_N, t) \,; \quad s = 1, \cdot\cdot, N \tag{2.6}$$

Note the interpretations of m_s and F_s; for example, m_1, m_2, and m_3 are all the mass of the first particle and F_1, F_2, and F_3 are the three components of the resultant force acting on the first particle.

2.2 Motion Representation

Configuration Space. By the correspondence between n-tuples and vectors in Euclidean spaces, the components of displacement, u_1, $\cdot\cdot$, u_N can be thought of as forming a vector in a subset of $\mathbb{E}^N$, the N-dimensional Euclidean space:

$$\underline{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{pmatrix} \in C \subset \mathbb{E}^N \tag{2.7}$$

We call C the *configuration space.* As the motion of the system proceeds, a path is generated in this space called a C trajectory.

Event Space. The combination of the configuration components and time is called an event; an event is a vector in the *event space* E:

$$(\underline{u}, t) = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \\ t \end{pmatrix} \in E \subset \mathbb{E}^{N+1} \tag{2.8}$$

Paths in this space are called E trajectories.

State Space. The combination of the configuration components and the components of velocity defines a point in *state space* S:

$$(\underline{u}, \dot{\underline{u}}) = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \\ \dot{u}_1 \\ \dot{u}_2 \\ \vdots \\ \dot{u}_N \end{pmatrix} \in S \subset \mathbb{E}^{2N} \tag{2.9}$$

Paths in this space are S trajectories.

State-Time Space. The combination of states and time gives a point in *state-time space*

$$(\underline{u}, \dot{\underline{u}}, t) = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \\ \dot{u}_1 \\ \dot{u}_2 \\ \vdots \\ \dot{u}_N \\ t \end{pmatrix} \in T \subset \mathbb{E}^{2N+1} \tag{2.10}$$

2.3 Holonomic Constraints

Introduction. The motion of a particle system is frequently subject to constraints. As an example, suppose the motion of a single particle is constrained to be on a surface, as shown on Fig. 2-2. Note that now only two of the coordinates are independent; the third, say z, is determined by the constraint.

A special case is motion in the (x, y) plane, for which the constraint is:

$$f(x, y, z) = z = 0 \Longrightarrow \dot{z} = 0, \quad \ddot{z} = 0$$

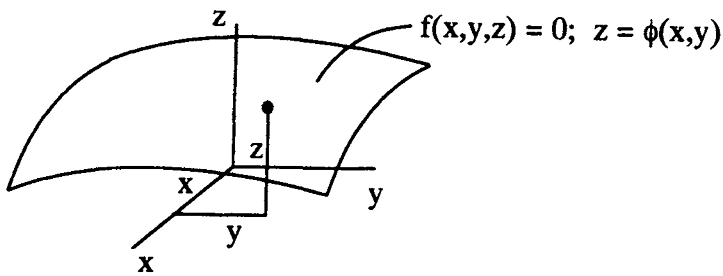


Fig. 2-2

Equations (2.6) become, for this case:

$$\begin{aligned} m\ddot{x} &= F_x(x,y,0,\dot{x},\dot{y},0,t) \\ m\ddot{y} &= F_y(x,y,0,\dot{x},\dot{y},0,t) \\ 0 &= F_z(x,y,0,\dot{x},\dot{y},0,t) \end{aligned}$$

These are the equations of planar motion as expected.

Of course constraints also may be prescribed functions of time, for example:

$$f(x,y,z,t) = 0$$

Definitions. Consider a system of n particles; such a system has a $N = 3n$ dimensional configuration space C. A *holonomic* constraint on the motion of the particles is one that can be expressed in the form

$$f(u_1, u_2, \cdots, u_N, t) = 0 \tag{2.11}$$

Otherwise the constraint is *nonholonomic.* As a special case, if a holonomic constraint can be expressed as

$$f(u_1, u_2, \cdots, u_N) = 0 \tag{2.12}$$

then it is *scleronomic*; otherwise it is *rheonomic.* If all constraints are holonomic we say the system is holonomic, and if all holonomic constraints are scleronomic the system is scleronomic.

The constraint equation is an $N-1$ dimensional surface in the configuration space $C \subset \mathbb{E}^N$; the C trajectories must lie on this surface. Of course there may be several such constraints.

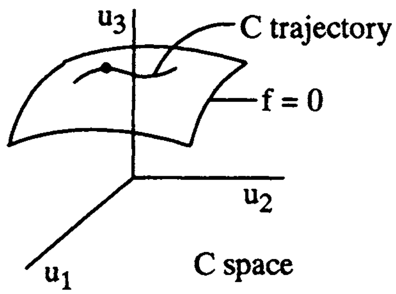


Fig. 2-3

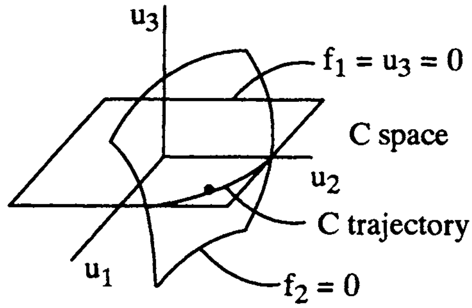


Fig. 2-4

Figures 2-3 and 2-4 show the cases of a single particle subject to one and two constraints, respectively. In E space, for the case of two components, the E trajectories lie on a right cylindrical surface, if the constraints are scleronomic (Fig. 2-5).

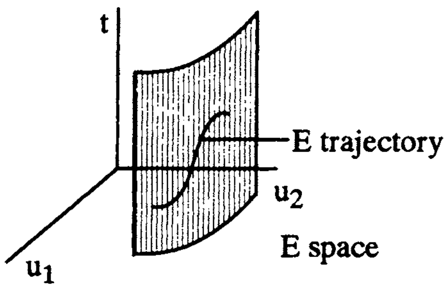


Fig. 2-5

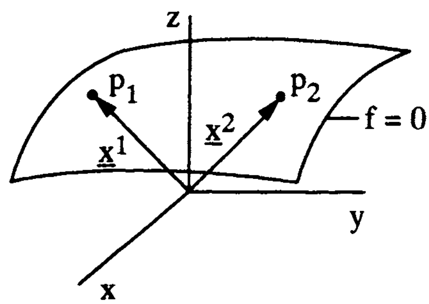


Fig. 2-6

Next consider two particles constrained to move on a single surface (Fig. 2-6). The position vectors of the particles resolved into components are:

$$\underline{x}^1 = x_1^1\hat{e}_1 + x_2^1\hat{e}_2 + x_3^1\hat{e}_3$$
$$\underline{x}^2 = x_1^2\hat{e}_1 + x_2^2\hat{e}_2 + x_3^2\hat{e}_3$$

Relabeling to put in component form:

$$u_1 = x_1^1\,, \quad u_2 = x_2^1\,, \quad u_3 = x_3^1$$
$$u_4 = x_1^2\,, \quad u_5 = x_2^2\,, \quad u_6 = x_3^2$$

If the surface is given by $f(x, y, z) = 0$, then there are *two* constraints in configuration space:

$$f_1 = f(u_1, u_2, u_3) = 0 \; ; \qquad f_2 = f(u_4, u_5, u_6) = 0$$

The unconstrained particles had a total of six independent components, but the two constraints have reduced this number to four.

Every C trajectory satisfying all constraints is a *possible motion*, but is not necessarily an *actual motion* because actual motions also obey Newton's Laws.

Degrees of Freedom. Given a system of $n = N/3$ particles suppose there are L independent constraints. The number of *degrees of freedom* of the system is then

$$DOF = N - L > 0 \tag{2.13}$$

If $N = L$ the system is fixed in space if all constraints are scleronomic and moves with prescribed motion if at least one is rheonomic.

For a single particle, if there are no constraints $DOF = N - L = 3 - 0 = 3$ and it takes three independent parameters to specify position in configuration space (Fig. 2-7). If there is one holonomic constraint, $DOF = 2$ and it takes two (motion on a surface). If there are two holonomic constraints, $DOF = 1$ and it takes one (motion on a line) and if there are three holonomic constraints, $DOF = 0$ and the particle is fixed.

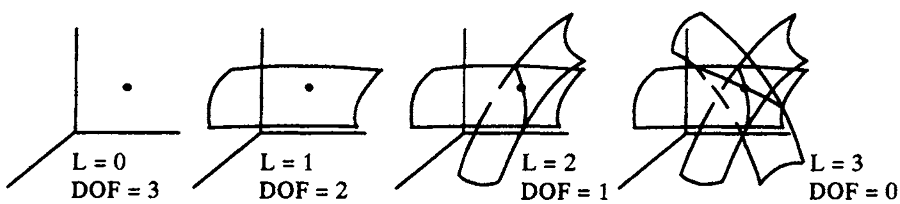


Fig. 2-7

The situation for a rigid body is more difficult. We establish the number of DOF of a rigid body in 3-D unconstrained motion in two different ways. First, fix a body-fixed reference frame with axes ζ, η, ν at the body's center of mass and let the coordinates of the origin of this frame be $\bar{x}, \bar{y}, \bar{z}$ with respect to a non-body-fixed frame with axes x, y, z (Fig. 2-8). Let the direction cosines of ζ, η, ν relative to x, y, z be

$$\begin{pmatrix} \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \\ \ell_3 & m_3 & n_3 \end{pmatrix}$$

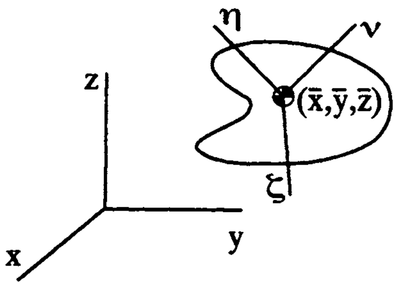


Fig. 2-8

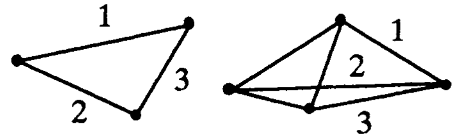


Fig. 2-9

(For example, ℓ_1 is the cosine of the angle between ζ and x). The coordinates of the r^{th} particle of the rigid body are then

$$\begin{aligned} x_r &= \bar{x} + \ell_1\zeta_r + \ell_2\eta_r + \ell_3\nu_r \\ y_r &= \bar{y} + m_1\zeta_r + m_2\eta_r + m_3\nu_r \\ z_r &= \bar{z} + n_1\zeta_r + n_2\eta_r + n_3\nu_r \end{aligned}$$

Due to the orthogonality of the direction cosine matrix, the direction cosines are functions of three independent angles, say θ_1, θ_2, θ_3. Thus the location of all the points of the rigid body are specified by $(\bar{x}, \bar{y}, \bar{z}, \theta_1, \theta_2, \theta_3)$ relative to the other frame, and therefore the body has 6 DOF.

Alternatively, we may view the rigid body as a system of constrained particles. It is clear that the first three particles take 3 constraints, and that each additional particle takes 3 more (Fig. 2-9). Thus if the rigid body has n particles, the total number of constraints is $3+3(n-3)$. But since a particle without constraints has 3 DOF, the DOF for the rigid body is

$$DOF = 3n - [3 + 3(n-3)] = 6$$

Infinitesimal Displacements. The constraint as specified by Eqn. (2.11) or (2.12) is for arbitrarily large displacements. We now derive local conditions, that is, conditions on small displacements. Suppose we have L holonomic constraints and let $u_s = u_s(\alpha)$ and $t = t(\alpha)$ where α is a parameter:

$$f_r[u_1(\alpha),\ u_2(\alpha),\ \cdots,\ u_N(\alpha),\ t] = 0\ ; \qquad r = 1,\ 2,\ \cdot\cdot,\ L \tag{2.14}$$

Differentiating w.r.t. α:

$$\sum_{s=1}^{N} \frac{\partial f_r}{\partial u_s}\frac{du_s}{d\alpha} + \frac{\partial f_r}{\partial t}\frac{dt}{d\alpha} = 0\ ; \qquad r = 1,\ 2,\ \cdot\cdot,\ L \tag{2.15}$$

An important special case is $\alpha = t$:

$$\sum_{s=1}^{N} \frac{\partial f_r}{\partial u_s} \dot{u}_s + \frac{\partial f_r}{\partial t} = 0 ; \qquad r = 1,\ 2,\ \cdot\cdot,\ L \tag{2.16}$$

In differential form

$$\sum_{s=1}^{N} \frac{\partial f_r}{\partial u_s} du_s + \frac{\partial f_r}{\partial t} dt = 0 ; \qquad r = 1,\ 2,\ \cdot\cdot,\ L \tag{2.17}$$

For the special case of *all* constraints scleronomic, Eqns. (2.16) and (2.17) become

$$\sum_{s=1}^{N} \frac{\partial f_r}{\partial u_s} \dot{u}_s = 0 ; \qquad r = 1,\ 2,\ \cdot\cdot,\ L \tag{2.18}$$

$$\sum_{s=1}^{N} \frac{\partial f_r}{\partial u_s} du_s = 0 ; \qquad r = 1,\ 2,\ \cdot\cdot,\ L \tag{2.19}$$

The first of these is a local condition on velocities and displacements and the latter is a local condition that small displacements must remain in the tangent plane of the constraint. If u_s^* is a position on the constraint, then infinitesimal displacements du_s satisfying

$$\sum_{s=1}^{N} \left. \frac{\partial f}{\partial u_s} \right|_{u_s^*} du_s = 0$$

are in the tangent plane (Fig. 2-10) of the constraint at u_s^*.

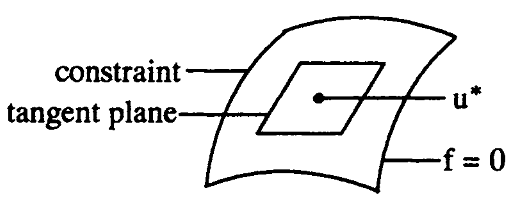


Fig. 2-10

2.4 Nonholonomic Constraints

Configuration Constraints. A nonholonomic constraint is one that is not holonomic. This can happen in several ways and we discuss two.

If a constraint can be reduced to an *inequality* in the configuration space,

$$f(u_1,\ u_2,\ \cdot\cdot,\ u_N,\ t) \leq 0 \tag{2.20}$$

it is called a *configuration* constraint. Such a constraint may depend explicitly on t (rheonomic), or not (scleronomic). An example of a configuration constraint is the requirement that an object must stay on or above a plane surface (Fig. 2-11).

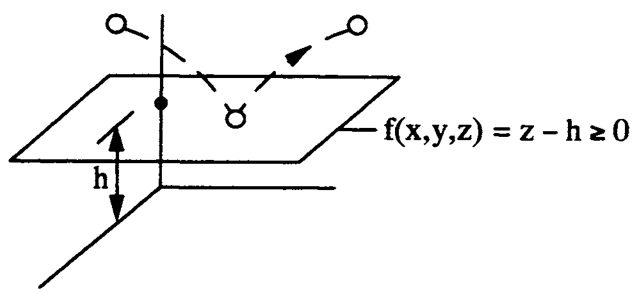


Fig. 2-11

Equality Constraints. These are differential relations among the $u_1,\ u_2,\ \cdot\cdot,\ u_N,\ t$ of the form

$$\sum_{s=1}^{N} A_{rs} du_s + A_r dt = 0\ ; \qquad r = 1,\ 2,\ \cdot\cdot,\ L \tag{2.21}$$

that are not integrable; that is, we cannot use this to get a relation between finite displacements.

Recall that starting with a holonomic constraint

$$f_r(u_1, u_2,\ \cdots,\ u_N, t) = 0\ ;$$

we differentiated to get

$$\sum_{s=1}^{N} \frac{\partial f_r}{\partial u_s}\, du_s + \frac{\partial f_r}{dt}\, dt = 0$$

Integrating, we can go back to the finite form. With a nonholonomic constraint of the type of Eqn. (2.21) this cannot be done.

From Eqn. (2.16), we see that one way nonholonomic constraints can occur is as constraints on the velocity components, and that such constraints are restricted to those in which this dependence is linear. The most common situations in which such constraints arise involve bodies rolling on other bodies without slipping; several example of this will be analyzed later.

Pfaffian Form. The differential form (whether integrable or not) of a constraint, Eqn. (2.21), is called the *Pfaffian form* and it is the most general type of constraint we will consider in this book. Thus the DOF of a system is the number of velocity components that can be given arbitrary values.

2.5 Catastatic Constraints

Definitions. Consider a system of constraints in Pfaffian form

$$\sum_{s=1}^{N} A_{rs} du_s + A_r dt = 0 \, ; \qquad r = 1, \cdot\cdot, L \tag{2.22}$$

Each constraint may be either holonomic (integrable) or nonholonomic. Each constraint may be either scleronomic ($A_{rs} \neq A_{rs}(t)$ and $A_r = 0$) or rheonomic. We make an additional distinction. A constraint is *catastatic* if $A_r = 0$ and *acatastatic* otherwise; if all constraints are catastatic, we say the system is catastatic. Note that the A_{rs} may be functions of t in a catastatic constraint.

Equation (2.22) implies that the condition of static equilibrium, $\dot{u}_s = 0$, $s = 1, \cdot\cdot, n$, is possible if and only if the system is catastatic.

2.6 Determination of Holonomic Constraints

Remarks. Holonomic constraints usually come in integrated form. However, sometimes they come in Pfaffian form. We must then be able to distinguish between holonomic and nonholonomic. If the Pfaffian form is an exact differential, then the constraint is integrable, but this is not necessary. The following theorem (without proof) is a very general result.

Theorem. Suppose an equation of independent variables $y_1, y_2, \cdots, y_M$ is given in differential form as:

$$\sum_{s=1}^{M} A_s(y_1, y_2, \cdots, y_M) dy_s = 0 \tag{2.23}$$

Then it is necessary and sufficient for the existence of an integral of this equation of the form

$$f(y_1, y_2, \cdots, y_M) = 0$$

that the equations

$$\begin{aligned} A_\gamma \left(\frac{\partial A_\beta}{\partial y_\alpha} - \frac{\partial A_\alpha}{\partial y_\beta} \right) + A_\beta \left(\frac{\partial A_\alpha}{\partial y_\gamma} - \frac{\partial A_\gamma}{\partial y_\alpha} \right) & \\ + A_\alpha \left(\frac{\partial A_\gamma}{\partial y_\beta} - \frac{\partial A_\beta}{\partial y_\gamma} \right) = 0 \, ; \qquad & \alpha, \beta, \gamma = 1, 2, \cdots, M \end{aligned} \tag{2.24}$$

be simultaneously and identically satisfied. There are $M(M-1)(M-2)/6$ such equations, of which $(M-1)(M-2)/2$ are independent.

For three variables, $M = 3$, Eqn. (2.23) is

$$A_1(y_1, y_2, y_3) dy_1 + A_2(y_1, y_2, y_3) dy_2 + A_3(y_1, y_2, y_3) dy_3 = 0 \tag{2.25}$$

and

$$\frac{M(M-1)(M-2)}{6} = 1 \, ; \qquad \frac{(M-1)(M-2)}{2} = 1$$

Thus for the equation to be integrable the only requirement is:

$$A_1 \left(\frac{\partial A_2}{\partial y_3} - \frac{\partial A_3}{\partial y_2} \right) + A_2 \left(\frac{\partial A_3}{\partial y_1} - \frac{\partial A_1}{\partial y_3} \right) + A_3 \left(\frac{\partial A_1}{\partial y_2} - \frac{\partial A_2}{\partial y_1} \right) = 0 \tag{2.26}$$

In the application of this theorem to the constraints on a dynamical system, the y_r may be displacement components, velocity components, or time.

As an example, suppose a particle moves in the (x, y) plane such that the slope of it's path is proportional to time, $dy/dx = Kt$. In Pfaffian form, this constraint is

$$Kt \, dx - dy = 0$$

This is a differential constraint of three variables; thus we take $y_1 = x$, $y_2 = y$, and $y_3 = t$ so that $A_1 = Ky_3$, $A_2 = -1$, and $A_3 = 0$. The left-hand side of Eqn. (2.26) equals K, so that the constraint is nonholonomic.

One consequence of the theorem is that any differential relationship between only two variables is always integrable. Indeed, in this case the constraint may be written

$$\frac{dy_2}{dy_1} = -\frac{A_1(y_1, y_2)}{A_2(y_1, y_2)} \tag{2.27}$$

This may be integrated under mild assumptions on the functions $A_1(\cdot)$ and $A_2(\cdot)$, if not analytically, then numerically. This means for example, that *any* time-independent constraint on the position of a particle in 2-D motion is holonomic.

2.7 Accessibility of Configuration Space

Definition. Recall that a holonomic constraint reduces the number of quantities required to define a point in configuration space. If there are n particles and L holonomic constraints this number is $N - L$, where $N = 3n$. We say that there is an $N-L$ *fold* ∞ *of motion* or, alternatively, that the *dimensionality of the space of accessible configurations (DSAC)* is $N - L$.

Nonholonomic equality constraints *do not* reduce the DSAC; consequently in general the DSAC is given by

$$DSAC = N - L' \tag{2.28}$$

where L' is the number of holonomic constraints ($L' \leq L$).[2] The fact that nonholonomic constraints do not reduce the DSAC will be shown by some of the following examples.

2.8 Examples

Example. Suppose a constraint on the motion of a particle is $z = dy/dx$. In Pfaffian form,

$$dy - z\,dx = 0$$

Relabeling:

$$\begin{array}{lll} y_1 = x\,, & y_2 = y\,, & y_3 = z \\ A_1 = -z\,, & A_2 = 1\,, & A_3 = 0 \end{array}$$

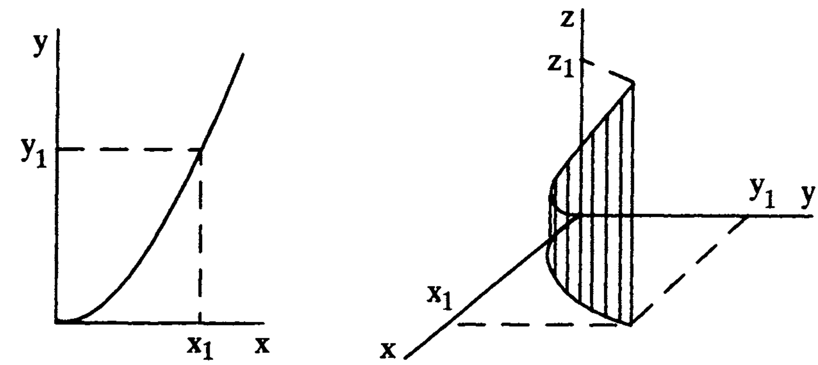


Fig. 2-12

The left-hand-side of Eqn. (2.26) is equal to one so that the constraint is nonholonomic and therefore the DSAC is not reduced. We now show this directly. We need to show that there is at least one path from the origin to any arbitrary fixed point (x_1, y_1, z_1) that satisfies the constraint. Consider any path such that (Fig. 2-12):

$$y = f(x)\,, \qquad z = \frac{df}{dx}\,, \qquad f(0) = f'(0) = 0\,,$$
$$f(x_1) = y_1\,, \quad f'(x_1) = z_1$$

The following shows that the constraint is satisfied and that the endpoint is reached.

$$dy - zdx = f'dx - f'dx = 0$$
$$\text{at } x = x_1\,, \qquad y = y_1 \quad \text{and} \quad z = z_1$$

Example. Two particles p_1 and p_2 moving in the (x, y) plane are connected by a light rod of length a which changes as a prescribed function of time, $a(t) \in C^1$. Let the coordinates of the two particles be (x_1, y_2) and (x_2, y_2). Then the constraint is

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 = a^2$$

In Pfaffian and velocity forms, respectively,

$$(x_2 - x_1)(dx_2 - dx_1) + (y_2 - y_1)(dy_2 - dy_1) - a\dot{a}dt = 0$$
$$(x_2 - x_1)\left(\frac{dx_2}{dt} - \frac{dx_1}{dt}\right) + (y_2 - y_1)\left(\frac{dy_2}{dt} - \frac{dy_1}{dt}\right) - a\frac{da}{dt} = 0$$

It is clear that the constraint is holonomic rheonomic and that DOF = DSAC = 3.

Now suppose the rod has constant length a, but there is additionally the constraint that the velocity of p_1 is always directly along the rod. The two constraints are

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 = a^2$$

$$(y_2 - y_1)\frac{dx_1}{dt} - (x_2 - x_1)\frac{dy_1}{dt} = 0$$

or, in Pfaffian form,

$$(x_2 - x_1)(dx_2 - dx_1) + (y_2 - y_1)(dy_2 - dy_1) = 0$$

$$(y_2 - y_1)dx_1 - (x_2 - x_1)dy_1 = 0$$

The first of these is holonomic scleronomic and the second is nonholonomic. Thus DOF = 2 and DSAC = 3.

Example – Disk Rolling on Plane. A knife-edged disk rolls without slipping on a horizontal plane (Fig. 2-13). There are two constraints – (i) the edge remains in contact with the plane, and (ii) the no slipping condition. The first is *holonomic* and reduces the DSAC from six (the general number for a rigid body) to five, say the (x, y) coordinates of the contact point and three angles usually taken as Euler's angles. The second constraint is a relation between velocities (the contact point must have instantaneous zero velocity relative to the surface) and is in general *nonholonomic*.

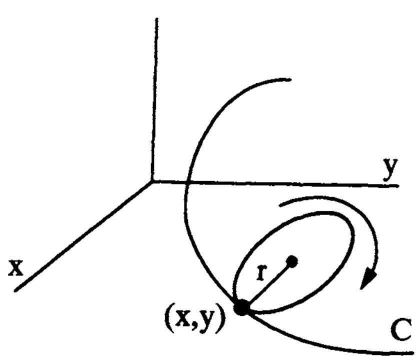
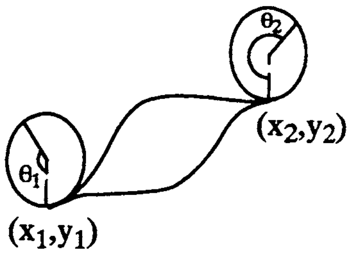
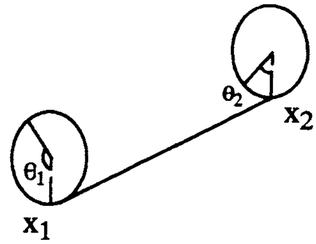


Fig. 2-13



rolling without slipping on a plane.

Fig. 2-14



rolling without slipping on a line.

Fig. 2-15

One of the Euler angles is the angle of rotation of some fixed line in the plane of the disk, say θ, Fig. 2-14. It is clear that there is no finite relation between θ and the coordinates of the contact point, (x, y), because the path can vary and still satisfy the constraint. If the rolling is confined to be on a line (Fig. 2-15), however, there is such a relationship, namely $(x_2 - x_1) = r(\theta_2 - \theta_1)$, and therefore the constraint is holonomic. In Chapter 7, we will return to the problem of a disk rolling on a plane and obtain the equations of motion.

Notes

1 Pars shows that otherwise the law of vector addition of forces would be violated.

2 It is clear that inequality constraints of the type of Eqn. (2.20) do not decrease the DSAC but rather restrict configurations to regions, but we are not considering this type of constraint.

PROBLEMS

2/1. Two particles having Cartesian coordinates $(x_1,\ y_1,\ z_1)$ and $(x_2,\ y_2,\ z_2)$, respectively, are attached to the extremities of a bar whose length $l(t)$ changes with time in a prescribed fashion. Give the equations of constraint on the finite and infinitesimal displacements of the Cartesian coordinates.

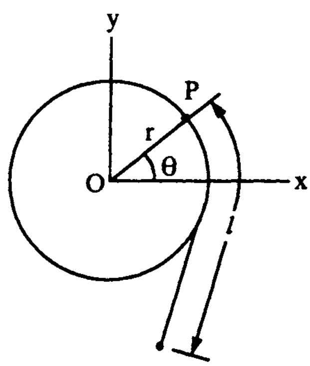
2/2. What are the equations of constraint on the finite and infinitesimal coordinates $(x_1,\ y_1,\ z_1)$ and $(x_2,\ y_2,\ z_2)$ of the bobs of a double spherical pendulum of lengths l_1 and l_2, respectively?

2/3. A thin bar of length $l < 2r$ can move in a plane in such a way that its endpoints are always in contact with a circle of radius r. If the Cartesian coordinates of its endpoints are (x_1, y_1) and (x_2, y_2), respectively, what constraints on finite and infinitesimal displacements must these coordinates satisfy?

2/4. Answer the same questions as in Problem 2/3 if the circle is replaced by an ellipse having major axis $2a$ and minor axis $2b$, and $l < 2b$. Are these constraints holonomic?

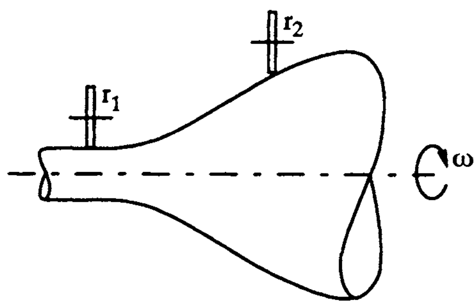
2/5. The motion of an otherwise unconstrained particle is subject to the conditions $\dot{z} = \dot{x}\dot{y}$. Discuss the constraint on the infinitesimal and finite displacements.

2/6. A particle moving in the xy plane is connected by an inextensible string of length l to a point P on the rim of a fixed disk of radius r, as shown. The line PO makes the angle θ with the x axis. What are the constraints on the finite and infinitesimal displacements of the point at the free end of the string having the position (x, y)?



Problem 2/6

2/7. A circular shaft of variable radius $r(x)$ rotates with angular velocity $\omega(t)$ about its centerline, as shown. The shaft is translated along its centerline in a prescribed fashion $f(t)$. Two disks of radii r_1 and r_2, respectively, roll without slipping on the shaft. A mechanism permits the disks to rise and fall in such a way that the disk rims never lose contact with the shaft. Show that the relation, free of ω, between the angular displacement ψ_1 and ψ_2 of the disks is in general nonholonomic. State the general condition that must be satisfied in the exceptional case that the constraint is holonomic and give an example.



Problem 2/7

2/8. A particle moving in the vertical plane is steered in such a way that the slope of its trajectory is proportional to its height. Formulate this constraint mathematically and classify it.

2/9. Write down and classify the equation of constraint of a particle moving in a plane if its slope is always proportional to the time.

2/10. A particle P moving in 3-space is steered in such a way that its velocity is directed for all time toward a point O which has a prescribed motion in space and time. Formulate and classify the equation(s) of constraint of the particle motion under the assumption that the positions of P and O never coincide.

2/11. A dynamic system is subject to the constraint

$$(\cos\theta)dx + (\sin\theta)dy + (y\cos\theta - x\sin\theta)d\theta = 0$$

Is this constraint holonomic? Prove your answer.