Chapter 17

Hamilton-Jacobi Equation

17.1 The Principal Function

Hamilton's Principle Again. Consider the motion of a holonomic conservative system in configuration space and consider a varied path such that the δq_r occur at a fixed time. Figure 17-1 shows the situation for two generalized coordinates. In this case,

$$\frac{d}{dt}(\delta q) = \delta \dot{q}$$

and Eqn. (15.5) may be written:

$$\delta L = \frac{d}{dt}\left(\sum_r p_r \delta q_r\right) \tag{17.1}$$

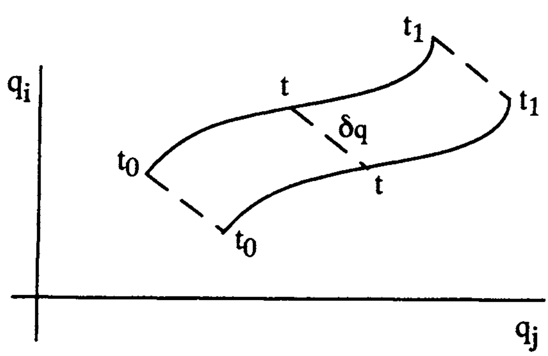

Fig. 17-1

This leads directly to Hamilton's principle, as follows. Integrating Eqn. (17.1) from t_0 to t_1 with the variations δq_r zero at the endpoints gives

$$\int_{t_0}^{t_1} \delta L dt = \sum_r p_r \delta q_r \Big|_{t_0}^{t_1}$$

$$\delta \int_{t_0}^{t_1} L dt = 0$$

which is the third form of Hamilton's principle (see Section 4.3).

Principal Function. Define *Hamilton's principal function* by

$$S = \int_{t_0}^{t_1} L dt \tag{17.2}$$

Now suppose that all n integrals of Lagrange's equations are known; they will be functions of the form

$$q_s(t) = \rho_s(q_r^0, \omega_r^0, t_0, t) \ ; \quad s = 1, \cdot\cdot, n \tag{17.3}$$

where $q_r^0 = q_r(t_0)$ and $\omega_r^0 = \dot{q}_r(t_0)$. Then the Lagrangian will be of the form $L = L(q_r^0, \omega_r^0, t_0, t)$ and from Eqn. (17.2) the principal function will be of the form

$$S = S(q_r^0, \omega_r^0, t_0, t_1) \tag{17.4}$$

We want, instead, to express S in terms of boundary conditions

$$S = S(q_r^0, q_r^1, t_0, t_1) \tag{17.5}$$

where $q_r^1 = q_r(t_1)$. We see that the solution may be thought of as a $2n$ parameter family of functions, the parameters being the q_r^0 and the ω_r^0. Alternatively, the solutions may be parameterized by the q_r^0 and the q_r^1, and we now proceed to replace the dependence on the ω_r^0 by dependence on the q_r^1. In effect, this replaces a point-slope specification of the solution curves by a point-point specification (see Fig. 17-2).

From Eqn. (17.3),

$$q_s^1 = \rho_s(q_r^0, \omega_r^0, t_0, t_1) \ ; \quad s = 1, \cdot\cdot, n$$

If the Jacobian of this transformation is non-zero, this relation may be inverted to give

$$\omega_s^0 = \psi_s(q_r^0, q_r^1, t_0, t_1) \ ; \quad s = 1, \cdot\cdot, n \tag{17.6}$$

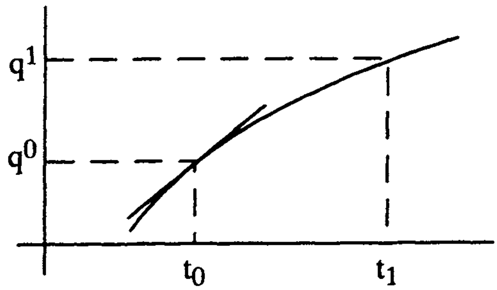

Fig. 17-2

Substitution of these relations in Eqns. (17.4) gives Eqns. (17.5).

Variation of the Principal Function. First, fix t_0 and t_1 and vary the q_s^1, which also varies the q_s^0, as shown on Fig. 17-3 for one q. Since t_0 and t_1 are fixed, Eqn. (17.1) applies

$$\begin{aligned} \delta S &= \delta \int_{t_0}^{t_1} L dt = \int_{t_0}^{t_1} \delta L dt \\ &= \sum p_r^1 \delta q_r^1 - \sum p_r^0 \delta q_r^0 \end{aligned} \tag{17.7}$$

Further, by Eqn. (17.5) with t_0 and t_1 fixed,

$$\delta S = \sum_r \frac{\partial S}{\partial q_r^0} \delta q_r^0 + \sum_r \frac{\partial S}{\partial q_r^1} \delta q_r^1 \tag{17.8}$$

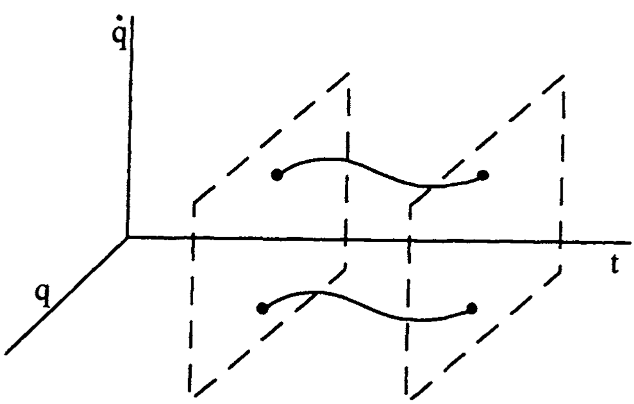

Fig. 17-3

Comparing Eqns. (17.7) and (17.8) and recalling that the δq_r^0 and the δq_r^1 are being regarded as independent, their coefficients must be equal,

$$p_r^0 = -\frac{\partial S}{\partial q_r^0}\ ; \quad p_r^1 = \frac{\partial S}{\partial q_r^1}\ ; \quad r = 1,\cdot\cdot,n \tag{17.9}$$

From these equations, we see that if S is known, all the integrals of the motion are known; that is, the dynamics problem is completely solved. Indeed, the first set of Eqns. (17.9) provides the q_r^1 in terms of the q_r^0 and p_r^0, and t_0 and t_1. Together, they provide the solution (i.e. all the integrals) of Hamilton's equations. We note that if $L \neq L(t)$ then the q_r are functions of time only of the form $(t - t_0)$ and hence S is a function of time only of the form $(t_1 - t_0)$.

Next, fix the q_r^0 and ω_r^0 and vary t_1 (and hence also the q_r^1). From Eqn. (17.2),

$$L_1 = \frac{dS}{dt_1} = \frac{\partial S}{\partial t_1} + \sum_r \frac{\partial S}{\partial q_r^1}\frac{\partial q_r^1}{\partial t_1} \tag{17.10}$$

Thus, using the second set of Eqns. (17.9),

$$\frac{\partial S}{\partial t_1} = L_1 - \sum_r p_r^1 \omega_r^1 = -H_1 \tag{17.11}$$

Similarly, it may be shown that

$$\frac{\partial S}{\partial t_0} = H_0 \tag{17.12}$$

Finally, then, from Eqns. (17.7), (17.11), and (17.12) the *total variation* in S due to variations in all the $2n+2$ arguments of S is

$$dS = \sum_r p_r^1 dq_r^1 - \sum_r p_r^0 dq_r^0 - H_1 dt_1 + H_0 dt_0 \tag{17.13}$$

Thus the transformation $(q_r^0, p_r^0) \to (q_r^1, p_r^1)$ is a contact transformation (CT) with generating function S.

Remarks

1. The goal has been to construct a unique trajectory through any two points in the event (q_r, t) space; if this can be done, S exists.

2. We have started by assuming that the integrals of motion, Eqns. (17.3), are known. Thus none of what we have done indicates how to find S. We have shown only that if S can be found, the dynamics problem is solved. We will turn to the problem of finding S in Section 17.2. Thus S is another descriptive function, but one with an important difference from L and H.

Example. Consider a particle of unit mass moving in a plane under constant gravity (Fig. 17-4). The solution is known to be

$$\begin{aligned} x &= x_0 + u_0(t - t_0) \\ y &= y_0 + v_0(t - t_0) - \frac{1}{2}g(t - t_0)^2 \end{aligned}$$

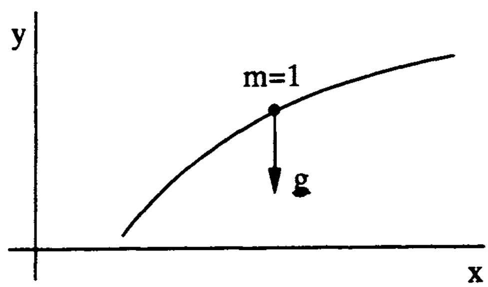

Fig. 17-4

where $x_0 = x(t_0)$, $y_0 = y(t_0)$, $u_0 = \dot{x}(t_0)$ and $v_0 = \dot{y}(t_0)$. From this solution, S may be computed directly from Eqn. (17.2) as follows

$$\begin{aligned} L &= T - V = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - gy \\ &= \frac{1}{2}\left(u_0^2 + (v_0 - g(t - t_0))^2\right) - gy_0 - gv_0(t - t_0) + \frac{1}{2}g^2(t - t_0)^2 \\ S &= \int_{t_0}^{t_1} L dt = S(x_0, u_0, y_0, v_0, t_0, t_1) \end{aligned}$$

This corresponds to Eqn. (17.4). Next we do the inversion, Eqn. (17.6):

$$u_0 = \frac{x_1 - x_0}{t_1 - t_0}; \qquad v_0 = \frac{y_1 - y_0 + \frac{1}{2}g(t_1 - t_0)^2}{t_1 - t_0}$$

Substitution of these into the expression for S gives (the details are left as an exercise)

$$S(x_0, x_1, y_0, y_1, t_0, t_1) = \frac{(x_1 - x_0)^2 + (y_1 - y_0)^2}{2(t_1 - t_0)} - \frac{1}{2}g(t_1 - t_0)(y_1 - y_0) - \frac{1}{24}g^2(t_1 - t_0)^3$$

Now apply the first set of Eqns. (17.9) to get

$$u_0 = -\frac{\partial S}{\partial x_0} = \frac{x_1 - x_0}{t_1 - t_0}$$

$$v_0 = -\frac{\partial S}{\partial y_0} = \frac{y_1 - y_0}{t_1 - t_0} + \frac{1}{2}g(t_1 - t_0)$$

This is the solution we started out with; this demonstrates that if S is known, then the solution (all integrals of the motion) is readily obtained. Clearly, this is valid for all cases except the trivial one, $t_1 - t_0 = 0$.

Example – Harmonic Oscillator. The equation of motion and its solution are

$$\ddot{x} + n^2 x = 0$$

$$x = x_0 \cos n(t - t_0) + \frac{u_0}{n} \sin n(t - t_0)$$

Computing S as before

$$L = \frac{1}{2}\dot{x}^2 - \frac{1}{2}n^2 x^2$$

$$S(x_0, u_0, t_0, t_1) = \frac{1}{2}\int_{t_0}^{t_1} (\dot{x}^2 - n^2 x^2)dt$$

We can solve for $u_0 = u_0(x_0, x_1, t_0, t_1)$ uniquely provided $n(t_1 - t_0) \neq r\pi$, r an integer (see Fig. 17-5). Under this restriction, the result of putting S in the form of Eqn. (17.5) is

$$S = \frac{1}{2}n(x_1^2 + x_0^2)\cot n(t_1 - t_0) - \frac{n x_1 x_0}{\sin n(t_1 - t_0)}$$

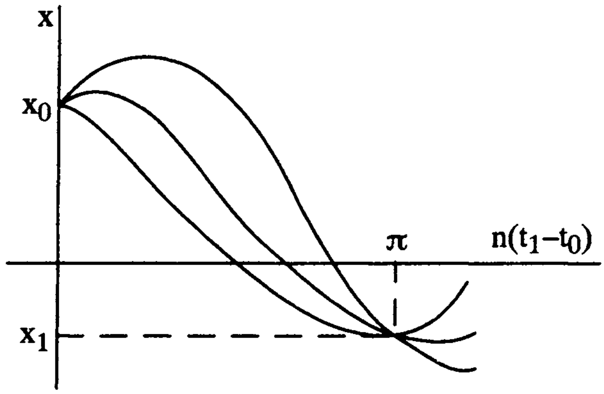

Fig. 17-5

so that from Eqns. (17.9)

$$u_0 = -\frac{\partial S}{\partial x_0} = -nx_0 \cot n(t_1 - t_0) + \frac{nx_1}{\sin n(t_1 - t_0)}$$

which is the solution we started out with.

17.2 Hamilton-Jacobi Theorem

The Hamilton-Jacobi Equation. The solution of Eqns. (17.9) gives the q_r^1 and p_r^1 as functions of the $2n$ parameters q_r^0 and p_r^0. It is often convenient, however, to use other parameters α_r and β_r related to the q_r^0 and p_r^0 by a HCT (see Section 16.3). We seek such a transformation that leaves S invariant:

$$S(q_r^0, q_r^1, t_0, t_1) = \tilde{S}(\alpha_r, q_r^1, t_0, t_1) \tag{17.14}$$

Using Eqns. (17.13) and (16.19) results in

$$\begin{aligned} d\tilde{S} &= dS = \sum p_r^1 dq_r^1 - \sum p_r^0 dq_r^0 - H_1 dt_1 + H_0 dt_0 \\ &= \sum p_r^1 dq_r^1 - \sum \beta_r d\alpha_r - H_1 dt_1 + H_0 dt_0 \end{aligned} \tag{17.15}$$

so that

$$\frac{\partial \tilde{S}}{\partial \alpha_r} = -\beta_r \tag{17.16}$$

These equations are the solutions to Lagrange's equations. Also,

$$\frac{\partial \tilde{S}}{\partial q_r^1} = p_r^1 \tag{17.17}$$

Previously, we have shown that if we could find the principal function, S, then the solution to the dynamics problem is easily obtained. We now turn to the task of finding an equation for S. Let α_r, β_r define a HCT; that is, from Eqn. (16.19),

$$\sum_r \beta_r d\alpha_r = \sum_r p_r^0 dq_r^0 \tag{17.18}$$

In this section, we shall take $t_0 = 0$, write $S = \tilde{S}$, and suppress the superscript 1; thus

$$S = \tilde{S}(q_r^1, \alpha_r, t_0, t_1) = S(q_r, \alpha_r, t) \tag{17.19}$$

With this new notation, Eqn. (17.15) becomes

$$dS = \sum_r p_r dq_r - \sum_r \beta_r d\alpha_r - H dt \tag{17.20}$$

so that

$$\frac{\partial S}{\partial q_r} = p_r \; ; \quad r = 1, \cdot\cdot, n \tag{17.21}$$

$$\frac{\partial S}{\partial \alpha_r} = -\beta_r \; ; \quad r = 1, \cdot\cdot, n \tag{17.22}$$

$$\frac{\partial S}{\partial t} = -H \tag{17.23}$$

where $H = H(q_r, p_r, t)$. Substituting Eqn. (17.21) into (17.23), we arrive at

$$\frac{\partial S}{\partial t} + H(q_r,\, p_r,\, t) = 0$$

$$\frac{\partial S}{\partial t} + H\left(q_r,\, \frac{\partial S}{\partial q_r},\, t\right) = 0$$

$$\frac{\partial S}{\partial t} + H\left(q_1, \cdot\cdot, q_n, \frac{\partial S}{\partial q_1}, \cdot\cdot, \frac{\partial S}{\partial q_n}, t\right) = 0 \tag{17.24}$$

This first-order, non-linear, partial differential equation is known as the *Hamilton-Jacobi equation*,[1] or sometimes as Hamilton's equation.

We know that the principal function, Eqn. (17.19), is a solution (complete integral) of Eqn. (17.24). The solutions of Eqn. (17.24), however, are not unique, raising the question of which of the solutions solve the dynamics problem. The following theorem establishes that *any* solution of Eqn. (17.24) also satisfies Eqns. (17.21) and (17.22), and thus solves the problem.

Hamilton-Jacobi Theorem. If $S = S(q_r, \alpha_r, t)$ is a complete integral of Eqn. (17.24) then the integral's of Hamilton's equations are given by Eqns. (17.21) and (17.22). Hence we have replaced the problem of solving a $2n$-order system of ordinary differential equations (Hamilton's equations) by the problem of solving one first order partial differential equation (the Hamilton-Jacobi equation).

The proof proceeds as follows. By definition, a complete integral of Eqn. (17.24) is a function of class C^2 containing n arbitrary constants $\alpha_1, \cdot\cdot, \alpha_n$ such that

$$\begin{vmatrix} \dfrac{\partial^2 S}{\partial q_1 \partial \alpha_1} & \cdots & \dfrac{\partial^2 S}{\partial q_1 \partial \alpha_n} \\ \vdots & & \vdots \\ \dfrac{\partial^2 S}{\partial q_n \partial \alpha_1} & \cdots & \dfrac{\partial^2 S}{\partial q_n \partial \alpha_n} \end{vmatrix} \neq 0 \tag{17.25}$$

Regard the q_r and α_r as independent parameters and differentiate Eqn. (17.24) w.r.t. α_1:

$$\frac{\partial^2 S}{\partial \alpha_1 \partial t} + \sum_r \frac{\partial H}{\partial p_r} \frac{\partial^2 S}{\partial \alpha_1 \partial q_r} = 0 \tag{17.26}$$

where Eqn. (17.21) was used. Also, from Eqns. (17.21) and (17.22),

$$\frac{\partial S}{\partial \alpha_1} = -\beta_1$$

$$\frac{\partial^2 S}{\partial t \partial \alpha_1} = -\frac{\partial \beta_1}{\partial t} = -\sum_r \frac{\partial \beta_1}{\partial q_r} \frac{\partial q_r}{\partial t}$$

$$\frac{\partial^2 S}{\partial t \partial \alpha_1} + \sum \frac{\partial^2 S}{\partial q_r \partial \alpha_1} \frac{\partial q_r}{\partial t} = 0 \tag{17.27}$$

Since $S \in C^2$,

$$\frac{\partial^2 S}{\partial t \partial \alpha_1} = \frac{\partial^2 S}{\partial \alpha_1 \partial t_1} ; \qquad \frac{\partial^2 S}{\partial \alpha_1 \partial q_r} = \frac{\partial^2 S}{\partial q_r \partial \alpha_1}$$

so that Eqns. (17.26) and (17.27) combine to give

$$\sum_r \frac{\partial^2 S}{\partial q_r \partial \alpha_1} \left[\frac{\partial q_r}{\partial t} - \frac{\partial H}{\partial p_r} \right] = 0 \tag{17.28}$$

If this procedure is repeated for $\alpha_2, \cdot\cdot, \alpha_n$, the following matrix equation results

$$\left\| \frac{\partial^2 S}{\partial q \partial \alpha} \right\| \left| \frac{\partial q}{\partial t} - \frac{\partial H}{\partial p} \right| = 0 \tag{17.29}$$

The first of these factors in an $n \times n$ matrix and the second is $n \times 1$. In view of Eqn. (17.25), Eqn. (17.29) implies

$$\frac{\partial q_r}{\partial t} = \frac{\partial H}{\partial p_r} ; \quad r = 1, \cdot\cdot, n \tag{17.30}$$

which are the first n of Hamilton's equations. Note that we have written $\partial q_r / \partial t$ here instead of dq_r/dt because we are considering the family of trajectories generated by independently varying α, β, and t, and not the time rate of change along a trajectory.

To get the other n of Hamilton's equations, we proceed much as before. Differentiate Eqn. (17.24) w.r.t. q_1 and use Eqns. (17.21) to obtain

$$\frac{\partial^2 S}{\partial q_1 \partial t} + \frac{\partial H}{\partial q_1} + \sum_r \frac{\partial^2 S}{\partial q_1 \partial q_r} \frac{\partial H}{\partial p_r} = 0 \tag{17.31}$$

Also, from Eqns. (17.21) and (17.22),

$$\frac{\partial p_1}{\partial t} = \frac{\partial^2 S}{\partial t \partial q_1} + \sum_r \frac{\partial^2 S}{\partial q_r \partial q_1} \frac{\partial q_r}{\partial t} \tag{17.32}$$

Combining these two equations, repeating this for $q_2, \cdot\cdot, q_n$, and forming a matrix equation as before, we arrive at

$$\frac{\partial p_r}{\partial t} = -\frac{\partial H}{\partial q_r} ; \quad r = 1, \cdot\cdot, n \tag{17.33}$$

and the theorem is proved.

Historical Remarks. Hamilton, born in Ireland, in 1805, was the ultimate child prodigy. Attracted to foreign languages as a child, by the age of 10 he was proficient in writing Latin, Greek, Hebrew, Italian, French, Arabic, and Sanskrit, and was learning a half a dozen others. He then became interested in mathematics, teaching himself the known mathematics of the time, by the age of 17. He was by then a student at Trinity College, Dublin, and had started his revolutionary research in optics.

Hamilton's goal was to bring the theory of optics to the same "state of perfection" that Lagrange had brought dynamics. By the age of 22 his work on optics was complete; it succeeded in resolving the most outstanding problem of mathematical physics of his time – unifying the particle and wave concepts of light into one elegant, comprehensive theory. Already he was called "the first mathematician of the age" and it was said that "a second Newton has arrived". At this time, while still a student, he was appointed professor of astronomy at Trinity, not having even applied for the position.

Hamilton then turned his attention back to dynamics, applying the methods he had developed in optics. In his *First Essay on a General Method in Dynamics* (1834) he introduced the "characteristic function", $\int_0^t 2T dt$, clearly motivated by the Principle of Least Action, and used it to formulate the dynamics problem. In this work he also introduced the principal function, $S = \int_0^t L\, dt$. In the *Second Essay on a General Method in Dynamics* (1835), he derived both what we now call the Hamilton-Jacobi equation, Eqn. (17.24) and Hamilton's canonical equations, Eqns. (15.11). He fully realized the importance of finding the function S as well as the difficulty in doing so (he gave an approximation method). As Hamilton stated in his paper, in the impersonal style then in vogue:

> "Professor Hamilton's solution of this long celebrated problem contains, indeed, one unknown function, namely, the *principal function S*, to the search and the study of which he has reduced mathematical dynamics. This function must not be confounded with that so beautifully conceived by Lagrange for the more simple and elegant expression of the known differential equations. Lagrange's function *states*, Mr. Hamilton's function would *solve* the problem."

Beginning in his late 20's, Hamilton suffered severe psychological problems. He became reclusive, alcoholic, and irregular in his eating and sleeping habits. As a consequence, he was not productive during these years. Later in life he spent all of his mathematical energies on the development of quaternians, which he regarded as his greatest achievement.

During his life, Hamilton was awarded every honor possible to a scientist, including being knighted and being named the first foreign member of the U.S. Academy of Sciences. When he died at age 61, his study was found piled high with mathematical papers, interspersed with plates of partially finished meals.

Jacobi's life was apparently relatively settled. He was born in Prussia in 1804 and spent most of his professional life as a professor (he was by all accounts an excellent teacher). His main contributions to dynamics were the proof of the Hamilton-Jacobi theorem and putting Hamilton-Jacobi theory into its modern form. These contributions were given in a series of lectures in 1842 and 1843, which were not published until 1866. Jacobi was a first-rate mathematician and he is perhaps best known for his contributions outside of dynamics, specifically to the fields of elliptic functions, solution of algebraic equations, number theory, and differential equations.

Hamiltonian dynamics has had a far-reaching impact on all of mathematics and physical science. As Bell states, "it is the aim of many workers in particular branches of theoretical physics to sum up the whole of a theory in a Hamiltonian principle." Most remarkable, is that when, about 100 years ago, experiments began to reveal the nature of the motion of atomic particles, the tools of Hamiltonian dynamics (Hamilton-Jacobi equation, canonical equations, contact transformations, and Poisson brackets) proved to be ideal as the basis for the modern theory of quantum mechanics.

17.3 Integration of the Hamilton-Jacobi Equation

Natural Systems. Consider a natural system. For such a system

$$H(q_r, \dot{q}_r) = T(q_r, \dot{q}_r) + V(q_r) = h = \text{constant} \qquad (17.34)$$

We see by direct substitution that in this case a solution of Eqn. (17.24) is of the form

$$S = -ht + K \tag{17.35}$$

where

$$K = K(q_1, \cdot\cdot, q_n, h, \alpha_2, \cdot\cdot, \alpha_n)$$

and where we have taken $\alpha_1 = h$. The function $K(\cdot)$ is called *Hamilton's characteristic function.*[2] Substitution of Eqn. (17.35) into (17.24) gives

$$H\left(q_r, \frac{\partial K}{\partial q_r}\right) = h \tag{17.36}$$

By the Hamilton-Jacobi Theorem, the integrals of motion are given by Eqns. (17.21) and (17.22); the first of Eqns. (17.22) yields:

$$\frac{\partial S}{\partial \alpha_1} = \frac{\partial S}{\partial h} = -\beta_1$$

$$-t + \frac{\partial K}{\partial h} = -\beta_1 = -t_0$$

$$\frac{\partial K}{\partial h} = t - t_0 \tag{17.37}$$

where t_0 is written in place of β_1. The rest of the equations are

$$\frac{\partial K}{\partial \alpha_r} = -\beta_r \ ; \quad r = 2, \cdot\cdot, n \tag{17.38}$$

$$\frac{\partial K}{\partial q_r} = p_r \ ; \quad r = 1, \cdot\cdot, n \tag{17.39}$$

The remaining problem is to find the function K. Note that Eqns. (17.38) determine the path in configuration space, and Eqn. (17.37) then gives time elapsed along the path.

Natural System with Ignorable Coordinate. Now, in addition, suppose q_n is ignorable with corresponding momentum integral $p_n = \gamma =$ constant. Then

$$H(q_1, \cdot\cdot, q_{n-1}, \dot{q}_1, \cdot\cdot, \dot{q}_n) = T(q_1, \cdot\cdot, q_{n-1}, \dot{q}_1, \cdot\cdot, \dot{q}_n) + V(q_1, \cdot\cdot, q_{n-1})$$

In this case we write

$$K = \gamma q_n + K' \tag{17.40}$$

where

$$K' = K'(q_1, \cdot\cdot, q_{n-1}, h, \alpha_2, \cdot\cdot, \alpha_{n-1}, \gamma)$$

where we have taken $\alpha_n = \gamma$. The last of Eqns. (17.22) gives

$$\frac{\partial S}{\partial \alpha_n} = -\beta_n$$

$$q_n + \frac{\partial K'}{\partial \gamma} = -\beta_n = q_n^0$$

where we have taken $\beta_n = -q_n^0$. Thus from Eqns. (17.22) the first n integrals are given by

$$\begin{aligned} &\frac{\partial K'}{\partial h} = t - t_0 \\ &\frac{\partial K'}{\partial \alpha_r} = -\beta_r \; ; \quad r = 2, \cdot\cdot, n-1 \\ &\frac{\partial K'}{\partial \gamma} = q_n^0 - q_n \end{aligned} \tag{17.41}$$

Equations (17.21) give the other n integrals:

$$\begin{aligned} &\frac{\partial K'}{\partial q_r} = p_r \; ; \quad r = 1, \cdot\cdot, n-1 \\ &\frac{\partial K'}{\partial q_n} = p_n = \gamma \end{aligned} \tag{17.42}$$

The remaining problem is now to find the function K'.

17.4 Examples

Example. We now return to the two examples of Section 17.1. Consider again a particle of unit mass in 2-D motion in a uniform gravitational field (Fig. 17-4). We have

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - gy$$

$$p_x = \frac{\partial L}{\partial \dot{x}} = \dot{x} \; ; \qquad p_y = \frac{\partial L}{\partial \dot{y}} = \dot{y}$$

$$H = \sum_{r=1}^{2} p_r \dot{q}_r - L = \dot{x}^2 + \dot{y}^2 - \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + gy$$
$$= \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + gy = T + V = \frac{1}{2}(p_x^2 + p_y^2) + gy$$

We see that x is ignorable. Hamilton's equation is

$$\frac{\partial S}{\partial t} + H\left(x,\ y,\ \frac{\partial S}{\partial x},\ \frac{\partial S}{\partial y}\right) = 0$$
$$\frac{\partial S}{\partial t} + \frac{1}{2}\left(\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2\right) + gy = 0$$

Since $H = h =$ constant and x is ignorable, Eqns. (17.35) and (17.40) apply:

$$S = -ht + \gamma x + K'(y)$$

Substitution into Hamilton's equation gives

$$-h + \frac{1}{2}\left(\gamma^2 + \left(\frac{\partial K'}{\partial y}\right)^2\right) + gy = 0$$

Letting $gk = h - \frac{1}{2}\gamma^2$, the solution of this equation is

$$K' = \int_y^k \sqrt{2g(k-\eta)}\, d\eta$$
$$K' = \sqrt{2g}\int_0^{k-y} \sqrt{\nu}\, d\nu$$

Thus

$$S = -\left(\frac{1}{2}\gamma^2 + gk\right)t + \gamma x + \sqrt{2g}\int_0^{k-y} \sqrt{\nu}\, d\nu$$

Now apply Eqns. (17.21) and (17.22)

$$\frac{\partial S}{\partial x} = \gamma = p_x = \dot{x}$$
$$\frac{\partial S}{\partial y} = -\sqrt{2g(k-y)} = p_y = \dot{y}$$
$$\frac{\partial S}{\partial \gamma} = -\gamma t + x = -\beta_1$$
$$\frac{\partial S}{\partial k} = -gt + \sqrt{2g(k-y)} = -\beta_2$$

The last two of these may be written as

$$x + \beta_1 = \gamma t$$
$$2g(k - y) = \beta_2^2 + g^2t^2 - 2\beta_2 gt$$

If a projectile is launched at position (x_0, y_0) at time $t_0 = 0$ with velocity components (u_0, v_0), these equations give at time t_0

$$\gamma = u_0$$
$$-\sqrt{2g(k - y_0)} = v_0$$
$$x_0 + \beta_1 = 0$$
$$2g(k - y_0) = \beta_2^2$$

which give

$$\gamma = u_0\,, \qquad \beta_1 = -x_0$$
$$k = y_0 + \frac{v_0^2}{2g}\,, \qquad \beta_2 = v_0$$

so that the solution may be written as

$$x - x_0 = u_0 t$$
$$2g(y_0 - y) = g^2t^2 - 2v_0 gt$$

which is the well-known solution to this problem. We note that h is the energy integral and k is the maximum height of the projectile. The problem has been completely solved.[3]

Example – Harmonic Oscillator. For this problem,

$$T = \frac{1}{2}\dot{x}^2\,, \quad V = \frac{1}{2}n^2x^2\,, \quad p = \frac{\partial L}{\partial \dot{x}} = \dot{x}$$
$$H = \frac{1}{2}\dot{x}^2 + \frac{1}{2}n^2x^2 = \frac{1}{2}(p^2 + n^2x^2) = h$$

Thus Hamilton's equation is

$$\frac{\partial S}{\partial t} + \frac{1}{2}\left(\left(\frac{\partial S}{\partial x}\right)^2 + n^2x^2\right) = 0$$

We try a solution of the form

$$S = -\frac{1}{2}n^2\alpha^2 t + \rho(x)$$

Substitution gives

$$-\frac{1}{2}n^2\alpha^2 + \frac{1}{2}\left(\left(\frac{\partial\rho}{\partial x}\right)^2 + n^2x^2\right) = 0$$

$$\rho = n\int_0^x \sqrt{\alpha^2 - \eta^2}\, d\eta$$

where η is a dummy integration variable. Then

$$S = -\frac{1}{2}n^2\alpha^2 t + n\int_0^x \sqrt{\alpha^2 - \eta^2}\, d\eta$$

Equation (17.22) gives the solution of the problem as:

$$\frac{\partial S}{\partial \alpha} = -\beta$$

$$-n^2\alpha t + n\alpha\int_0^x (\alpha^2 - \eta^2)^{-1/2} d\eta = -\beta$$

where Liebnitz' rule for differentiating under the integral sign has been used; this may be written in the more familiar form[3]

$$x = \alpha \sin n(t - t_0)$$

where $\beta = n^2\alpha t_0$. The constants α and t_0 may be expressed in terms of initial conditions if desired.

In Section 17.1, we found (by starting with a known solution) a different function S that satisfies Hamilton's equation for this problem than the one found here. This shows that the solution of Hamilton's partial differential equation is not unique. Both functions, however, provide the complete solution to the problem.

17.5 Separable Systems

Separability. In this section we continue to consider natural systems. In Section 17.3, we gave partial solutions of the Hamilton-Jacobi equation by writing the principal function as the sum of two or more parts. For example, in Eqn. (17.35) S was written as the sum of a function depending only on t and a function depending only on the q_r. Such a separation is always possible when $H \neq H(t)$. More generally, it may be possible to write S as a sum of functions, each containing just one of the q_i or just t. In this case we say the problem is *completely separable.* Both of the examples of Section 17.4 were completely separable.

In practice, the Hamilton-Jacobi equation is only useful when there is some degree of separability. Some problems, for example the famous three-body problem, are not separable. For other problems, separability depends on the choice of coordinates. For example, the central force problem is not separable in rectangular coordinates but is in polar (because in the latter case one coordinate is ignorable). A classic and important special case of complete separability is that of linear systems written in terms of modal coordinates.

Conditions for Separability. General conditions for complete separability are not known, but Pars (who devotes two chapters to the subject of separability), gives some results for systems for which the kinetic energy contains only squared terms:

$$T = \frac{1}{2}\sum_r a_r \dot{q}_r^2 \tag{17.43}$$

Not surprisingly, the same systems for which Lagrange's equations are separable (see Section 8.4) are separable in the Hamiltonian sense. Thus Liouville systems, defined by Eqns. (8.4), are completely separable.

The most general separable system of the type Eqn. (17.43) is given by Stackel's theorem, not stated here.

Solution of Separable Systems. General methods have been developed for solving completely separable systems; see for example Pars, Goldstein, or McCuskey. These methods depend on the theory of contact transformations, developed in the previous chapter. They will not be reviewed here.

As a practical matter, the most common case of separability occurs when there are ignorable coordinates. We have already seen that if a coordinate, say q_n, is ignorable, then a partial separation occurs, as expressed by Eqn. (17.40). In general, if coordinates $q_m, \cdot\cdot, q_n$ are ignorable then the characteristic function may be written as

$$K = \alpha_m q_m + \cdot\cdot + \alpha_n q_n + K'(q_1, \cdot\cdot, q_{m-1}, h, \alpha_2, \cdot\cdot, \alpha_n) \tag{17.44}$$

where the α_i are constants. In particular, if all coordinates but q_1 are ignorable, the problem is completely separable, because then

$$K = \alpha_2 q_2 + \cdots + \alpha_n q_n + K'(q_1, h, \alpha_2, \cdots, \alpha_n) \tag{17.45}$$

In this case the Hamilton-Jacobi equation reduces to an equation in q_1, which is always reducible to quadratures; the problem has been completely solved.

Example.[4] As an example of a non-trivial problem, consider again the heavy symmetrical top analyzed in Section 11.2 and shown on Fig. 11-4. From Eqns. (8.24) and (11.10),

$$\begin{aligned} p_\theta &= I\dot{\theta} \\ p_\phi &= I\dot{\phi}\sin^2\theta + J(\dot{\psi} + \dot{\phi}\cos\theta)\cos\theta \\ p_\psi &= J(\dot{\psi} + \dot{\phi}\cos\theta) \end{aligned}$$

Using Eqns. (11.11) and (15.15), we arrive at

$$H = \frac{1}{2}\left[\frac{p_\theta^2}{I} + \frac{(p_\phi - p_\psi\cos\theta)^2}{I\sin^2\theta} + \frac{p_\psi^2}{J}\right] + mg\ell\cos\theta$$

First, we see that $H \neq H(t)$ so that Eqns. (17.35) and (17.36) apply. Second, we see that ϕ and ψ are ignorable so that K has the form

$$K = \alpha_2\phi + \alpha_3\psi + K'(\theta)$$

The problem is thus completely separable.

The Hamilton-Jacobi equation, Eqn. (17.36), is

$$\begin{aligned} &\frac{1}{2I}\left(\frac{\partial K}{\partial\theta}\right)^2 + \frac{1}{2I\sin^2\theta}\left(\frac{\partial K}{\partial\phi} - \frac{\partial K}{\partial\psi}\cos\theta\right)^2 + \frac{1}{2J}\left(\frac{\partial K}{\partial\psi}\right)^2 \\ &\quad + mg\ell\cos\theta = h \end{aligned}$$

Substituting for K, we obtain

$$\frac{1}{2I}\left(\frac{dK'}{d\theta}\right)^2 + \frac{1}{2I\sin^2\theta}(\alpha_2 - \alpha_3\cos\theta)^2 + \frac{1}{2J}\alpha_3^2 + mg\ell\cos\theta = h$$

so that

$$\frac{dK'}{d\theta} = \sqrt{F(\theta)}$$

and

$$K' = \int \sqrt{F(\theta)} d\theta$$

where

$$F(\theta) = 2Ih - \frac{I}{J}\alpha_3^2 - 2Img\ell\cos\theta - \frac{1}{\sin^2\theta}(\alpha_2 - \alpha_3\cos\theta)^2$$

Thus

$$K = \alpha_2\phi + \alpha_3\psi + \int \sqrt{F(\theta)} d\theta$$

and S is given by Eqn. (17.35).

Now apply Eqns. (17.22); the first of these is given by Eqn. (17.37):

$$\int \frac{I d\theta}{\sqrt{F(\theta)}} = t - t_0$$

and the other two are

$$\phi - \int \frac{(\alpha_2 - \alpha_3\cos\theta)d\theta}{\sin^2\theta\sqrt{F(\theta)}} = -\beta_2$$

$$\psi - \frac{I}{J}\alpha_3 \int \frac{d\theta}{\sqrt{F(\theta)}} + \int \frac{(\alpha_2 - \alpha_3\cos\theta)\cos\theta d\theta}{\sin^2\theta\sqrt{F(\theta)}} = -\beta_3$$

The constants h, α_2, α_3 may be determined by initial conditions.

Notes

1 The same equation plays a central role in the subject of *dynamic programming*, where it is called the Hamilton-Jacobi-Bellman equation.

2 It may be shown that K is equivalent to the action integral, Eqn. (4.40), see Goldstein.

3 The details of the solution to this problem are left as an exercise.

4 McCuskey.

PROBLEMS

Solve the following three problems by the Hamilton-Jacobi method.

17/1. Problem 4/2.

17/2. Problem 6/7.

17/3. Problem 10/1.

17/4. Show that linear systems written in terms of modal coordinates are completely separable.

17/5. Fill in the details of the solution to the first example of Section 17.4.

17/6. Fill in the details of the solution to the second example of Section 17.4.

17/7. Fill in the details of the solution to the example of Section 17.5.