Chapter 16

Contact Transformations

16.1 Introduction

The Nature of Hamiltonian Dynamics. In the last chapter we have seen that for solving specific problems, use of Hamilton's equations offers no particular advantage over Lagrange's equations; the procedures and the amount of work required are essentially the same. Rather, the Hamiltonian formulation offers a new point of view, one that will be exploited in this and the following chapters.

In the Lagrangean formulation, q_i and $\dot{q}_i$ are regarded as the independent variables. There is an obvious connection between the two sets q_i and $\dot{q}_i$, however, the latter being the derivatives of the former. In the Hamiltonian formulation, the variables q_i and p_i must be regarded as truly independent. The motion is now envisioned as the motion of a point in phase space (see Eqn. (15.12)). In modern mathematical terms, the motion of a dynamical system defines a *continuous group* of transformations in phase space that carry the point

$$(q_1^0, \cdot\cdot, q_n^0,\ p_1^0, \cdot\cdot, p_n^0)$$

at $t = 0$ to the point

$$(q_1, \cdot\cdot, q_n,\ p_1, \cdot\cdot, p_n)$$

at time t. The equations defining this transformation are the solutions of Hamilton's equations, say

$$q_r = \phi_r(q_r^0, p_r^0, t)\ ; \qquad p_r = \phi_{n+r}(q_r^0, p_r^0, t)\ ; \qquad r = 1, \cdot\cdot, n \tag{16.1}$$

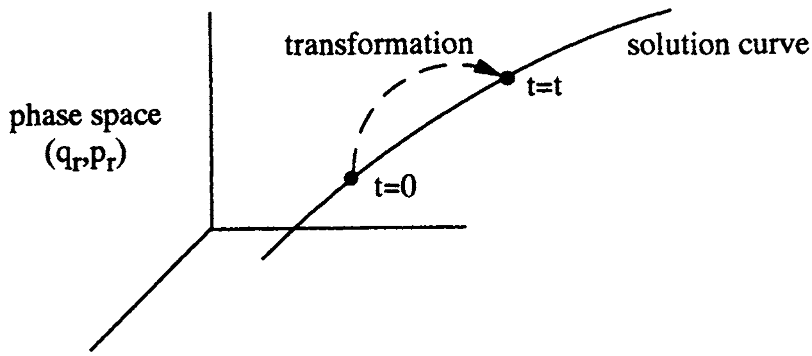

Fig. 16-1

Thus the dynamics problem becomes the study of transformations (Fig. 16-1).

Ignorable Coordinates. Other types of transformations are also of interest, for example a transformation of coordinates. Consider the case of a natural system, for which H = constant, and suppose all of the coordinates are ignorable. Then $L \neq L(q_k)$ and from Eqn. (15.6) $H = H(p_k)$. Equations (15.11) then give

$$\dot{q}_r = \frac{\partial H}{\partial p_r} ; \qquad \dot{p}_r = 0 ; \qquad r = 1, \cdot\cdot, n \tag{16.2}$$

These have solutions

$$p_r = p_r^0 = \text{constant} ; \qquad q_r = \omega_r t + q_r^0 ; \qquad r = 1, \cdot\cdot, n \tag{16.3}$$

where $\omega_r = \omega_r\ (p_s^0)$ are constants. Thus in this special case the dynamics problem is easily completely solved.

This case is not quite as restricted as it first appears. A dynamic system may be described by any set of suitable generalized coordinates, and for some choices more of the coordinates may be ignorable than for others. For example, for the problem of Fig. 8-3 none of the rectangular coordinates, x, y, z, are ignorable but one of the cylindrical coordinates, ϕ, is. As another example, for the problem of Fig. 8-4 neither x_1, x_2 are ignorable but for the choice x, y one is, namely y.

Idea of Contact Transformations. Based on the preceeding observations, it would be of great value if we could find transformations such that either the new variables were constants, say the initial conditions q_k^0, p_k^0, or the new coordinates were ignorable. In either case, the problem would be completely solved. Such transformations must have the property that if the original variables are Hamiltonian (i.e. satisfy Hamilton's equations) then the new ones must be as well. It must be remembered that not only the generalized coordinates but also their corresponding generalized momenta must be transformed. Such transformations are called *contact transformations.*[1] We will first define and study contact transformations in general, and then prove that such transformations have the desired property of preserving the Hamiltonian structure. In the following chapter, the most important application of contact transformations will be covered.

16.2 General Contact Transformations

Definition. Now consider a general transformation of $2n$ variables, one not necessarily giving the motion of a dynamic system:

$$(q_1,\cdot\cdot,q_n,\ p_1,\cdot\cdot,p_n) \longrightarrow (Q_1,\cdot\cdot,Q_n,\ P_1,\cdot\cdot,P_n)$$

such that the differential relation is true:

$$\sum_r P_r dQ_r = \sum_r p_r dq_r + Rdt - dW \tag{16.4}$$

This equation defines a *contact transformation* (CT). The function W is called the *generating function* of the CT. The CT generates the transformation

$$Q_r = \phi_r(q_s, p_s, t)\ ; \qquad P_r = \phi_{r+n}(q_s, p_s, t)\ ; \quad r = 1,\cdot\cdot,n \tag{16.5}$$

These are $2n$ equations in $4n$ variables. Because any set of the q_k, p_k combined with any set of the Q_k, P_k may be regarded as the $2n$ independent variables of the transformation, there are four possibilities: $W = W_1(q_r, Q_r, t)$, $W = W_2(q_r, P_r, t)$, $W = W_3(p_r, Q_r, t)$, and $W = W_4(p_r, P_r, t)$.

Case $W = W_1(q_r, Q_r, t)$. First suppose that the Jacobian

$$\frac{\partial(\phi_1,\cdot\cdot,\phi_n)}{\partial(p_1,\cdot\cdot,p_n)} \neq 0 \tag{16.6}$$

In this case, by the Implicit Function Theorem, we can solve the first set of Eqns. (16.5) for the p_r in terms of the q_r, Q_r, and t. The functions R

and W can then be expressed in terms of the q_r, Q_r, and t and there is no relation connecting these variables. Taking the differential of W_1:

$$dW_1 = \sum_r \frac{\partial W_1}{\partial q_r} dq_r + \sum_r \frac{\partial W_1}{\partial Q_r} dQ_r + \frac{\partial W_1}{\partial t} dt \tag{16.7}$$

Since the q_r, Q_r, t are independent, comparison of Eqns. (16.4) and (16.7) gives

$$\begin{aligned} p_r &= \frac{\partial W_1}{\partial q_r}\,; \qquad P_r = -\frac{\partial W_1}{\partial Q_r}\,; \quad r = 1, \cdot\cdot, n \\ R &= \frac{\partial W_1}{\partial t} \end{aligned} \tag{16.8}$$

These are the explicit equations of the CT.

If the Jacobian defined in Eqn. (16.6) is zero, we proceed as follows. Suppose, for definiteness, that the rank is $n-1$; then there is one relation

$$\theta(q_r, Q_r, t) = 0 \tag{16.9}$$

Take the differential of this:

$$d\theta = \sum_r \frac{\partial \theta}{\partial q_r} dq_r + \sum_r \frac{\partial \theta}{\partial Q_r} dQ_r + \frac{\partial \theta}{\partial t} dt = 0 \tag{16.10}$$

Now, the q_r, Q_r, t are not independent but are constrained by Eqn. (16.10). Using a Lagrange multiplier λ to account for the constraint,

$$\begin{aligned} p_r &= \frac{\partial W_1}{\partial q_r} + \lambda \frac{\partial \theta}{\partial q_r}\,; \quad r = 1, \cdot\cdot, n \\ P_r &= -\frac{\partial W_1}{\partial Q_r} - \lambda \frac{\partial \theta}{\partial Q_r}\,; \quad r = 1, \cdot\cdot, n \\ R &= \frac{\partial W_1}{\partial t} + \lambda \frac{\partial \theta}{\partial t} \end{aligned} \tag{16.11}$$

There will be a Lagrange multiplier for each such relation θ.

The Other Cases. Next consider the case $W = W_2(q_r, P_r, t)$. Now take as the generating function

$$W_2 = W_1 + \sum_r P_r Q_r \tag{16.12}$$

Forming dW_2 and using Eqns. (16.7) and (16.8) gives:

$$\begin{aligned} dW_2 &= dW_1 + \sum_r P_r dQ_r + \sum_r Q_r dP_r \\ &= \sum_r p_r dq_r + \sum_r Q_r P_r + R dt \end{aligned} \tag{16.13}$$

Since in this case the q_r, P_r, t are independent, this implies

$$\begin{aligned} &p_r = \frac{\partial W_2}{\partial q_r}\,; \qquad Q_r = \frac{\partial W_2}{\partial P_r}\,; \quad r = 1, \cdot\cdot, n \\ &R = \frac{\partial W_2}{\partial t} \end{aligned} \tag{16.14}$$

The next case[2] is $W = W_3(p_r, Q_r, t)$. The appropriate generating function is

$$W_3 = W_1 - \sum_r q_r p_r \tag{16.15}$$

and this gives

$$\begin{aligned} &q_r = -\frac{\partial W_3}{\partial p_r}\,; \qquad P_r = -\frac{\partial W_3}{\partial Q_r}\,; \qquad r = 1, \cdot\cdot, n \\ &R = \frac{\partial W_3}{\partial t} \end{aligned} \tag{16.16}$$

The final case is $W = W_4(p_r, P_r, t)$. The generating function is

$$W_4 = W_1 + \sum_r P_r Q_r - \sum_r p_r q_r \tag{16.17}$$

which gives

$$\begin{aligned} &q_r = -\frac{\partial W_4}{\partial p_r}\,; \qquad Q_r = \frac{\partial W_4}{\partial P_r}\,; \qquad r = 1, \cdot\cdot, n \\ &R = \frac{\partial W_4}{\partial t} \end{aligned} \tag{16.18}$$

Of course, in all these cases if there are relations among the variables one Lagrange multiplier will have to be introduced for each such relation.

Example. Consider the transformation described by the generating function of the first kind:

$$W_1 = \sum_r q_r Q_r$$

From Eqns. (16.8),

$$p_r = \frac{\partial W_1}{\partial q_r} = Q_r \; ; \;\; r = 1, \cdot\cdot, n$$

$$P_r = -\frac{\partial W_1}{\partial Q_r} = -q_r \; ; \;\; r = 1, \cdot\cdot, n$$

Thus this transformation interchanges the q_r and the p_r (except for a change of sign).

Example. Next consider an example of a CT with a generating function of the second type given by

$$W_2 = \sum_r q_r P_r$$

Applying Eqns. (16.14),

$$p_r = \frac{\partial W_2}{\partial q_r} = P_r$$

$$Q_r = \frac{\partial W_2}{\partial P_r} = q_r$$

Thus the old and new variables are the same; W_2 generates the identity transformation.

16.3 Homogeneous Contact Transformations

Definition. The special case of a CT with $Rdt - dW = 0$ is called a *homogeneous contact transformation* (HCT). From Eqn. (16.4) this is defined by

$$\sum_r P_r dQ_r = \sum_r p_r dq_r \tag{16.19}$$

It is assumed that the Q_r, P_r are independent functions of the $q_r, p_r \in C^1$; that is, the Jacobian of the transformation is not zero:

$$\left| \frac{\partial(Q_r, P_r)}{\partial(q_r, p_r)} \right| \neq 0$$

Transformation of Coordinates. One application of HCT's is the transformation of one set of generalized coordinates, q_r, to another, Q_r. Let the inverse transformation be

$$q_r = F_r(Q_s) \; ; \qquad r = 1, \cdot\cdot, n \tag{16.20}$$

where $F_r \in C^2$. Such a transformation is sometimes referred to as a *continuous point transformation.* The corresponding generalized momenta are denoted p_r and P_r. We assume a natural system so that the transformation does not explicitly contain time. For such a system (see Sections 6.1 and 15.1),

$$T = \frac{1}{2}\sum_r p_r \dot{q}_r = \frac{1}{2}\sum_r P_r \dot{Q}_r \tag{16.21}$$

Thus

$$\sum_r P_r dQ_r = \sum_r p_r dq_r$$

so that the transformation is a HCT.

We now obtain the explicit relations for the transformation of the corresponding generalized momenta. From Eqn. (6.3), for a natural system,

$$2T = \sum_r \sum_s a_{rs} \dot{q}_r \dot{q}_s = \sum_r \sum_s A_{rs} \dot{Q}_r \dot{Q}_s$$

so that, using Eqn. (16.20),

$$P_r = \frac{\partial T}{\partial \dot{Q}_r} = \sum_i \sum_j \sum_s a_{ij} \frac{\partial F_i}{\partial Q_r} \frac{\partial F_j}{\partial Q_s} \dot{Q}_s = \sum_i \sum_j a_{ij} \frac{\partial F_i}{\partial Q_r} \dot{q}_j$$

$$P_r = \sum_i p_i \frac{\partial F_i}{\partial Q_r}\ ; \qquad r = 1, \cdot\cdot, n \tag{16.22}$$

Equations (16.20) and (16.22) define the CT. Note that the P_r are homogeneous linear functions of the p_r.

Example. Consider the transformation from rectangular coordinates to polar coordinates for a particle moving in the (x, y) plane. The transformation is given by

$$x = r\cos\theta\ , \qquad y = r\sin\theta$$

Letting $q_1 = x$, $q_2 = y$, $Q_1 = r$, and $Q_2 = \theta$, this is

$$q_1 = Q_1 \cos Q_2 = F_1(Q_1, Q_2)\ , \qquad q_2 = Q_1 \sin Q_2 = F_2(Q_1, Q_2)$$

These correspond to Eqns. (16.20). Equations (16.22) give expressions for the new generalized momenta in terms of the old:

$$P_1 = p_1\frac{\partial F_1}{\partial Q_1} + p_2\frac{\partial F_2}{\partial Q_1} = p_1 \cos Q_2 + p_2 \sin Q_2$$

$$P_2 = p_1\frac{\partial F_1}{\partial Q_2} + p_2\frac{\partial F_2}{\partial Q_2} = Q_1(-p_1 \sin Q_2 + p_2 \cos Q_2)$$

where $p_1 = p_x$, $p_2 = p_y$, $P_1 = p_r$, and $P_2 = p_\theta$.

16.4 Conditions for a Contact Transformation

Remarks. It may be required to determine whether or not a given transformation is a CT. If we can find a function W such that Eqn. (16.4) is satisfied, then the transformation is a CT, but this is often difficult to do. In this section, we derive tests that are usually easier to apply.

Liouville's Theorem. Suppose the transformation $(q_r, p_r) \to (Q_r, P_r)$ is a CT with no relations between the Q_r and q_r. Consider Case 1 of Section 16.2 for which Eqns. (16.8) define the transformation. Forming the Jacobian $\partial(Q_r, P_r)/\partial(q_r, p_r)$,

$$\frac{\partial(Q_r, P_r)}{\partial(q_r, p_r)} = \frac{\frac{\partial(Q_r,P_r)}{\partial(q_r,Q_r)}}{\frac{\partial(q_r,p_r)}{\partial(q_r,Q_r)}} = (-1)^n\frac{\frac{\partial(P_r)}{\partial(q_r)}}{\frac{\partial(p_r)}{\partial(Q_r)}} = \frac{\frac{\partial\left(\frac{\partial W_1}{\partial Q_r}\right)}{\partial(q_r)}}{\frac{\partial\left(\frac{\partial W_1}{\partial q_r}\right)}{\partial(Q_r)}} = 1 \qquad (16.23)$$

The same result holds for the other cases. This is sometimes called Liouville's theorem. Equation (16.23) expresses the fact that the transformation is *measure preserving.*

Lagrange Brackets. Let $q_1, \cdot\cdot, q_n$, $p_1, \cdot\cdot, p_n$ be C^2 functions of variables u and v. Then the *Lagrange bracket* of u and v is

$$[u, v] = \sum_r \left(\frac{\partial q_r}{\partial u}\frac{\partial p_r}{\partial v} - \frac{\partial p_r}{\partial u}\frac{\partial q_r}{\partial v}\right) = \sum_r \frac{\partial(q_r, p_r)}{\partial(u, v)} \qquad (16.24)$$

Theorem. The transformation from (q_r, p_r) to (Q_r, P_r) is a CT if and only if,

$$\begin{aligned}
&(1)\ \ [Q_r, Q_s] = 0 \\
&(2)\ \ [P_r, P_s] = 0 \\
&(3)\ \ [Q_r, P_s] = \delta_{rs} = \begin{Bmatrix} 0 \text{ if } s \neq r \\ 1 \text{ if } s = r \end{Bmatrix}
\end{aligned} \tag{16.25}$$

for all $r,\ s = 1, \cdot\cdot, n$ and fixed t.

The proof is as follows. Treating the Q_r and P_r as independent variables, with t fixed, in Eqn. (16.4), one obtains

$$\begin{aligned}
dW &= \sum_r p_r dq_r - \sum_r P_r dQ_r \\
&= \sum_r p_r \left(\sum_s \frac{\partial q_r}{\partial Q_s} dQ_s + \sum_s \frac{\partial q_r}{\partial P_s} dP_s \right) - \sum_r P_r dQ_r \\
&= \sum_s \left(\sum_r p_r \frac{\partial q_r}{\partial Q_s} - P_s \right) dQ_s + \sum_s \left(\sum_r p_r \frac{\partial q_r}{\partial P_s} \right) dP_s
\end{aligned} \tag{16.26}$$

For such a generating function to exist, this must be a perfect differential. The necessary and sufficient conditions for this to be a perfect differential is that Eqn. (2.24) be satisfied. But Eqns. (16.25) are just the conditions for this to be true, and the theorem is proved.

Example. Consider the rectilinear motion of a particle in a uniform gravitational field. The familiar solution of the equation of motion is:

$$x = x_0 + \dot{x}_0 t + \frac{1}{2} g t^2 , \qquad \dot{x} = \dot{x}_0 + g t$$

In our current viewpoint, this is regarded as the transformation of the generalized coordinates and momenta at time zero to those at time t, $(q, p) \to (Q, P)$, given by

$$Q = q + pt + \frac{1}{2} g t^2 , \qquad P = p + g t$$

We check to see if this is a CT by the previous two methods.

First,

$$\frac{\partial(Q, P)}{\partial(q, p)} = \begin{vmatrix} 1 & t \\ 0 & 1 \end{vmatrix} = 1$$

so that Eqn. (16.23) is satisfied and the transformation is a CT.

Second, we check Eqns. (16.25). Inverting the transformation,

$$q = Q - Pt + \frac{1}{2}gt^2\,, \qquad p = P - gt$$

Thus conditions (1) and (2) are satisfied trivially and condition (3) is

$$[Q, P] = \frac{\partial q}{\partial Q}\frac{\partial p}{\partial P} - \frac{\partial p}{\partial Q}\frac{\partial q}{\partial P} = 1$$

again verifying that the transformation is a CT.

Relation Between Poisson and Lagrange Brackets. Comparison of Eqns. (15.28) and (16.24) seems to indicate that there is some sort of inverse relationship between Poisson and Lagrange brackets. This is indeed the case. Consider independent functions $u_1(q_r, p_r), \cdot\cdot, u_{2n}(q_r, p_r)$. Then it can be proved that[3]

$$\sum_{s=1}^{2n}[u_s, u_i](u_s, u_j) = \delta_{ij} \tag{16.27}$$

The necessary and sufficient conditions for the transformation $q_r, p_r \rightarrow Q_r P_r$ to be a CT in terms of Poisson brackets are

$$\begin{aligned} (Q_r, Q_s) &= 0 \\ (P_r, P_s) &= 0 \\ (Q_r, P_s) &= \delta_{rs} \end{aligned} \tag{16.28}$$

for all $r, s = 1, \cdot\cdot, n$ and fixed t.

16.5 Jacobi's Theorem

Remarks. Up to now, we have been considering general contact transformations. Now we will consider the transformation of the generalized coordinates and momenta of a dynamics problem. One of the reasons CT's are important is because if the original variables satisfy Hamilton's equations, then the transformed ones do also, as will be proved shortly.

In the original variables, the q_r and the p_r, $H = H(q_r, p_r, t)$ and Hamilton's equations are Eqns. (15.11), repeated here:

$$\dot{q}_r = \frac{\partial H}{\partial p_r}\,; \qquad \dot{p}_r = -\frac{\partial H}{\partial q_r}\,; \qquad r = 1, \cdot\cdot, n \tag{16.29}$$

Consider a CT defining new variables P_r, Q_r:

$$\sum_r P_r dQ_r = \sum_r p_r dq_r + Rdt - dW \tag{16.30}$$

Jacobi's Theorem. If the equations of motion for q_r, p_r are Hamiltonian (that is, if they satisfy Eqns. (16.29)) then they are also Hamiltonian for P_r, Q_r as given by Eqn. (16.30). To prove this, two lemmas are needed.

Lemma 1. If

$$\begin{aligned} q_r &= \rho_r(\gamma_1, \cdot\cdot, \gamma_{2n}, t) \; ; \qquad r = 1, \cdot\cdot, n \\ p_r &= \rho_{r+n}(\gamma_1, \cdot\cdot, \gamma_{2n}, t) \; ; \qquad r = 1, \cdot\cdot, n \end{aligned} \tag{16.31}$$

are the general solutions of Eqns. (16.29) and if we substitute Eqns. (16.31) in H to get $H = G(\gamma_1, \cdot\cdot, \gamma_{2n}, t)$ then $G \in C^2$ and

$$\frac{\partial G}{\partial \gamma_i} = [t, \gamma_i] \; ; \qquad i = 1, \cdot\cdot, 2n \tag{16.32}$$

To prove this, we use Eqns. (16.29) and (16.24):

$$\begin{aligned} \frac{\partial G}{\partial \gamma_i} &= \sum_r \frac{\partial H}{\partial q_r}\frac{\partial q_r}{\partial \gamma_i} + \sum_r \frac{\partial H}{\partial p_r}\frac{\partial p_r}{\partial \gamma_i} \\ &= \sum_r \left(\dot{q}_r \frac{\partial p_r}{\partial \gamma_i} - \dot{p}_r \frac{\partial q_r}{\partial \gamma_i} \right) = [t, \gamma_i] \; ; \qquad i = 1, \cdot\cdot, 2n \end{aligned}$$

Lemma 2. This is the converse of Lemma 1. If q_r, p_r are $2n$ independent functions of $\gamma_1, \cdot\cdot, \gamma_{2n}, t$ and if there exists a function $G(\gamma_1, \cdot\cdot, \gamma_{2n}, t)$ such that

$$\frac{\partial G}{\partial \gamma_i} = [t, \gamma_i] \; ; \qquad i = 1, \cdot\cdot, 2n$$

then q_r, p_r satisfy Eqn. (16.29).

To prove this, think of the functions of Eqns. (16.31) as the functions just described and solve them for the $\gamma_1, \cdot\cdot, \gamma_{2n}$. Substitute the result into G to get a function H such that

$$H(q_r, p_r, t) = G(\gamma_r, t)$$

Now differentiate

$$\begin{aligned}\frac{\partial G}{\partial \gamma_i} &= \sum_r \left(\frac{\partial H}{\partial q_r}\frac{\partial q_r}{\partial \gamma_i} + \frac{\partial H}{\partial p_r}\frac{\partial p_r}{\partial \gamma_i}\right) = [t, \gamma_i] \\ &= \sum_r \left(\frac{\partial q_r}{\partial t}\frac{\partial p_r}{\partial \gamma_i} - \frac{\partial p_r}{\partial t}\frac{\partial q_r}{\partial \gamma_i}\right) ; \qquad i = 1, \cdot\cdot, 2n\end{aligned}$$

where Eqns. (16.32) and (16.24) were used. Since the $\partial p_r/\partial \gamma_i$ and $\partial q_r/\partial \gamma_i$ are all independent, this equation implies Eqns. (16.29).

Proof of Jacobi's Theorem. Choose generating function

$$W = W_1(q_r, Q_r, t) = F(\gamma_i, t) \tag{16.33}$$

Using Eqns. (16.8),

$$\begin{aligned}\frac{\partial F}{\partial \gamma_i} &= \sum_r \left(\frac{\partial W_1}{\partial q_r}\frac{\partial q_r}{\partial \gamma_i} + \frac{\partial W_1}{\partial Q_r}\frac{\partial Q_r}{\partial \gamma_i}\right) \\ &= \sum_r \left(p_r\frac{\partial q_r}{\partial \gamma_i} - P_r\frac{\partial Q_r}{\partial \gamma_i}\right) ; \qquad i = 1, \cdot\cdot, 2n\end{aligned} \tag{16.34}$$

$$\begin{aligned}\frac{\partial F}{\partial t} &= \sum_r \left(\frac{\partial W_1}{\partial q_r}\frac{\partial q_r}{\partial t} + \frac{\partial W_1}{\partial Q_r}\frac{\partial Q_r}{\partial t}\right) + \frac{\partial W_1}{\partial t} \\ &= \sum_r \left(p_r\frac{\partial q_r}{\partial t} - P_r\frac{\partial Q_r}{\partial t}\right) + \frac{\partial W_1}{\partial t}\end{aligned} \tag{16.35}$$

The function $F(\gamma_i, t)$ is of class C^2 so that

$$\frac{\partial^2 F}{\partial \gamma_i \partial t} = \frac{\partial^2 F}{\partial t \partial \gamma_i} ; \qquad i = 1, \cdot\cdot, 2n \tag{16.36}$$

From Eqns. (16.34) - (16.36):

$$\frac{\partial}{\partial t}\sum_r \left(p_r\frac{\partial q_r}{\partial \gamma_i} - P_r\frac{\partial Q_r}{\partial \gamma_i}\right) = \frac{\partial}{\partial \gamma_i}\left[\sum_r \left(p_r\frac{\partial q_r}{\partial t} - P_r\frac{\partial Q_r}{\partial t}\right) + \frac{\partial W}{\partial t}\right] ;$$

$$\sum_r \left(\frac{\partial Q_r}{\partial t}\frac{\partial P_r}{\partial \gamma_i} - \frac{\partial P_r}{\partial t}\frac{\partial Q_r}{\partial \gamma_i}\right) = \sum_r \left(\frac{\partial q_r}{\partial t}\frac{\partial p_r}{\partial \gamma_i} - \frac{\partial p_r}{\partial t}\frac{\partial q_r}{\partial \gamma_i}\right) + \frac{\partial^2 W}{\partial \gamma_i \partial t} ;$$

$$[t, \gamma_i] = \frac{\partial G}{\partial \gamma_i} \qquad i = 1, \cdot\cdot, 2n \tag{16.37}$$

where, using Eqns. (16.29),

$$\begin{aligned}\frac{\partial H}{\partial \gamma_i} &= \sum_r \left(\frac{\partial H}{\partial q_r}\frac{\partial q_r}{\partial \gamma_i} + \frac{\partial H}{\partial p_r}\frac{\partial p_r}{\partial \gamma_i} \right) \\ &= \sum_r \left(\frac{\partial q_r}{\partial t}\frac{\partial p_r}{\partial \gamma_i} - \frac{\partial p_r}{\partial t}\frac{\partial q_r}{\partial \gamma_i} \right) ; \qquad i = 1, \cdot\cdot, 2n\end{aligned}$$

was used, and where $G = H + \partial W/\partial t$ and $[t, \gamma_i]$ is the Lagrange bracket of t and γ_i in terms of the Q_r, P_r.

By Lemma 2, P_r, Q_r satisfy Hamilton's equations with Hamiltonian function G, that is,

$$\frac{\partial Q_r}{\partial t} = \frac{\partial H^*}{\partial P_r} ; \qquad \frac{\partial P_r}{\partial t} = -\frac{\partial H^*}{\partial Q_r} ; \qquad r = 1, \cdot\cdot, n \tag{16.38}$$

where the new Hamiltonian is

$$H^* = H + \frac{\partial W}{\partial t} \tag{16.39}$$

Example.[4] Consider a linear harmonic oscillator with linear restoring force constant k. Then

$$\begin{aligned}T &= \frac{1}{2}m\dot{q}^2 , \qquad V = \frac{kq^2}{2} , \qquad p = \frac{\partial L}{\partial \dot{q}} = m\dot{q} \\ L &= \frac{1}{2}m\dot{q}^2 - \frac{kq^2}{2} \\ H &= \frac{p^2}{2m} + \frac{m\omega^2}{2}q^2\end{aligned}$$

where $\omega^2 = k/m$. Now consider a CT from q, p to new variables Q, P as defined by the generating function

$$W = W_1(q, Q) = \frac{m}{2}\omega q^2 \cot Q$$

Equations (16.8) give

$$\begin{aligned}p &= \frac{\partial W_1}{\partial q} = m\omega q \cot Q \\ P &= -\frac{\partial W_1}{\partial Q} = \frac{m\omega q^2}{2\sin^2 Q}\end{aligned}$$

Solving for q, p in terms of Q, P,

$$q = \sqrt{\frac{2P}{m\omega}} \sin Q$$
$$p = \sqrt{2m\omega P} \cos Q$$

Since the transformation does not contain t explicitly, Eqn. (16.39) gives $H^*(Q, P) = H(p, q)$; thus

$$H = \omega P \cos^2 Q + \omega P \sin^2 Q = \omega P$$

Therefore, Q is an ignorable coordinate with momentum integral

$$P = \frac{H}{\omega} = \frac{E}{\omega} = \text{constant}$$

The equation of motion for Q is now easy to solve:

$$\dot{Q} = \frac{\partial H}{\partial P} = \omega$$
$$Q = \omega t + \alpha$$

In terms of the original generalized coordinate, the well-known solution to this problem is obtained as

$$q = \sqrt{\frac{2E}{m\omega^2}} \sin(\omega t + \alpha)$$

This is an example of how a CT can be used to obtain a Hamiltonian for which all coordinates are ignorable.

Notes

1 The term originated in optics where it has to do with preserving the contact point between wave fronts; they are called *canonical transformations* in some texts.

2 The details of the analysis of the last two cases may be found in Goldstein.

3 See Pars or Goldstein.

4 Goldstein.

PROBLEMS

In the two following problems, show that the indicated transformation is a contact transformation by three methods: (i) Directly using the definition of a CT, (ii) Using Liouville's theorem, and (iii) Using the Lagrange brackets or Poisson brackets tests.

16/1. $Q = e^{kp}\sqrt{q+a}\,, \qquad P = -\left(\frac{1}{k}\right) e^{-kp}\sqrt{q+a}$

16/2. $Q = \ln\left(\frac{1}{q}\sin p\right)\,, \qquad P = q\cot p$

16/3. For what values of α and β is

$$Q = q^\alpha \cos\beta p\,, \qquad P = q^\alpha \sin\beta p$$

a CT?

16/4. Prove Liouville's theorem for the case $n = 1$.

16/5. Consider the transformation described by the generating function $W_2 = \sum_i f_i(q_1, \cdot\cdot, q_n, t)P_i$ where the f_i are any smooth functions. Show that the new coordinates depend only on the old coordinates and time and thus W_2 generates a continuous point transformation.

16/6. Show that the generating function $W_2 = \sum_{i,k} a_{ik}q_k p_i$ generates a linear transformation of coordinates, $Q_i = \sum_k a_{ik}q_k$ and that the generalized momenta also transform linearly.