Chapter 15

Hamilton's Equations

15.1 Derivation of Hamilton's Equations

Another Fundamental Equation. Consider a holonomic system with all forces embodied in a function $V = V(q_r, t)$. (Strictly speaking, the system is not conservative because in that case $V = V(q_r)$ is required.) From Eqn. (6.28), the fundamental equation in this case is

$$\sum_{r=1}^{n} \left[\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_r} \right) - \frac{\partial L}{\partial q_r} \right] \delta q_r = 0 \tag{15.1}$$

where n (the number of generalized coordinates)$=k$ (the degrees of freedom). Because the system is holonomic, the δq_r are independent, leading directly to Lagrange's equations, Eqns. (6.35).

Since $L = L(q_r, \dot{q}_r, t)$, the variation of L is[1]

$$\delta L = \sum_r \left(\frac{\partial L}{\partial q_r} \delta q_r + \frac{\partial L}{\partial \dot{q}_r} \delta \dot{q}_r \right) \tag{15.2}$$

with, as usual, $\sum\limits_r = \sum\limits_{r=1}^{n}$. Recall that the generalized momenta are defined by Eqn. (8.25), repeated here:

$$p_r = \frac{\partial L}{\partial \dot{q}_r} \; ; \quad r = 1, \cdot\cdot, n \tag{15.3}$$

Thus Eqn. (15.1) implies

$$\dot{p}_r - \frac{\partial L}{\partial q_r} = 0 \; ; \quad r = 1, \cdot\cdot, n \tag{15.4}$$

Substituting Eqns. (15.3) and (15.4) into (15.2) gives

$$\delta L = \sum_r (\dot{p}_r \delta q_r + p_r \delta \dot{q}_r) \tag{15.5}$$

Pars calls this the sixth form of the fundamental equation. Note that Eqns. (15.4) are just Lagrange's equations.

Hamilton's Equations. Define the Hamiltonian function by

$$H = \sum_r p_r \dot{q}_r - L \tag{15.6}$$

Form the variation of this function and use Eqn. (15.5) to get

$$\begin{aligned} \delta H &= \sum_r \dot{q}_r \delta p_r + \sum_r p_r \delta \dot{q}_r - \delta L \\ \delta H &= \sum_r (\dot{q}_r \delta p_r - \dot{p}_r \delta q_r) \end{aligned} \tag{15.7}$$

Next recall that the kinetic energy in generalized coordinates is given by Eqn. (6.3) and therefore

$$p_r = \frac{\partial L}{\partial \dot{q}_r} = \frac{\partial T}{\partial \dot{q}_r} = \sum_s a_{rs} \dot{q}_s + b_r \ ; \quad r = 1, \cdot\cdot, n \tag{15.8}$$

Since a_{rs} is nonsingular, we may solve these equations for the $\dot{q}_s$ in terms of the q_s and p_s. If the result is substituted into Eqn. (15.6), H will be of the form $H = H(q_r, p_r, t)$. Now take the variation of this function:

$$\delta H = \sum_r \frac{\partial H}{\partial q_r} \delta q_r + \sum_r \frac{\partial H}{\partial p_r} \delta p_r \tag{15.9}$$

Combining Eqns. (15.7) and (15.9) gives

$$\sum_r \left[\left(\frac{\partial H}{\partial q_r} + \dot{p}_r \right) \delta q_r + \left(\frac{\partial H}{\partial p_r} - \dot{q}_r \right) \delta p_r \right] = 0 \tag{15.10}$$

But because the δq_r and δp_r are independent, this gives

$$\dot{q}_r = \frac{\partial H}{\partial p_r} \ ; \quad \dot{p}_r = -\frac{\partial H}{\partial q_r} \ ; \quad r = 1, \cdot\cdot, n \tag{15.11}$$

These are *Hamilton's equations.* Their special form is called *canonical.* They give the motion in the *phase space* P, defined by:

$$(\underline{q}, \underline{p}) = \begin{pmatrix} q_1 \\ \vdots \\ q_n \\ p_1 \\ \vdots \\ p_n \end{pmatrix} \in P \subset I\!\!E^{2n} \tag{15.12}$$

Note that the first set of Eqns. (15.11) follows directly from the definition of H and is equivalent to the set Eqns. (15.3), which is the definition of the p_r. The second set expresses the dynamics. Although these two sets of equations have different meanings, they are to be treated mathematically as of equal stature.

Next we compute the total time derivative of H; using Eqns. (15.11) this is

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} + \sum_r \frac{\partial H}{\partial q_r}\dot{q}_r + \sum_r \frac{\partial H}{\partial p_r}\dot{p}_r = \frac{\partial H}{\partial t} \tag{15.13}$$

Natural Systems. Recall that (Section 3.5) a natural system is one that is holonomic, scleronomic, and conservative. In such a system, $T = T(q_r, \dot{q}_r)$ and $V = V(q_r)$. Thus, $H \neq H(t)$ and from Eqn. (15.13) $H =$ constant. Also, for such a system Eqns. (6.4) give $b_r = 0$ so that Eqns. (15.8) become

$$p_r = \sum_s a_{rs}\dot{q}_s \; ; \quad r = 1, \cdot\cdot, n \tag{15.14}$$

Consequently, using this and Eqns. (15.6) and (6.6), the H function becomes

$$\begin{aligned} H &= \sum_r \sum_s a_{rs}\dot{q}_s\dot{q}_r - (T - V) = 2T - T + V \\ &= T + V = E = h = \text{constant}. \end{aligned} \tag{15.15}$$

Thus for a natural system, the Hamiltonian is equal to the system mechanical energy and is an integral of the motion.

An explicit form of H for a natural system[2] may be obtained as follows. If Eqns. (15.14) are inverted, there results

$$\dot{q}_s = \sum_r c_{sr}p_r \tag{15.16}$$

Substitute this into Eqn. (15.6) to obtain

$$\begin{aligned} H &= \sum_r \sum_s p_r c_{rs} p_s - \left(\frac{1}{2} \sum_r \sum_s a_{rs} \dot{q}_r \dot{q}_s - V \right) \\ &= \frac{1}{2} \sum_r \sum_s p_r c_{rs} p_s + V \end{aligned} \tag{15.17}$$

General Systems. If there are nonholonomic constraints and forces not derivable from a function $V(q_r, t)$, Eqns. (6.34), (8.25), and (15.6) lead to

$$\dot{q}_r = \frac{\partial H}{\partial p_r} \, ; \qquad \dot{p}_r = -\frac{\partial H}{\partial q_r} + Q_r^{nc} - \sum_{s=1}^{\ell} \lambda_s B_{sr} \, ; \quad r = 1, \cdot\cdot, n \tag{15.18}$$

which are no longer of canonical form. In what follows, we will consider only holonomic systems with forces derivable from $V(q_r, t)$, that is the canonical Eqns. (15.11).

Remarks.

1. The functions L and H are sometimes termed *descriptive functions*, because once they are known for a given system the equations of motion of the system can be produced. In subsequent chapters we will identify other such functions.

2. The important difference between L and H is that we regard $L = L(q_r, \dot{q}_r, t)$ and $H = H(q_r, p_r, t)$.

3. As remarked in Section 1.6, one of the main aims of analytical dynamics is to find integrals of the motion (i.e. to "solve" the dynamics problem in whole or in part). This will be the primary motivation for the developments in the remaining chapters.

15.2 Hamilton's Equations as a First Order System

Remarks. Recall from Section 12.1 that Lagrange's equations always may be written in state variable form, that is as a system of uncoupled first order differential equations. It is clear that Hamilton's equations, Eqns. (15.11), come naturally in this form. This will now be made explicit.

Hamilton's Equations in First Order Form. Set

$$\begin{array}{ll} x_1 = q_1 & x_{n+1} = p_1 \\ \vdots & \vdots \\ x_n = q_n & x_{2n} = p_n \end{array} \tag{15.19}$$

$$\begin{array}{ll} X_1 = \dfrac{\partial H}{\partial p_1} & X_{n+1} = -\dfrac{\partial H}{\partial q_1} \\ \vdots & \vdots \\ X_n = \dfrac{\partial H}{\partial p_n} & X_{2n} = -\dfrac{\partial H}{\partial q_n} \end{array} \tag{15.20}$$

Then Hamilton's equations are in state variable form.

We can also write these equations compactly in matrix form. Let

$$x = \begin{pmatrix} q_1 \\ \vdots \\ q_n \\ p_1 \\ \vdots \\ p_n \end{pmatrix}; \qquad H_Z = \begin{pmatrix} \partial H/\partial q_1 \\ \vdots \\ \partial H/\partial q_n \\ \partial H/\partial p_1 \\ \vdots \\ \partial H/\partial p_n \end{pmatrix}$$

and define the $2n \times 2n$ matrix Z by

$$Z = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

where I and 0 are the $n \times n$ identity and zero matrices, respectively. Then Eqns. (15.11) are

$$\dot{x} = ZH_Z \tag{15.21}$$

15.3 Examples

Example. Use of Hamilton's equations is often a convenient method for solving specific problems, and we first illustrate this use by obtaining the

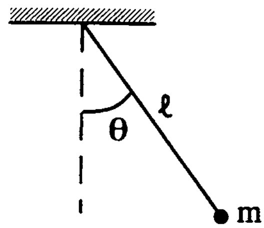

Fig. 15-1

equation of motion of the simple pendulum (Fig. 15-1). For this system,

$$\begin{aligned} T &= \frac{1}{2}m\ell^2\dot{\theta}^2\,, \quad V = mg\ell(1-\cos\theta) \\ L &= \frac{1}{2}m\ell^2\dot{\theta}^2 - mg\ell(1-\cos\theta) \\ p_\theta &= \frac{\partial L}{\partial \dot{\theta}} = m\ell^2\dot{\theta} \\ H &= p_\theta\dot{\theta} - L = \frac{1}{2}m\ell^2\dot{\theta}^2 + mg\ell(1-\cos\theta) \\ &= \frac{1}{2}\frac{p_\theta^2}{m\ell^2} + mg\ell(1-\cos\theta) \end{aligned}$$

Applying Eqns. (15.11),

$$\begin{aligned} \dot{\theta} &= \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{m\ell^2} \Longrightarrow \dot{p}_\theta = m\ell^2\ddot{\theta} \\ \dot{p}_\theta &= -\frac{\partial H}{\partial \theta} = -mg\ell\sin\theta \end{aligned}$$

Thus,

$$\ddot{\theta} + \frac{g}{\ell}\sin\theta = 0$$

Example.[3] We next consider a more substantial example. Figure 15-2 shows two bodies of equal mass connected by a rigid, massless tether. The system is traveling in a planar earth orbit. Only the motion of the mass centers of the bodies is of interest. It is desired to obtain the equations of motion of the system.

The system is natural with three degrees of freedom; we choose $(r,\ \rho,\ \theta)$ as generalized coordinates. The length z is a known function

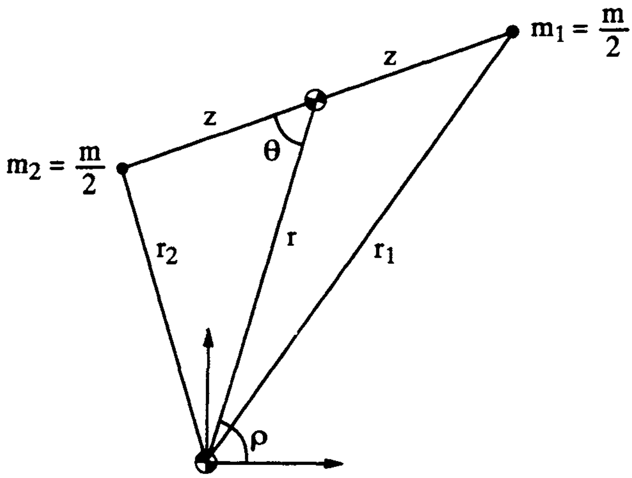

Fig. 15-2

of time. The potential and kinetic energies are (using Koenig's theorem for the latter),

$$V = -\frac{m}{2}\frac{\mu}{r_1} - \frac{m}{2}\frac{\mu}{r_2}$$

$$T = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\rho}^2\right) + \frac{m}{2}\left(\dot{z} + z^2(\dot{\rho} - \dot{\theta})^2\right)$$

where

$$r_1^2 = r^2 + z^2 + 2zr\cos\theta$$
$$r_2^2 = r^2 + z^2 - 2zr\cos\theta$$

The L function and the generalized momenta are

$$L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\rho}^2 + \dot{z}^2 + z^2(\dot{\rho} - \dot{\theta})^2\right) + \frac{m\mu}{2}\left(\frac{1}{r_1} + \frac{1}{r_2}\right)$$

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = -mz^2(\dot{\rho} - \dot{\theta})$$

$$p_\rho = \frac{\partial L}{\partial \dot{\rho}} = m\left(r^2\dot{\rho} + z^2(\dot{\rho} - \dot{\theta})\right)$$

Consequently, the Hamiltonian function is

$$\begin{aligned} H &= \sum_{r=1}^{3} p_r \dot{q}_r - L = p_r \dot{r} + p_\theta \dot{\theta} + p_\rho \dot{\rho} - L \\ &= \frac{1}{2m}\left(p_r^2 + \frac{1}{r^2}(p_\rho + p_\theta)^2 + \frac{p_\theta^2}{z^2}\right) - \frac{m}{2}\dot{z}^2 - \frac{m\mu}{2}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \end{aligned}$$

Hamilton's equations then give the equations of motion:

$$\begin{aligned} \dot{r} &= \frac{\partial H}{\partial p_r} = \frac{1}{m} p_r \\ \dot{\rho} &= \frac{\partial H}{\partial p_\rho} = \frac{1}{mr^2}(p_\rho + p_\theta) \\ \dot{\theta} &= \frac{\partial H}{\partial p_\theta} = \frac{1}{mr^2}(p_\rho + p_\theta) + \frac{1}{mz^2} p_\theta \\ \dot{p}_r &= -\frac{\partial H}{\partial r} = \frac{1}{mr^3}(p_\rho + p_\theta)^2 - \frac{m\mu}{2}\left[\frac{1}{r_1^3}(r + z\cos\theta)\right. \\ &\quad \left. + \frac{1}{r_2^3}(r - z\cos\theta)\right] \\ \dot{p}_\rho &= -\frac{\partial H}{\partial \rho} = 0 \qquad (\rho \text{ is ignorable}) \\ \dot{p}_\theta &= -\frac{\partial H}{\partial \theta} = \frac{m\mu z r \sin\theta}{2}\left(\frac{1}{r_1^3} - \frac{1}{r_2^3}\right) \end{aligned}$$

Of particular interest in applications is the "spoke equilibrium", defined by

$$r = r_0\,, \quad \rho = \omega t\,, \quad \theta = 0\,, \quad z = z_0$$

where r_0, ω and z_0 are constants. It may be shown (see Problems) that this equilibrium is possible only for a certain specific value of ω.

15.4 Stability of Hamiltonian Systems

Variational Equations. Here we write Eqns. (15.11) in vector form as

$$\dot{q} = H_p\,; \qquad \dot{p} = -H_q \tag{15.22}$$

where

$$q = \begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix} ; p = \begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix} ; \quad H_p = \begin{pmatrix} \dfrac{\partial H}{\partial p_1} \\ \vdots \\ \dfrac{\partial H}{\partial p_n} \end{pmatrix} ; H_q = \begin{pmatrix} \dfrac{\partial H}{\partial q_1} \\ \vdots \\ \dfrac{\partial H}{\partial q_n} \end{pmatrix}$$

Only the case of H not an explicit function of t will be considered.

Now suppose $q^*(t)$, $p^*(t)$ is a reference motion satisfying Hamilton's equations, and that a perturbed motion is

$$q(t) = q^*(t) + \alpha(t) ; \qquad p(t) = p^*(t) + \beta(t)$$

where $\alpha(t)$ and $\beta(t)$ are small perturbations. Substituting these into Eqns. (15.22), expanding, and retaining only first order terms,[4]

$$\begin{aligned} \dot{q}^* + \dot{\alpha} &= H_p(q^* + \alpha, p^* + \beta) = H_p(q^*, p^*) + H_{pq}(q^*, p^*)\alpha \\ &\quad + H_{pp}(q^*, p^*)\beta \end{aligned}$$

$$\begin{aligned} \dot{p}^* + \dot{\beta} &= -H_q(q^* + \alpha, p^* + \beta) = -H_q(q^*, p^*) - H_{qq}(q^*, p^*)\alpha \\ &\quad - H_{qp}(q^*, p^*)\beta \end{aligned}$$

But since q^*, p^* satisfy Eqns. (15.22),

$$\begin{aligned} \dot{\alpha} &= H^*_{pq}\alpha + H^*_{pp}\beta \\ \dot{\beta} &= -H^*_{qq}\alpha - H^*_{qp}\beta \end{aligned} \tag{15.23}$$

These equations are of Hamiltonian form with Hamiltonian

$$H' = \frac{1}{2}\alpha^T H^*_{qq}\alpha + \beta^T H^*_{pq}\alpha + \frac{1}{2}\beta^T H^*_{pp}\beta$$

Stability of Motion. Recognizing that $H^*_{qp} = H^{*T}_{pq}$, where T denotes transpose, Eqns. (15.23) may be written

$$\begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \end{pmatrix} = \begin{pmatrix} H^*_{pq} & H_{pp} \\ -H^*_{qq} & -H^{*T}_{pq} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{15.24}$$

It may be shown that the eigenvalues of such a matrix occur in positive and negative pairs. Further, since the coefficients of the matrix in Eqn.

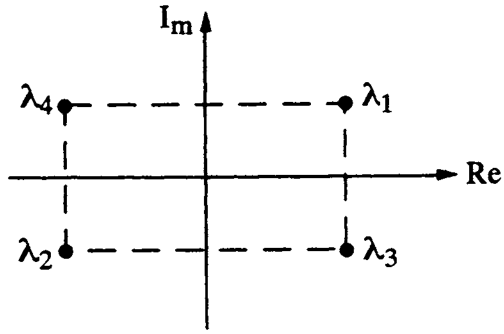

Fig. 15-3

(15.24) are real, the eigenvalues occur in complex conjugate pairs. Thus if $\lambda_1 = a + ib$ is one eigenvalue then so are $\lambda_2 = -a - ib$, $\lambda_3 = a - ib$, and $\lambda_4 = -a + ib$ (some of these may not be distinct; for example if $b = 0$ then $\lambda_1 = \lambda_3$ and $\lambda_2 = \lambda_4$). Plotted in the complex plane, these eigenvalues exhibit a "box form" (Fig. 15-3). Since eigenvalues with positive real parts and those with negative real parts denote unstable and stable modes, respectively, speaking loosely we may say that the system is "half stable and half unstable", unless, of course, some eigenvalues have zero real parts. This stability property may cause numerical problems when integrating Hamilton's equations.

15.5 Poisson Brackets

Definitions. Recall from Section 8.1 that an *integral of the motion* is a function that remains constant along a solution in state-time space (x, t), where we now take the state x as $x = (q_r, p_r)$:

$$F(x, t) = \text{constant}. \tag{15.25}$$

Taking the total derivative of this:

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \sum_{r=1}^{2n} \frac{\partial F}{\partial x_r}\dot{x}_r = 0 \tag{15.26}$$

$$\frac{\partial F}{\partial t} + \sum_{r=1}^{n} \left(\frac{\partial F}{\partial q_r}\frac{\partial H}{\partial p_r} - \frac{\partial F}{\partial p_r}\frac{\partial H}{\partial q_r} \right) = 0 \tag{15.27}$$

where Eqns. (15.11) were used.

Define the *Poisson bracket* of F and H by

$$(F, H) = \sum_{r=1}^{n} \left(\frac{\partial F}{\partial q_r} \frac{\partial H}{\partial p_r} - \frac{\partial F}{\partial p_r} \frac{\partial H}{\partial q_r} \right) = \sum_{r=1}^{n} \frac{\partial(F, H)}{\partial(q_r, p_r)} \tag{15.28}$$

Thus Eqn. (15.27) may be written as

$$\frac{\partial F}{\partial t} + (F, H) = 0 \tag{15.29}$$

This equation is satisfied by any function F that is an integral of the motion. Consider (x_r, H); by definition:

$$(x_r, H) = \sum_{s} \left(\frac{\partial x_r}{\partial q_s} \frac{\partial H}{\partial p_s} - \frac{\partial x_r}{\partial p_s} \frac{\partial H}{\partial q_s} \right)$$

Since $x_r = q_r$ for $r = 1, \cdot\cdot, n$ and $x_r = p_r$ for $r = n + 1, \cdot\cdot, 2n$, and the q_r and p_r are independent, this reduces to

$$(x_r, H) = \frac{\partial H}{\partial p_r} = \dot{q}_r \; ; \quad r = 1, \cdot\cdot, n$$

$$(x_r, H) = -\frac{\partial H}{\partial q_{r-n}} = \dot{p}_{r-n} \; ; \quad r = n + 1, \cdot\cdot, 2n$$

where Eqns. (15.11) were used. Consequently Hamilton's equations in terms of Poisson's brackets are

$$\dot{x}_r = (x_r, H) \; ; \quad r = 1, \cdot\cdot, 2n \tag{15.30}$$

Properties. Let u, v, and w be class C^2 functions of (q_r, p_r, t) and let c be a constant. Then the following properties of the Poisson brackets follow directly from Eqn. (15.28):

(i) $(u, u) = (u, c) = (c, u) = 0$

(ii) $(v, u) = (-u, v) = (u, -v) = -(u, v)$

(iii) $\dfrac{\partial}{\partial t}(u, v) = \left(\dfrac{\partial u}{\partial t}, v \right) + \left(u, \dfrac{\partial v}{\partial t} \right)$

(iv) $(u, (v, w)) + (v, (w, u)) + (w, (u, v)) = 0$

Poisson's Theorem. If ϕ and ψ are functions of class C^2 and are integrals of Hamilton's equations then (ϕ, ψ) is also an integral.

This result provides a means of constructing a new integral of the motion if at least two are already known. This new integral, however, may or may not be independent of the two used to generate it. It is obvious that independent integrals cannot be constructed indefinitely by this method, because only $2n$ such integrals exist. Furthermore, sometimes the new integral produced is identically zero.

We now prove the theorem. Since ϕ and ψ are integrals, Eqn. (15.29) gives

$$\frac{\partial \phi}{\partial t} + (\phi, H) = 0\,, \qquad \frac{\partial \psi}{\partial t} + (\psi, H) = 0$$

We need to show that

$$\frac{\partial}{\partial t}(\phi, \psi) + ((\phi, \psi), H) = 0$$

Using the properties of the Poisson bracket stated above and Eqn. (15.29),

$$\begin{aligned} \frac{\partial}{\partial t}(\phi, \psi) + ((\phi, \psi), H) &= \left(\frac{\partial \phi}{\partial t}, \psi\right) + \left(\phi, \frac{\partial \psi}{\partial t}\right) + ((\phi, \psi), H) \\ &= -((\phi, H), \psi) - (\phi, (\psi, H)) + ((\phi, \psi), H) \\ &= (\psi, (\phi, H)) + (\phi, (H, \psi)) + (H, (\psi, \phi)) = 0 \end{aligned}$$

An important special case is when the system is natural. Then $H \neq H(t)$ and, in view of property (i), $F = H$ satisfies Eqn. (15.29) and consequently H is a constant of the motion, which of course we already knew.

Example – Central Force Motion. Consider a particle of unit mass subject to a conservative central force (Fig. 15-4). For this system,

$$r^2 = x^2 + y^2 + z^2$$

$$T = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

$$H = \sum_{i=1}^{3} p_i \dot{q}_i - L = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) + V(r)$$

It is known that in this system angular momentum is conserved but linear momentum is not. We show this by using Poisson's brackets. The

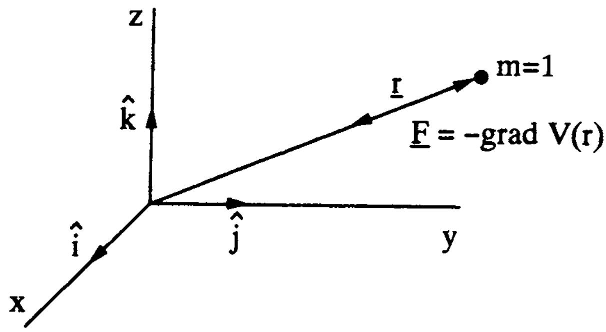

Fig. 15-4

angular momentum and its z component are

$$\underline{\ell} = \underline{r} \times \underline{v} = (x\hat{i} + y\hat{j} + z\hat{k}) \times (\dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k})$$

$$\ell_z = \underline{\ell} \cdot \hat{k} = x\dot{y} - y\dot{x} = xp_y - yp_x$$

This is an integral of the motion because

$$\begin{aligned}\frac{\partial \ell_z}{\partial t} + (\ell_z, H) &= 0 + \sum_{r=1}^{3} \left(\frac{\partial \ell_z}{\partial q_r}\frac{\partial H}{\partial p_r} - \frac{\partial \ell_z}{\partial p_r}\frac{\partial H}{\partial q_r} \right) \\ &= p_y p_x + y\frac{\partial V}{\partial x} - p_x p_y - x\frac{\partial V}{\partial y} = y\frac{\partial V}{\partial x} - x\frac{\partial V}{\partial y} \\ &= y\frac{x}{r}\frac{dV}{dr} - x\frac{y}{r}\frac{dV}{dr} = 0\end{aligned}$$

where the properties of the Poisson brackets were used and where

$$\frac{\partial V}{\partial x} = \frac{\partial V}{\partial r}\frac{\partial r}{\partial x} = \frac{x}{r}\frac{dV}{dr}, \qquad \frac{\partial V}{\partial y} = \frac{\partial V}{\partial r}\frac{\partial r}{\partial y} = \frac{y}{r}\frac{dV}{dr}$$

We can show that, similarly, ℓ_y, the y component of $\underline{\ell}$, is an integral of the motion. Next Poisson's theorem is used to get a third constant of the motion:

$$(\ell_z, \ell_y) = \sum_{r=1}^{3} \left(\frac{\partial \ell_z}{\partial q_r}\frac{\partial \ell_y}{\partial p_r} - \frac{\partial \ell_z}{\partial p_r}\frac{\partial \ell_y}{\partial q_r} \right) = p_y z - y p_z = \ell_x$$

In fact, taking the Poisson bracket of any two components of $\underline{\ell}$ gives the third.

Finally we show that the components of the linear momentum are not integrals of the motion. The linear momentum is $\underline{h} = \underline{v}$ and its z component is

$$h_z = \underline{v} \cdot \hat{k} = \dot{z} = p_z$$

Thus

$$\frac{\partial h_z}{\partial t} + (h_z, H) = \sum_{r=1}^{3} \left(\frac{\partial h_z}{\partial q_r} \frac{\partial H}{\partial p_r} - \frac{\partial h_z}{\partial p_r} \frac{\partial H}{\partial q_r} \right) = -\frac{z}{r} \frac{dV}{dr} \neq 0$$

and similarly for the other two components.

15.6 Reduction of System Order

Use of the Energy Integral. For a natural system, the Hamiltonian does not depend on time explicitly and $H = T + V$ is an integral of the motion:

$$H(q_1, \cdot\cdot, q_n, p_1, \cdot\cdot, p_n) = h = \text{constant.} \tag{15.31}$$

Suppose that we solve this equation for one of the q_r, say q_1:

$$q_1 = \phi(q_2, \cdot\cdot, q_n, p_1, \cdot\cdot, p_n, h) \tag{15.32}$$

Now substitute this into Eqn. (15.31) and take the derivatives with respect to the last $n-1$ of the p_r:

$$\frac{\partial H}{\partial p_r} + \frac{\partial H}{\partial q_1} \frac{\partial \phi}{\partial p_r} = 0 \, ; \qquad r = 2, \cdot\cdot, n \tag{15.33}$$

Therefore, using Eqns. (15.11),

$$\frac{dq_r}{dp_1} = \frac{\dot{q}_r}{\dot{p}_1} = -\frac{\frac{\partial H}{\partial p_r}}{\frac{\partial H}{\partial q_1}} = \frac{\partial \phi}{\partial p_r} \, ; \qquad r = 2, \cdot\cdot, n \tag{15.34}$$

Similarly, substituting Eqn. (15.32) into Eqn. (15.31) and differentiating with respect to the last $n-1$ of the q_r:

$$\frac{\partial H}{\partial q_r} + \frac{\partial H}{\partial q_1} \frac{\partial \phi}{\partial q_r} = 0 \tag{15.35}$$

which gives

$$\frac{dp_r}{dp_1} = \frac{\dot{p}_r}{\dot{p}_1} = \frac{\frac{\partial H}{\partial q_r}}{\frac{\partial H}{\partial q_1}} = -\frac{\partial \phi}{\partial q_r}\,; \qquad r = 2, \cdot\cdot, n \tag{15.36}$$

Equations (15.34) and (15.36) are a new Hamiltonian system with p_1 the independent variable (taking the role formerly played by t) and with ϕ the Hamiltonian function (taking the role formerly played by H). Note that the new Hamiltonian system is of order $2(n-1)$ and that the new Hamiltonian system is nonautonomous because ϕ is a function of independent variable p_1.

Example. Consider a system for which the Hamiltonian is

$$H = \frac{1}{2}(p_x^2 + p_y^2) - kyp_x + \left(\frac{1}{2}k^2y^2 - gx\right)$$

where x and y are generalized coordinates and k and g are positive constants. Solving $H = h$ for x we have

$$x = \phi(y, p_x, p_y, h) = \frac{1}{2g}\left[(p_x - ky)^2 + p_y^2\right] - \frac{h}{g}$$

Using ϕ as the new Hamiltonian, p_x as the independent variable, and y and p_y as the remaining dependent variables, Eqns. (15.34) and (15.36) give

$$\frac{dy}{dp_x} = \frac{\partial \phi}{\partial p_y} = \frac{p_y}{g}$$

$$\frac{dp_y}{dp_x} = -\frac{\partial \phi}{\partial y} = \frac{k}{g}(p_x - ky)$$

The solution of these equations is obtained as follows:

$$\frac{d^2p_y}{dp_x^2} = \frac{k}{g} - \frac{k^2}{g}\frac{dy}{dp_x} = \frac{k}{g} - \frac{k^2}{g^2}p_y$$

$$p_y = \frac{g}{k} + A\cos\frac{k}{g}p_x + B\sin\frac{k}{g}p_x$$

$$y = \frac{1}{g}\int\left(\frac{g}{k} + A\cos\frac{k}{g}p_x + B\sin\frac{k}{g}p_x\right)dp_x + C$$

$$y = \frac{p_x}{k} + \frac{A}{k}\sin\frac{k}{g}p_x - \frac{B}{k}\cos\frac{k}{g}p_x + C$$

The variable p_x is obtained from Eqns. (15.11) as

$$\dot{p}_x = -\frac{\partial H}{\partial x} = g$$

$$p_x = gt + D$$

We now have all the equations necessary to express the solution of the problem as

$$x = x(t; A, B, C, D) \; ; \qquad y = y(t; A, B, C, D)$$

Use of a Momentum Integral. When there is an ignorable coordinate, the corresponding momentum integral may be used in the same way as the energy integral to reduce the order of a Hamiltonian system.[5]

Theorem of the Last Multiplier. If we have found $2n-2$ integrals, this theorem tells us how to find the last two. Consider a system of first order differential equations:

$$\frac{dx_r}{dt} = X_r(x_r) \; ; \qquad r = 1, \cdot\cdot, m \tag{15.37}$$

or equivalently

$$\frac{dx_1}{X_1} = \frac{dx_2}{X_2} = \cdot\cdot = \frac{dx_m}{X_m} \tag{15.38}$$

Suppose we have found $(m-2)$ independent integrals of the motion given by

$$f_r(x_r, t) = c_r \; ; \qquad r = 1, \cdot\cdot, m-2 \tag{15.39}$$

Then to complete the solution for the trajectories we need only to integrate

$$\frac{dx_{m-1}}{X'_{m-1}} = \frac{dx_m}{X'_m}$$

which is equivalent to

$$X'_m dx_{m-1} - X'_{m-1} dx_m = 0 \tag{15.40}$$

where $X'_{m-1}(x_{m-1}, x_m) = X_{m-1}(c_1, \cdot\cdot, c_{m-2}, x_{m-1}, x_m)$ and $X'_m(x_{m-1}, x_m) = X_m(c_1, \cdot\cdot, c_{m-2}, x_{m-1}, x_m)$. *Jacobi's theorem of the last multiplier* (TLM)[6] then states that an additional integral of motion is given by

$$f_{m-1} = \int \frac{M'}{K'}(X'_m dx_{m-1} - X'_{m-1} dx_m) = c_{m-1} \tag{15.41}$$

where

$$K = \frac{\partial(f_1, \cdot\cdot, f_{m-2})}{\partial(x_1, \cdot\cdot, x_{m-2})}$$

and M is any solution of the partial differential equation

$$\frac{\partial}{\partial x_1}(MX_1) + \frac{\partial}{\partial x_2}(MX_2) + \cdot\cdot + \frac{\partial}{\partial x_m}(MX_m) = 0 \tag{15.42}$$

The functions $M(x_1, \cdot\cdot, x_m)$ satisfying this equation are called the *multipliers*[7] for the system of Eqns. (15.37).

The final integral, to obtain the time, is then obtained by one more application of the theorem. The final system equation is

$$\frac{dx_m}{X_m} = \frac{dt}{1} \tag{15.43}$$

The TLM then provides an integrating factor for

$$dx_m - X'_m dt = 0 \tag{15.44}$$

where $X'_m(x_m) = X_m(c_1, \cdot\cdot, c_{m-1}, x_m)$. Although we have considered an autonomous system, the TLM works for nonautonomous systems as well.

Application of the TLM to Hamiltonian Systems. Now consider an autonomous Hamiltonian system with two degrees of freedom. Then $m = 2n = 4$ and Eqns. (15.38) are

$$\frac{dq_1}{\partial H/\partial p_1} = \frac{dq_2}{\partial H/\partial p_2} = \frac{dp_1}{-\partial H/\partial q_1} = \frac{dp_2}{-\partial H/\partial q_2} \tag{15.45}$$

where $H = H(q_1, q_2, p_1, p_2)$. Assume that two integrals of the motion are known, the energy integral plus one other; thus

$$\begin{aligned} H(q_1, q_2, p_1, p_2) &= h \\ F(q_1, q_2, p_1, p_2) &= \alpha \end{aligned} \tag{15.46}$$

It is assumed that the Jacobian

$$J = \frac{\partial(H,F)}{\partial(p_1,p_2)} = \frac{\partial H}{\partial p_1}\frac{\partial F}{\partial p_2} - \frac{\partial H}{\partial p_2}\frac{\partial F}{\partial p_1} \tag{15.47}$$

is not zero. For this system, $M = 1$ is a multiplier and we use it. Then Eqn. (15.41) gives a third integral as

$$\int \frac{1}{J}\left(\frac{\partial H}{\partial p_2}dq_1 - \frac{\partial H}{\partial p_1}dq_2\right) = \text{constant} \tag{15.48}$$

where the coefficients are expressed in terms of q_1, q_2, h, and α.

The integral may be put into a different form. Suppose Eqns. (15.46) are solved for p_1 and p_2:

$$p_1 = f_1(q_1, q_2, h, \alpha)\ ; \qquad p_2 = f_2(q_1, q_2, h, \alpha)$$

Now differentiate Eqns. (15.46) with respect to α:

$$\begin{aligned} \frac{\partial H}{\partial p_1}\frac{\partial f_1}{\partial \alpha} + \frac{\partial H}{\partial p_2}\frac{\partial f_2}{\partial \alpha} &= 0 \\ \frac{\partial F}{\partial p_1}\frac{\partial f_1}{\partial \alpha} + \frac{\partial F}{\partial p_2}\frac{\partial f_2}{\partial \alpha} &= 1 \end{aligned} \tag{15.49}$$

Using Eqns. (15.47) and (15.49) in (15.48), we arrive at

$$\int \left(\frac{\partial f_1}{\partial \alpha}dq_1 + \frac{\partial f_2}{\partial \alpha}dq_2\right) = \text{constant} \tag{15.50}$$

Thus $f_1\, dq_1 + f_2 dq_2$ is a perfect differential and there is a function $K(q_1, q_2, h, \alpha)$ such that $dK = f_1 dq_1 + f_2 dq_2$ and the third integral, Eqn. (15.50), may be written as

$$\int \frac{\partial K}{\partial \alpha} = \text{constant} \tag{15.51}$$

Equating the expressions in Eqn. (15.45) to dt and using the same procedure, we arrive at the fourth integral,

$$\int \frac{\partial K}{\partial h} = t + \text{constant} \tag{15.52}$$

In summary, the four integrals of the motion are

$$\begin{aligned} H &= h \\ F &= \alpha \\ \int \frac{\partial K}{\partial \alpha} &= -\beta \\ \int \frac{\partial K}{\partial h} &= t + t_0 \end{aligned} \tag{15.53}$$

In many specific problems, the second integral will be a momentum integral corresponding to an ignorable coordinate. Suppose q_2 is ignorable; then two known integrals are

$$H(q_1, p_1, p_2) = h \; ; \qquad p_2 = \alpha \tag{15.54}$$

In this case, $f_2 = \alpha$ and $f_1(q_1, h, \alpha)$ is obtained by solving $H(q_1, p_1, \alpha) = h$ for p_1. Now $dK = f_1 dq_1 + \alpha dq_2$ and the last two of Eqns. (15.53) give the third and fourth integrals as

$$\begin{aligned} \int \frac{\partial f_1}{\partial \alpha} dq_1 + q_2 &= -\beta \\ \int \frac{\partial f_1}{\partial h} dq_1 &= t - t_0 \end{aligned} \tag{15.55}$$

Example. Consider a particle of unit mass in a central force field with potential energy function $V(r)$. From Eqns. (7.11) and (8.24), $p_r = \dot{r}$ and $p_\theta = r^2\dot{\theta}$ so that Eqn. (15.15) gives

$$H = \frac{1}{2}\left(p_r^2 + \frac{1}{r^2}p_\theta^2\right) + V$$

Since energy is conserved and θ is ignorable, two integrals of the motion are

$$H(r, p_r, p_\theta) = h$$

$$p_\theta = \alpha$$

Solving the first of these for p_r gives

$$p_r = f_r(r, h, \alpha) = \sqrt{2h - 2V - \frac{\alpha^2}{r^2}}$$

Equations (15.55) then give the other two integrals as

$$-\int_{r_0}^{r} \frac{(\alpha/\zeta^2)}{f_r(\zeta)} d\zeta + \theta = -\beta$$

$$\int_{r_0}^{r} \frac{d\zeta}{f_r(\zeta)} = t - t_0$$

which are the same equations as were obtained in Chapter 10.

Notes

1 Recall from Section 3.3 that in the δ operation t is not varied.

2 Pars gives explicit forms of H for types of systems other than natural.

3 Anderson, K.S., and Hagedorn, P., "Control of Orbital Drift of Geostationary Tethered Satellites", *Journal of Guidance, Control, and Dynamics*, Vol. 17, No. 1, Jan–Feb 1994.

4 Subscripts here will denote partial derivatives; for example, H_{pq} is the matrix $\|\partial^2 H/\partial p_r \partial q_r\|$.

5 See Pars for the details.

6 See Pars or Whittaker for the proof.

7 Not to be confused with Lagrange multipliers.

PROBLEMS

Obtain the equations of motion using Hamilton's equations for the systems described in the following six problems:

15/1. Problem 4/2.

15/2. Problem 6/7.

15/3. Problem 6/10.

15/4. Problem 7/1, with $X_0 = Y_0 = Z_0 = 0$.

15/5. Problem 7/3.

15/6. Problem 10/1.

15/7. Prove properties (i) – (iii) of Poisson's brackets.

15/8. Prove property (iv) of Poisson's brackets.