Chapter 14

Gibbs-Appell Equations

14.1 Quasi-Coordinates

Introduction. Nonholonomic constraints are accounted for in Lagrange's equations by the use of Lagrange multipliers. We now develop an approach to systems with nonholonomic constraints that does not depend on multipliers – the use of quasi-coordinates and the Gibbs-Appell equations.

Quasi-coordinates are analogous to nonholonomic constraints in that they are defined by differential relations that are not integrable. Thus the requirement that the displacement components of the particles are explicit functions of the generalized coordinates is relaxed (see Eqns. (5.7)), and we consider coordinates such that the velocity components are explicit, linear, nonintegrable functions of the time derivatives of the generalized coordinates.

In this and the next four chapters, we will loosely follow Pars. Therefore, we make a notation change to bring our notation into line with that of Pars. The displacement components will now be denoted by x_r, that is

$$
\begin{aligned}
x_1 &= u_1 = x_1^1 \\
x_2 &= u_2 = x_2^1 \\
x_3 &= u_3 = x_3^1 \\
x_4 &= u_4 = x_1^2 \\
&\vdots \\
x_N &= u_N = x_3^{N/3}
\end{aligned}
$$

where $\underline{x}^r = (x_1^r, x_2^r, x_3^r)$; $r = 1, \cdot\cdot, \nu$ are the position vectors of the particles. Now, ν is the number of particles and $N = 3\nu$ is the number of displacement components; as before, n will be the number of generalized coordinates, L will be the total number of constraints, ℓ will be the number of nonholonomic constraints, and $k = N - L$ will be the degrees of freedom of the dynamic system. As usual, $\sum_s$ will denote $\sum_{s=1}^{n}$.

Quasi-Coordinates. Consider a dynamic system with ν particles, ℓ nonholonomic constraints, and no holonomic ones. Let a set of generalized coordinates be

$$q_r \; ; \; r = 1, \cdot\cdot, k + \ell = n = 3\nu = N$$

The ℓ constraints are

$$\sum_s B_{rs} dq_s + B_r dt = 0 \; ; \quad r = 1, \cdot\cdot, \ell \tag{14.1}$$

Introduce p new coordinates θ_r, called *quasi coordinates*, such that

$$d\theta_r = \sum_s C_{rs} dq_s + C_r dt \; ; \quad r = 1, \cdot\cdot, p \tag{14.2}$$

where C_{rs}, $C_r \in C^1(q, t)$. The total number of coordinates is now $k+\ell+p$ where

$$q_{k+\ell+r} = \theta_r \; ; \quad r = 1, \cdot\cdot, p$$

Relabel this new set of coordinates $q_1, \cdot\cdot, q_{k+\ell+p}$ so that Eqn. (14.2) may be rewritten as

$$dq_{k+\ell+r} = \sum_s C_{rs} dq_s + C_r dt \; ; \quad r = 1, \cdot\cdot, p \tag{14.3}$$

We now require that the matrix $\begin{bmatrix} B_{rs} \\ C_{rs} \end{bmatrix}$ have maximum rank. (Since B_{rs} is $(k + \ell) \times \ell$ and C_{rs} is $(k + \ell) \times p$, the matrix is $(k + \ell) \times (\ell + p)$.) Under this condition the implicit function theorem guarantees that we may solve for $\ell + p$ of the dq_r as functions of the remaining k dq_r. Call the remaining ones ρ_s. Then

$$dq_r = \sum_{s=1}^{k} D_{rs} d\rho_s + D_r dt \; ; \quad r = 1, \cdot\cdot, \ell + p \tag{14.4}$$

Equation (14.4) is equivalent to Eqns. (14.1) and (14.3). In general, the coefficients D_{rs} and D_r will be functions of all $k + \ell$ generalized coordinates q_r.

The displacement components in terms of the generalized coordinates are, as usual, given by Eqns. (5.9):

$$dx_r = \sum_{s=1}^{k+\ell} \frac{\partial x_r}{\partial q_s} dq_s + \frac{\partial x_r}{\partial t} dt \; ; \quad r = 1, \cdot\cdot, N \tag{14.5}$$

Next we relabel $\rho_s = q_s$; $r = 1, \cdot\cdot, k$ and use Eqn. (14.4) to eliminate the superfluous coordinates in Eqn. (14.5):

$$dx_r = \sum_{s=1}^{k} \alpha_{rs} dq_s + \alpha_r dt \; ; \quad r = 1, \cdot\cdot, N \tag{14.6}$$

Equation (14.4) becomes

$$dq_r = \sum_{s=1}^{k} \beta_{rs} dq_s + \beta_r dt \; ; \quad r = 1, \cdot\cdot, n = k + \ell \tag{14.7}$$

Comparing Eqns. (14.5) and (14.6), we see that we have reduced the number of dq_r upon which the dx_r depend to k, the degrees of freedom. Thus the system now *appears to be holonomic* because it takes k coordinates to specify the system, and no multipliers will be needed.

From Eqns. (14.6) and (14.7), virtual displacements satisfy

$$\delta x_r = \sum_{s=1}^{k} \alpha_{rs} \delta q_s \; ; \quad r = 1, \cdot\cdot, N \tag{14.8}$$

$$\delta q_r = \sum_{s=1}^{k} \beta_{rs} \delta q_s \; ; \quad r = 1, \cdot\cdot, n \tag{14.9}$$

Example. First consider a particle moving in a plane (Fig. 14-1). Two possible choices of generalized coordinates are rectangular, (x, y), and polar, (r, θ). A possible quasi-coordinate is q, defined by

$$dq = x dy - y dx$$

It is easy to show that this is nonintegrable (see Section 2.6). We have

$$\dot{q} = x\dot{y} - y\dot{x}$$

$$q = \int_{t_0}^{t} (x\dot{y} - y\dot{x}) dt$$

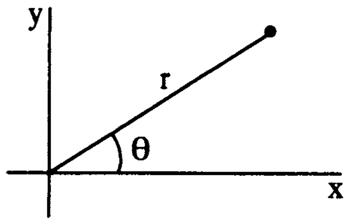

Fig. 14-1

so that q is twice the area swept out in time $(t - t_0)$ by the position vector. We will come back to this problem later.

As a second example, the total rotation about a given line of a rigid body is often a convenient quasi coordinate. For example, from Eqn. (11.8) the total rotation about the axis of a spinning top is q, where

$$dq = d\psi + \cos\theta d\phi$$

where ψ, θ, and ϕ are the spin, nutation, and precession angles, respectively.

14.2 Fundamental Equation

Fundamental Equation with Quasi-Coordinates. Recall the three forms of the fundamental equation established in Chapter 3, namely Eqns. (3.7), (3.38) and (3.39), repeated here in the new notation:

$$\sum_{r=1}^{N}(m_r\ddot{x}_r - F_r)\delta x_r = 0 \tag{14.10}$$

$$\sum_{r=1}^{N}(m_r\ddot{x}_r - F_r)\Delta\dot{x}_r = 0 \tag{14.11}$$

$$\sum_{r=1}^{N}(m_r\ddot{x}_r - F_r)\Delta\ddot{x}_r = 0 \tag{14.12}$$

We want to obtain a fundamental equation similar to Eqn. (14.12) in our new coordinates.

First, the virtual work is obtained using Eqn. (14.8):

$$\sum_{r=1}^{N} F_r\delta x_r = \sum_{r=1}^{N} F_r \sum_{s=1}^{k} \alpha_{rs}\delta q_s = \sum_{s=1}^{k} Q_s\delta q_s$$

so that

$$Q_s = \sum_{r=1}^{N} F_r \alpha_{rs} \tag{14.13}$$

From Eqn. (14.6),

$$\dot{x}_r = \sum_{s=1}^{k} \alpha_{rs}\dot{q}_s + \alpha_r \ ; \quad r = 1, \cdot\cdot, N$$

$$\ddot{x}_r = \sum_{s=1}^{k} \alpha_{rs}\ddot{q}_s + \text{terms without the } \ddot{q}_s \ ; \quad r = 1, \cdot\cdot, N \tag{14.14}$$

Consider another possible acceleration $\ddot{x}_r + \Delta\ddot{x}_r$; then

$$\ddot{x}_r + \Delta\ddot{x}_r = \sum_{s=1}^{k} \alpha_{rs}(\ddot{q}_s + \Delta\ddot{q}_s) + \text{terms without the } \ddot{q}_s \ ; \quad r = 1, \cdot\cdot, N$$

Thus

$$\Delta\ddot{x}_r = \sum_{s=1}^{k} \alpha_{rs}\Delta\ddot{q}_s \ ; \quad r = 1, \cdot\cdot, N \tag{14.15}$$

Substitute Eqn. (14.15) into (14.12) and use (14.13):

$$\sum_{r=1}^{N} m_r\ddot{x}_r\Delta\ddot{x}_r - \sum_{r=1}^{N} F_r \sum_{s=1}^{k} \alpha_{rs}\Delta\ddot{q}_s = 0$$

$$\sum_{r=1}^{N} m_r\ddot{x}_r\Delta\ddot{x}_r - \sum_{s=1}^{k} Q_s\Delta\ddot{q}_s = 0 \tag{14.16}$$

which is what we wanted to derive. Note that this involves a mixture of rectangular and generalized coordinates.

14.3 Gibbs' Theorem and the Gibbs-Appell Equations

Gibbs' Function. Define the acceleration, or Gibbs, function by

$$G = \frac{1}{2}\sum_{r=1}^{N} m_r\ddot{x}_r^2 \tag{14.17}$$

Note that this is similar to the definition of kinetic energy, except that accelerations are used instead of velocities. Substituting Eqn. (14.14) into (14.17) gives a function of the form

$$G = G_2 + G_1 + G_0$$

where G_2 is quadratic in the $\ddot{q}_r$, G_1 is linear in the $\ddot{q}_r$, and G_0 does not contain the $\ddot{q}_r$. We now state the following.

Gibb's Theorem. Given the displacements and velocities at some time t, the accelerations at that time are such that

$$G - \sum_{s=1}^{k} Q_s \ddot{q}_s$$

is a minimum with respect to the $\ddot{q}_r$.

The proof is as follows. Let $\ddot{q}_s$ be the actual accelerations and let $\ddot{q}_s + \Delta\ddot{q}_s$ be possible ones. Form the change in the function just above and use Eqn. (14.16):

$$\begin{aligned}
\Delta\left(G - \sum_{s=1}^{k} Q_s \ddot{q}_s\right) &= \frac{1}{2}\sum_{r=1}^{N} m_r(\ddot{x}_r + \Delta\ddot{x}_r)^2 - \sum_{s=1}^{k} Q_s(\ddot{q}_s + \Delta\ddot{q}_s) \\
&\quad -\frac{1}{2}\sum_{r=1}^{N} m_r \ddot{x}_r^2 + \sum_{s=1}^{k} Q_s \ddot{q}_s \\
&= \frac{1}{2}\sum_{r=1}^{N} m_r(\Delta\ddot{x}_r)^2 + \left(\sum_{r=1}^{N} m_r \ddot{x}_r \Delta\ddot{x}_r - \sum_{s=1}^{k} Q_s \Delta\ddot{q}_s\right) \\
&= \frac{1}{2}\sum_{r=1}^{N} m_r(\Delta\ddot{x}_r)^2 > 0
\end{aligned} \tag{14.18}$$

which proves the theorem.

Gibbs-Appell Equations. These equations are the first order necessary conditions associated with Gibbs' Theorem, namely,

$$Q_s = \frac{\partial G}{\partial \ddot{q}_s}\ ; \quad s = 1, \cdot\cdot, k \tag{14.19}$$

Also to be satisfied are the constraint equations, obtained from Eqns. (14.7):

$$\dot{q}_r = \sum_{s=1}^{k} \beta_{rs}\dot{q}_s + \beta_r\ ; \quad r = k+1, \cdot\cdot, n \tag{14.20}$$

Equations (14.19) and (14.20) serve to determine the equations of motion of a dynamic system.

Remarks.

1. The q_r are in general a mixture of generalized coordinates and quasi-coordinates.
2. Equations (14.19) were first discovered by Gibbs but attracted little attention. They were later discovered independently by Appell who first realized their full importance.
3. The Gibbs-Appell equations are equivalent to Kane's equations (see Baruh and Kane and Levinson)

Solution Procedure. To solve problems using Eqns. (14.19), the following steps are required.

1. Determine $k = N - \ell$, the degrees of freedom of the system.
2. Obtain G by expressing the $\ddot{x}_r^2$ in terms of k of the $\ddot{q}_r$ (see Eqn. (14.17)). Note that generally all the q_r and $\dot{q}_r$ will appear in G, but only k of the $\ddot{q}_r$. The k preferred q_r may be either generalized or quasi-coordinates.
3. Consider the work done in a virtual displacement to get $\sum_{s=1}^{k} Q_s \delta q_s$ and hence the Q_s.
4. Form the equations of motion from Eqns. (14.19) and (14.20).

14.4 Applications

Particle in a Plane. Let a particle in a plane be subjected to a force with radial and transverse components R and S, respectively, as shown on Fig. 14-2. As mentioned previously, either (x, y) or (r, θ) serve as generalized coordinates. We pick coordinates (r, q) defined by

$$r^2 = x^2 + y^2$$

$$dq = x dy - y dx$$

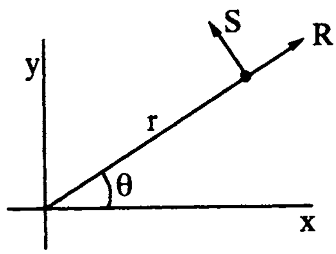

Fig. 14-2

Coordinates r and q are generalized and quasi-coordinates, respectively. To form the Gibbs function G from Eqn. (14.17), $\ddot{x}$ and $\ddot{y}$ are needed:

$$r\dot{r} = x\dot{x} + y\dot{y}$$
$$\dot{r}^2 + r\ddot{r} = \dot{x}^2 + x\ddot{x} + \dot{y}^2 + y\ddot{y}$$

$$\dot{q} = x\dot{y} - y\dot{x}$$
$$\ddot{q} = \dot{x}\dot{y} + x\ddot{y} - \dot{y}\dot{x} - y\ddot{x} = x\ddot{y} - y\ddot{x}$$

These expressions are to be solved for $\ddot{x} = \ddot{x}(\ddot{r}, \dot{r}, r, \ddot{q}, \dot{q}, q)$ and $\ddot{y} = \ddot{y}(\ddot{r}, \dot{r}, r, \ddot{q}, \dot{q}, q)$ and substituted into

$$G = \frac{1}{2}m(\ddot{x}^2 + \ddot{y}^2)$$

The result is

$$G = \frac{1}{2}m\left(\ddot{r}^2 - \frac{2}{r^3}\dot{q}^2\ddot{r} + \frac{1}{r^2}\ddot{q}^2\right)$$

where all terms not having the factors $\ddot{r}$ or $\ddot{q}$ have been omitted because in view of Eqn. (14.19) they do not enter into the equations of motion.

The generalized forces are obtained by considering the virtual work done by R and S. We have

$$\begin{aligned} \dot{q} &= x\dot{y} - y\dot{x} \\ &= (r\cos\theta)(\dot{r}\sin\theta + r\dot{\theta}\cos\theta) \\ -(r\sin\theta)(\dot{r}\cos\theta - r\dot{\theta}\sin\theta) &= r^2\dot{\theta} \end{aligned}$$

Thus

$$dq = r^2 d\theta$$

and

$$\delta q = r^2 \delta\theta$$

and therefore

$$\delta W = R\delta r + Sr\delta\theta = R\delta r + Sr\frac{\delta q}{r^2}$$

Consequently,

$$Q_r = R\,, \qquad Q_q = \frac{S}{r}$$

Now we apply the Gibbs-Appell equations, Eqns. (14.19),

$$\frac{\partial G}{\partial \ddot{r}} = Q_r\,, \qquad \frac{\partial G}{\partial \ddot{q}} = Q_q$$

to obtain

$$m\left(\ddot{r} - \frac{\dot{q}^2}{r^3}\right) = R\,, \qquad \frac{m\ddot{q}}{r^2} = \frac{S}{r}$$

Consider the special case of central force motion with conservative force; in this case

$$S = 0, \quad R = -m\frac{dV}{dr}$$

and

$$\dot{q} = \alpha = \text{constant}, \qquad m\left(\ddot{r} - \frac{\alpha^2}{r^3}\right) = -m\frac{dV}{dr}$$

Using the identity of Eqn. (9.14), the second of these integrates to

$$\dot{r}^2 + 2V + \frac{\alpha^2}{r^2} = 2h = \text{constant}$$

which is the energy integral.

Analogue of Koenig's Theorem. Let G be the center of mass of a rigid body and fix an axis system at G that does not rotate relative to an inertial frame (but the body may rotate) as shown on Fig. 14-3. Recall that Koenig's theorem states that the kinetic energy of the body is given by Eqn. (1.58). Since the Gibbs function is analogous to the kinetic energy, with accelerations replacing velocities, we have immediately, for the same situation,

$$G = \frac{1}{2}M(\ddot{\bar{x}}^2 + \ddot{\bar{y}}^2 + \ddot{\bar{z}}^2) + \frac{1}{2}\sum m_r(\ddot{\zeta}_r^2 + \ddot{\eta}_r^2 + \ddot{\nu}_r^2) \tag{14.21}$$

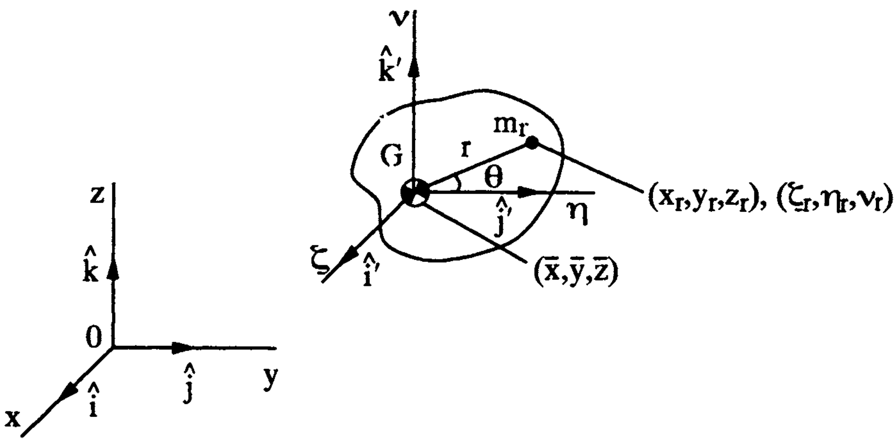

Fig. 14-3

where $M = \sum m_r$ is the mass of the rigid body.

Two-Dimensional Problems. Let a rigid body move in a plane (Fig. 14-4). We note that r is a constant for each particle, but θ varies with time. Consequently,

$$\begin{aligned} \zeta &= r\cos\theta \\ \dot{\zeta} &= -r\sin\theta\dot{\theta} \\ \ddot{\zeta} &= -r\cos\theta\dot{\theta}^2 - r\sin\theta\ddot{\theta} \end{aligned}$$

$$\begin{aligned} \eta &= r\sin\theta \\ \dot{\eta} &= r\cos\theta\dot{\theta} \\ \ddot{\eta} &= -r\sin\theta\dot{\theta}^2 + r\cos\theta\ddot{\theta} \end{aligned}$$

so that

$$\ddot{\zeta}^2 + \ddot{\eta}^2 = r^2\ddot{\theta}^2 + r^2\dot{\theta}^4$$

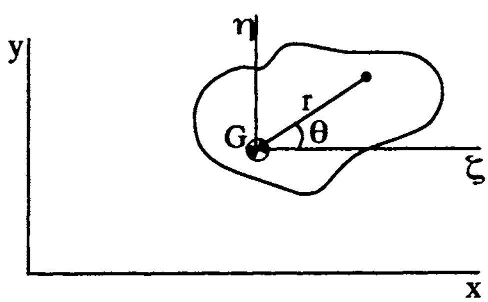

Fig. 14-4

Substituting into Eqn. (14.21),

$$G = \frac{1}{2}Mf^2 + \frac{1}{2}\bar{I}\ddot{\theta}^2 \tag{14.22}$$

where $f^2 = \ddot{\bar{x}}^2 + \ddot{\bar{y}}^2$ is the acceleration of G squared, $\bar{I} = \sum m_r r^2$ is the moment of inertia relative to an axis passing through G and perpendicular to the plane of the motion, and the $r^2\dot{\theta}^4$ term has been omitted because it does not contain any acceleration factors.

Cylinder Rolling in a Cylinder. We first get the rolling without slipping condition (Fig. 14-5) by noting that A' is at A when $\theta = 0$. Letting $c = b - a$,

$$\begin{aligned} AB &= A'B \\ b\theta &= a(\theta + \phi) \\ a\phi &= c\theta \\ a\dot{\phi} &= c\dot{\theta} \\ a\ddot{\phi} &= c\ddot{\theta} \end{aligned}$$

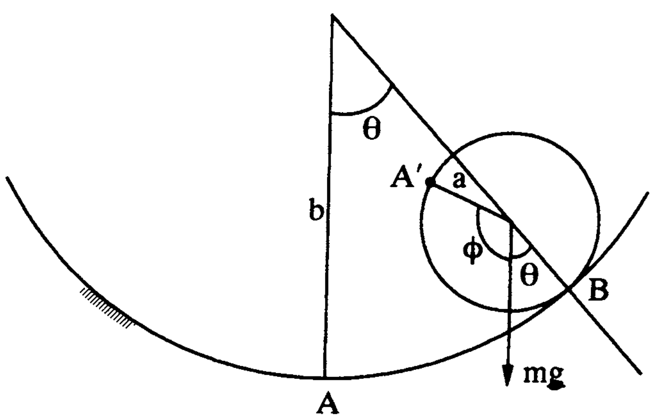

Fig. 14-5

From Eqn. (14.22),

$$G = \frac{1}{2}M(c^2\ddot{\theta}^2 + c^2\dot{\theta}^4) + \frac{1}{2}\left(\frac{1}{2}Ma^2\right)\ddot{\phi}^2$$

Because $k = 1$, we must write this in terms of only one acceleration component; the rolling constraint is used to do this:

$$G = \frac{1}{2}M(c^2\ddot{\theta}^2 + c^2\dot{\theta}^4) + \frac{1}{4}Ma^2\left(\frac{c}{a}\ddot{\theta}\right)^2$$

$$G = \frac{3}{4}Mc^2\ddot{\theta}^2 + \text{terms without } \ddot{\theta}$$

Since the contact force does no virtual work, the only given force doing virtual work is gravity:

$$\delta W = Mg\delta(c\cos\theta) = -Mgc\sin\theta\delta\theta$$

so that $Q_\theta = -Mgc\sin\theta$. Equation (14.19) then gives the equation of motion:

$$-Mgc\sin\theta = \frac{3}{2}Mc^2\ddot{\theta}$$

$$\ddot{\theta} + \frac{2}{3}\frac{g}{c}\sin\theta = 0$$

which is a form of the equation of a simple pendulum.

Sphere Rolling on a Rotating Plane. In the preceeding two examples, the systems were holonomic and the equations of motion could have been obtained by more elementary means. Now we consider a nonholonomic system, the situation in which the use of quasi-coordinates and the Gibbs-Appell equations is particularly advantageous.

Consider a spherical rigid body with radius a and radial mass symmetry (i.e. the mass density depends only on the distance from the center) rolling without slipping on a rotating plane (Fig. 14-6). The plane rotates with variable rate $\Omega(t) \in C^1$ about the z-axis. The $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ frame is fixed (inertial) with origin at the center of rotation and the $\{\hat{\imath}', \hat{\jmath}', \hat{k}'\}$ frame is parallel to the fixed frame with origin at G, the center of mass of the sphere. The rectangular coordinates of the center of mass relative to the fixed frame are (x, y, a). Let the angular velocity of the body be $\underline{\omega} = \omega_x\hat{\imath} + \omega_y\hat{\jmath} + \omega_z\hat{k}$.

If the plane were at rest, the rolling-without-slipping conditions would be $\dot{x} = a\omega_y$ and $\dot{y} = -a\omega_x$. If the sphere were at rest on the rotating

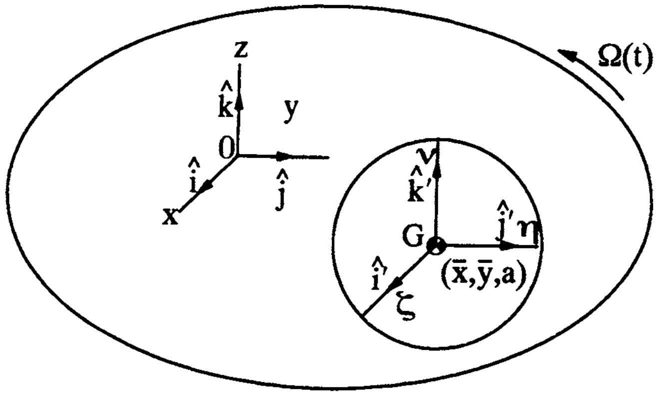

Fig. 14-6

plane, $\dot{x} = -\Omega y$ and $\dot{y} = \Omega x$. Combining the rotating and rolling gives the nonholonomic constraints:

$$\begin{aligned} \dot{x} - a\omega_y &= -\Omega y \\ \dot{y} + a\omega_x &= \Omega x \end{aligned} \tag{14.23}$$

Thus there are one holonomic ($z = a$) and two nonholonomic constraints on the motion so that $L' = 1$, $\ell = 2$ and $k = 3$. We choose the five coordinates x, y, q_x, q_y, q_z where x and y are generalized coordinates and the three quasi-coordinates are defined by

$$\dot{q}_x = \omega_x\,, \qquad \dot{q}_y = \omega_y\,, \qquad \dot{q}_z = \omega_z \tag{14.24}$$

Using the analoque of Koenig's theorem, the Gibbs function is[1]

$$G = \frac{1}{2}M\left(\ddot{x}^2 + \ddot{y}^2\right) + \frac{1}{2}I\left(\ddot{q}_x^2 + \ddot{q}_y^2 + \ddot{q}_z^2\right) \tag{14.25}$$

where all non-essential terms have been omitted, and where I is the moment of inertia of the body about any axis passing through G. (For a body with radial mass symmetry any axis passing through G is a principal axis of inertia and the moment of inertia about all such axes is the same.)

The Gibbs function must now be expressed in terms of the acceleration components of three of the coordinates; we choose x, y, and q_z. Differentiating Eqns. (14.23),

$$\begin{aligned} a\ddot{q}_y &= \ddot{x} + \Omega\dot{y} + \dot{\Omega}y \\ a\ddot{q}_x &= -\ddot{y} + \Omega\dot{x} + \dot{\Omega}x \end{aligned} \tag{14.26}$$

Substituting these relations into Eqn. (14.25) gives

$$\begin{aligned} G &= \frac{1}{2}M\left(\ddot{x}^2+\ddot{y}^2\right)+\frac{I}{2a^2}\left(\ddot{x}+\Omega\dot{y}+\dot{\Omega}y\right)^2 \\ &\quad +\frac{I}{2a^2}\left(-\ddot{y}+\Omega\dot{x}+\dot{\Omega}x\right)^2+\frac{1}{2}I\ddot{q}_z^2 \end{aligned} \tag{14.27}$$

Now suppose that the external force system acting on the body has been resolved into a force $\underline{F} = F_x\hat{\imath}+F_y\hat{\jmath}+F_z\hat{k}$ acting at the center of the sphere and a moment $\underline{M} = M_x\hat{\imath} + M_y\hat{\jmath} + M_z\hat{k}$ about the center. From Eqns. (14.23) and (14.24)

$$dx - a\,dq_y = -\Omega y\,dt$$

$$dy + a\,dq_x = \Omega x\,dt$$

so that virtual displacements satisfy

$$\delta x - a\,\delta q_y = 0$$

$$\delta y + a\,\delta q_x = 0$$

Consequently, the work done in a virtual displacement is

$$\begin{aligned} &F_x\delta x + F_y\delta y + F_z\delta z + M_x\delta q_x + M_y\delta q_y + M_z\delta q_z \\ &= \left(F_x+\frac{M_y}{a}\right)\delta x + \left(F_y-\frac{M_x}{a}\right)\delta y + M_z\delta q_z \end{aligned} \tag{14.28}$$

where, of course, $\delta z = 0$.

We are now in a position to apply the Gibbs-Appell equations, Eqns. (14.19); the result is

$$\begin{aligned} M\ddot{x}+\frac{I}{a^2}\left(\ddot{x}+\Omega\dot{y}+\dot{\Omega}y\right) &= F_x+\frac{M_y}{a} \\ M\ddot{y}+\frac{I}{a^2}\left(\ddot{y}-\Omega\dot{x}-\dot{\Omega}x\right) &= F_y-\frac{M_x}{a} \end{aligned} \tag{14.29}$$

$$I\ddot{q}_z = M_z$$

Consider the following special case: (i) the rotation Ω = const., (ii) the body is a homogeneous sphere, so that $I = \frac{2}{5}Ma^2$, and (iii) there is no external moment acting on the sphere. Then the equations of motion of the mass center reduce to:

$$\begin{aligned} 7\ddot{x} + 2\Omega\dot{y} &= \frac{5F_x}{M} \\ 7\ddot{y} - 2\Omega\dot{x} &= \frac{5F_y}{M} \end{aligned} \tag{14.30}$$

These linear equations are easily solved in terms of convolution integrals.

Notes

1 See Section 11.1

PROBLEMS

14/1. Consider a rigid body moving in space under the action of any given system of forces. Let $\{\hat{i}', \hat{j}', \hat{k}'\}$ be a body-fixed frame aligned with the principal axes of inertia, and $\{\hat{I}, \hat{J}, \hat{K}\}$ be non-moving (inertial) axes. Let the moments of inertia be I_x, I_y, and I_z and let the mass be m. Suppose the resultant force has components F_x, F_y, and F_z along the inertial axes and the resultant moment has components M_x, M_y, and M_y about the center of mass along the body-fixed axes. Then the Gibbs function is (Pars, pp. 216):

$$G = \frac{1}{2}M\left(\ddot{x}^2 + \ddot{y}^2 + \ddot{z}^2\right) + \frac{1}{2}\Big[I_x\dot{\omega}_x^2 - 2(I_y - I_z)\omega_y\omega_z\dot{\omega}_x + I_y\dot{\omega}_y^2 \\ -2(I_z - I_x)\omega_z\omega_x\dot{\omega}_y + I_z\dot{\omega}_z^2 - 2(I_x - I_y)\omega_x\omega_y\dot{\omega}_z\Big]$$

where (x, y, z) are the coordinates of the center of mass relative to $\{\hat{I}, \hat{J}, \hat{K}\}$, $\omega_x = \dot{q}_x$, $\omega_y = \dot{q}_y$, and $\omega_z = \dot{q}_z$, and where $\dot{q}_x$, $\dot{q}_y$, and $\dot{q}_z$ are the components along $\{\hat{i}', \hat{j}', \hat{k}'\}$ of the angular velocity of the body. The system is holonomic with 6 DOF. Choose as coordinates x, y, z, which are generalized coordinates, and q_1, q_2, q_3, which are quasi-coordinates.

Use the Gibbs-Appell Eqns. to generate the equations of motion, three of which are called in this case Euler's equations.

14/2. Fill in the details of the particle in a plane problem.

14/3. Fill in the details of the cylinder rolling in a cylinder problem.

14/4. Fill in the details of the sphere rolling on a turntable problem.