Chapter 12

Stability Of Motion

12.1 Introduction

First Order Form of Equations of Motion. As discussed in Section 6.5, application of Lagrange's equations, Eqns. (6.29), to a dynamic system results in a system of differential equations of the form

$$\begin{aligned}
&\sum_s a_{1s}\ddot{q}_s + \phi_1(q_1, \cdot\cdot, q_n, \dot{q}_1, \cdot\cdot, \dot{q}_n, t) = 0 \\
&\qquad\vdots \\
&\sum_s a_{ns}\ddot{q}_s + \phi_n(q_1, \cdot\cdot, q_n, \dot{q}_1, \cdot\cdot, \dot{q}_n, t) = 0
\end{aligned} \tag{12.1}$$

where $q_1, \cdot\cdot, q_n$ are suitable generalized coordinates and $\sum\limits_s = \sum\limits_{s=1}^{n}$. Equations (12.1) are called the mathematical model of the system. Note that: (1) These equations are linear in the acceleration components $\ddot{q}_1, \cdot\cdot, \ddot{q}_n$; (2) They are in general dynamically coupled; and (3) The matrix a_{rs} is positive definite.

In Section 8.1, the equations of motion were put into first order, generally coupled, form. We now do this by a different method that results in uncoupled equations. Because a_{rs} is positive definite, there exists a transformation to new generalized coordinates, say, $z_1, \cdot\cdot, z_n$,

$$q_r = \sum_s t_{rs} z_s \tag{12.2}$$

such that in the new variables the equations are dynamically uncoupled:

$$\begin{aligned} \ddot{z}_1 + \phi_1'(z_1, \cdot\cdot, z_n, \dot{z}_1, \cdot\cdot, \dot{z}_n, t) &= 0 \\ &\vdots \\ \ddot{z}_n + \phi_n'(z_1, \cdot\cdot, z_n, \dot{z}_1, \cdot\cdot, \dot{z}_n, t) &= 0 \end{aligned} \tag{12.3}$$

Now let

$$\begin{aligned} y_1 &= z_1 \\ &\vdots \\ y_n &= z_n \\ y_{n+1} &= \dot{z}_1 \\ &\vdots \\ y_{2n} &= \dot{z}_{2n} \end{aligned} \tag{12.4}$$

Then the equations of motion may be written as a system of $2n$ first order equations of the form

$$\begin{aligned} \dot{y}_1 &= Y_1(y_1, \cdot\cdot, y_{2n}, t) \\ &\vdots \\ \dot{y}_{2n} &= Y_{2n}(y_1, \cdot\cdot, y_{2n}, t) \end{aligned} \tag{12.5}$$

where $\underline{y} = (y_1, \cdot\cdot, y_{2n})$ is called the *state vector.*[1] The initial conditions are $\underline{y}(t_0) = (y_1(t_0), \cdot\cdot, y_{2n}(t_0))$. Equations (12.5) are said to be in *state variable form.* This is a convenient form for further analysis and computation.

Intuitive Notion of Stability. Stability has to do with the following question: Does the motion of a system stay close to the motion of some nominal (reference) motion if the conditions are somewhat perturbed? By motion, we mean the solution of Eqns. (12.5). There are generally three types of perturbations of interest:

1. *In initial conditions.* Frequently these are taken as current conditions in control applications. These perturbations may be due to sensor error or to disturbances.

2. *In parameters* (for example, mass, stiffness, or aerodynamic coefficients). This is sometimes called structural uncertainty.

3. *In the dynamic model.* Significant terms may have been neglected in formulating the equations, and there may be significant unmodeled dynamics (for example, the controller dynamics are neglected in many control problems).

Remarks.

1. The reference motion is frequently an equilibrium condition (no motion).

2. Only perturbations in initial conditions are usually considered in dynamics; all three are of importance in control system design.

3. The question of stability is usually of vital importance for a dynamic system because an unstable system is generally not usable.

4. Since any dynamic model is only an approximation of a physical system, there is always unmodeled dynamics.

Example. We investigate the stability of the motion of the harmonic oscillator, whose equation of motion is $\ddot{x} + n^2 x = 0$, when the initial conditions are perturbed. The unperturbed motion is given by

$$x = x_0 \cos nt + \frac{u_0}{n} \sin nt$$

where $x(0) = x_0$ and $\dot{x}(0) = u_0$. Let the perturbed initial conditions be $x(0) = x_0 + \eta_x$ and $\dot{x}(0) = u_0 + \eta_u$ so that the perturbed motion is

$$x' = (x_0 + \eta_x) \cos nt + \frac{(u_0 + \eta_u)}{n} \sin nt$$

The difference between the two is

$$x' - x = \eta_x \cos nt + \frac{\eta_u}{n} \sin nt$$

Since this will stay small if η_x and η_u are small, the motion is stable. Note that the perturbation in the motion does not tend to zero over time, but rather persists at a constant amplitude.

12.2 Definitions of Stability

Geometrical Representations of the Motion. Recall the representations of motion introduced in Section 2.2; in the present terms,

$\underline{z} \in C \subset \mathbb{E}^n$,	where C is the configuration space
$(\underline{z}, t) \in E \subset \mathbb{E}^{n+1}$,	where E is the event space
$\underline{y} \in S \subset \mathbb{E}^{2n}$,	where S is the state space
$(\underline{y}, t) \in T \subset \mathbb{E}^{2n+1}$,	where T is the state-time space

Liapunov Stability (L-Stability). Consider a motion (i.e. a solution of Eqns. (12.5)) $f_1(t), \cdot\cdot, f_{2n}(t)$. This motion is *L-stable* if for each $\epsilon > 0$ there exists a $\eta(\epsilon) > 0$ such that for all disturbed motions $y_1(t), \cdot\cdot, y_{2n}(t)$ with initial disturbances

$$\left| y_s(t_0) - f_s(t_0) \right| \le \eta(\epsilon) \tag{12.6}$$

we have

$$\left| y_s(t) - f_s(t) \right| < \epsilon \tag{12.7}$$

for all t and $s = 1, \cdot\cdot, 2n$. The situation is depicted in Fig. 12-1 in T space for the case of two states for the general case and in Fig. 12-2 for

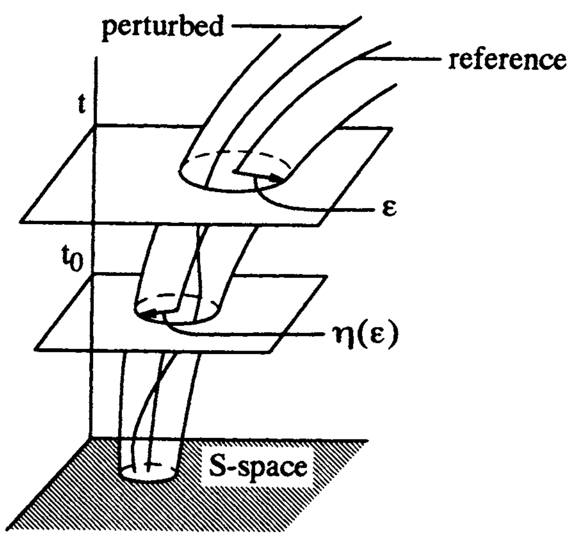


Fig. 12-1

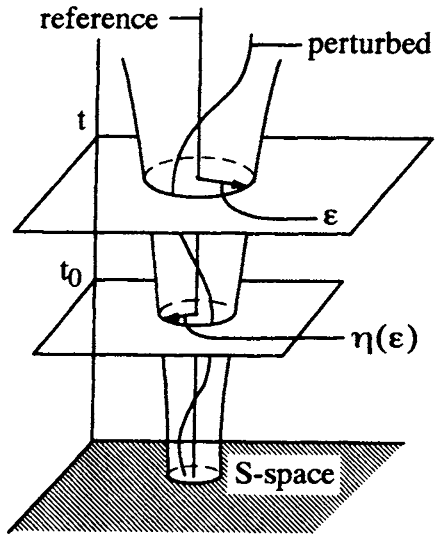


Fig. 12-2

the special case of the reference motion being equilibrium. In words, the motion is L-stable if when the perturbed motions are sufficiently close to the reference motion at some time, then they remain close thereafter.

If, further,

$$\lim_{t \to \infty} \left| y_s(t) - f_s(t) \right| = 0 \tag{12.8}$$

for all s then the motion is *asymptotically stable.*

Poincare Stability (P-Stability). In some cases, a type of stability other than L-Stability is of interest. For example, consider the motion of the harmonic oscillator expressed in the form:

$$x = A \sin(\omega t + B) \tag{12.9}$$

It is clear that this motion is L-Stable with respect to perturbations in both A and B, but is L-Unstable with respect to parameter ω (see Fig. 12-3). If the motion is plotted in the state space, however, we see that the perturbed motion remains close to the reference motion and it may be that this is all that's desired (Fig. 12-4). This motivates another definition of stability.

Consider a motion $f_1(\gamma), \cdot\cdot, f_{2n}(\gamma)$ where γ is an arc length parameter. Then the motion is *P-Stable* if for each $\epsilon > 0$ there exists a $\eta(\epsilon) > 0$ such

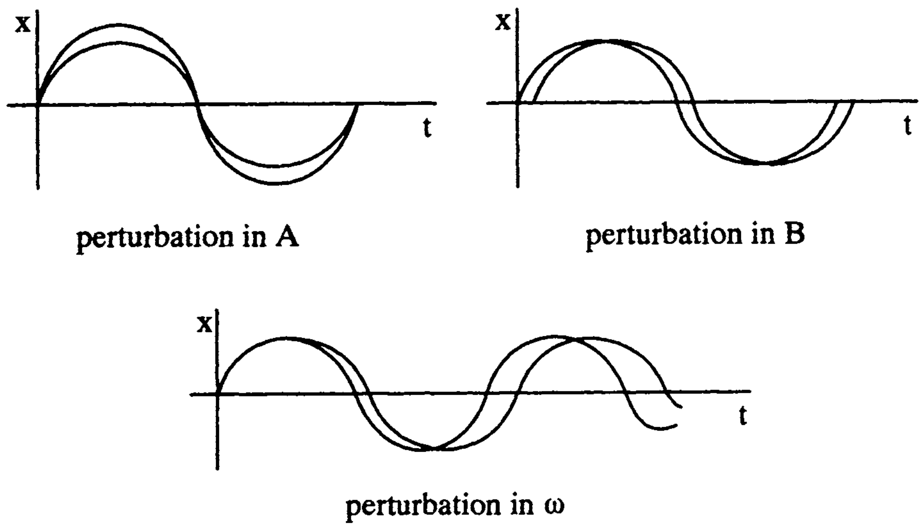


Fig. 12-3

that for all disturbed motions $y_s(\gamma)$ with initial disturbances

$$|y_s(\gamma_0) - f_s(\gamma_0)| \le \eta(\epsilon) \tag{12.10}$$

we have

$$|y_s(\gamma) - f_s(\gamma)| < \epsilon \tag{12.11}$$

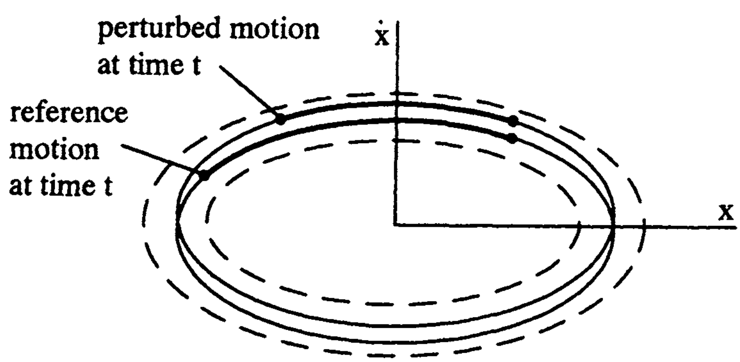


Fig. 12-4

for all γ and s. A motion that is L-Unstable and P-Stable is illustrated on Fig. 12-5 in T space. Note that if a motion is L-Stable, then it is always P-Stable, but not necessarily conversely.

Poincare stability is sometimes called *orbital stability*, for obvious reasons.

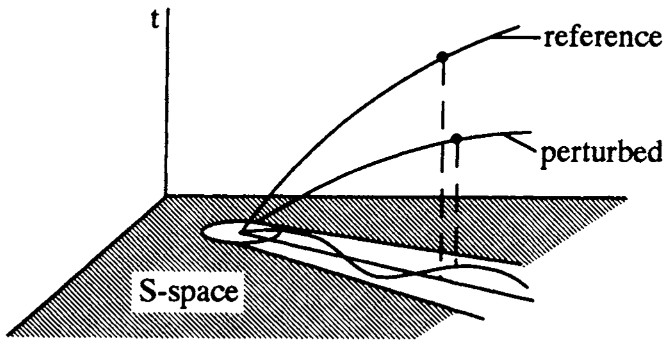


Fig. 12-5

12.3 Indirect Methods

Introductory Remarks. The stability properties of linear systems are well-known as compared with the stability properties of nonlinear systems. This observation suggests the following procedure. The nonlinear system is approximated by *linearizing* about a reference motion, that is, by expanding the disturbed motion in a Taylor series in the perturbations and retaining only the first (linear) terms. The stability of the linear system is then investigated.

There are two potential dangers in this approach:

1. If the disturbances become "large", the first order terms no longer dominate and the approximation is not valid. What constitutes "large", unfortunately, is not usually known.

2. In some exceptional cases, stability of the linear system does not guarantee stability of the nonlinear system, no matter how small the disturbances.

Variational Equations. Let the disturbed motion be equal to the reference motion plus a perturbation:

$$y_s(t) = f_s(t) + \eta_s(t) \; ; \quad s = 1, \cdot\cdot, 2n \tag{12.12}$$

Substitute this into Eqns. (12.5) and expand in a Taylor's series in the perturbations:

$$\dot{f}_s + \dot{\eta}_s \;\; = \;\; Y_s(f_1 + \eta_1, \cdot\cdot, f_{2n} + \eta_{2n}, t) \; ;$$

$$= \quad Y_s(f_1, \cdot\cdot, f_{2n}) + \sum_{r=1}^{2n} a_{sr}\eta_r$$
$$+\text{nonlinear terms in the } \eta_r; \; s = 1, \cdot\cdot, 2n \qquad (12.13)$$

where $a_{sr} = \dfrac{\partial Y_s}{\partial y_r}$ evaluated at $y_r = f_r$. But $\dot{f}_s = Y_s(f_1, \cdot\cdot, f_{2n}, t)$ because $f_s(t)$ is a motion; therefore, neglecting the nonlinear terms in Eqns. (12.13) gives

$$\dot{\eta}_s = \sum_{r=1}^{2n} a_{sr}\eta_r \; ; \quad s = 1, \cdot\cdot, 2n \qquad (12.14)$$

In general, the a_{sr} will be explicit functions of time. If, however, the reference motion is an equilibrium position ($\dot{f}_s = 0$ for all s, which implies that all velocities and accelerations are zero) and if the functions Y_s do not depend explicitly on time, then the a_{sr} do not depend on time and Eqns. (12.14) are a time-invarient linear system, the stability properties of which are well-known and easily stated.

Stability of Time-Invariant Linear Systems. The key results will be stated without proof. The *characteristic equation* associated with Eqns. (12.14) can be obtained by taking Laplace transforms or by substituting $\eta_1 = A_1 e^{st}, \cdot\cdot, \eta_{2n} = A_{2n}e^{st}$; the result is

$$|a_{sr} - \lambda I_{sr}| = 0 \qquad (12.15)$$

where I_{sr} is the identity matrix. The $2n$ roots of Eqn. (12.15) are called the *eigenvalues* of a_{sr} and their signs determine the stability of Eqns. (12.14) as follows:

(1) If all roots $\lambda_s \; ; s = 1, \cdot\cdot, 2n$ have negative real parts, Eqns. (12.14) are asymptotically stable.

(2) If one or more root has a positive real part, the equations are unstable.

(3) If all roots are distinct and some roots have zero and some negative real parts, the equations are stable but not asymptotically stable.

The characteristic equation, Eqn. (12.15) is a polynomial equation of order $2n$. Criteria, called the *Routh-Hurwitz Criteria,* have been developed to determine the stability of a system directly from the coefficients of the system's characteristic equation. This will not be pursued here.

Example – Double Plane Pendulum. Consider a double pendulum with both links of length ℓ and both bobs of mass m (Fig. 12-6). The masses of the links are negligible. In this case, Eqns. (7.25) reduce to

$$\begin{aligned} &2\ddot{\theta}_1 + \cos(\theta_1 - \theta_2)\ddot{\theta}_2 + \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) + 2\frac{g}{\ell}\sin\theta_1 = 0 \\ &\ddot{\theta}_2 + \cos(\theta_1 - \theta_2)\ddot{\theta}_1 - \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) + \frac{g}{\ell}\sin\theta_2 = 0 \end{aligned} \tag{12.16}$$

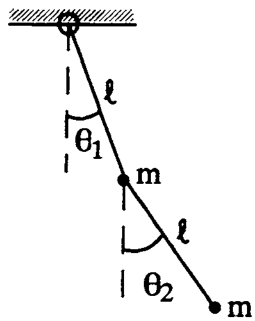


Fig. 12-6

We investigate the stability of the equilibrium positions; these are defined by $\dot{\theta}_1^* = \dot{\theta}_2^* = \ddot{\theta}_1^* = \ddot{\theta}_2^* = 0$. Substitution into Eqns. (12.16) gives $\sin\theta_1^* = 0$ and $\sin\theta_2^* = 0$ so that the four equilibrium positions are given by the combinations of (Fig. 12-7):

$$\theta_1^* = 0°, 180° \ ; \quad \theta_2^* = 0°, 180°$$

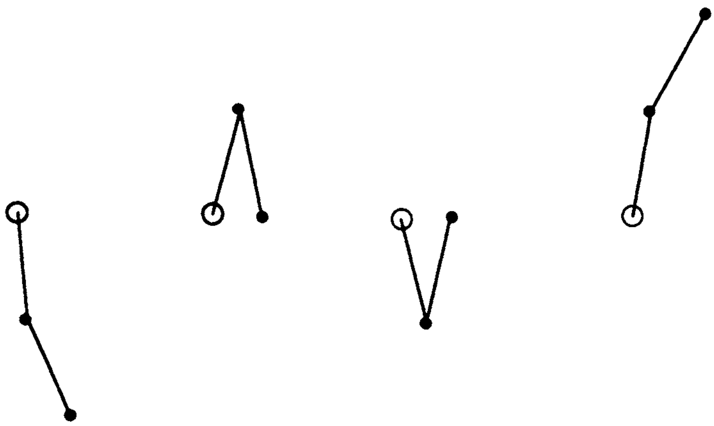
We investigate the stability of these positions by using the rules in the previous section.

First consider equilibrium position (a), $\theta_1^* = \theta_2^* = 0$. Let

$$\theta_1 = \eta_1, \quad \dot{\theta}_1 = \eta_2, \quad \theta_2 = \eta_3, \quad \dot{\theta}_2 = \eta_4$$

where the η_i are small perturbations from equilibrium. Substitution into Eqns. (12.16) and retaining only the first order terms gives:

$$\begin{aligned} &\dot{\eta}_1 = \eta_2 \\ &2\dot{\eta}_2 + \dot{\eta}_4 = -2\frac{g}{\ell}\eta_1 \\ &\dot{\eta}_3 = \eta_4 \\ &\dot{\eta}_4 + \dot{\eta}_2 = -\frac{g}{\ell}\eta_3 \end{aligned}$$



a) $\theta_1^* = 0$, $\theta_2^* = 0$ b) $\theta_1^* = 180^\circ$, $\theta_2^* = 0$ c) $\theta_1^* = 0$, $\theta_2^* = 180^\circ$ d) $\theta_1^* = 180^\circ$, $\theta_2^* = 180^\circ$

Fig. 12-7

(Note that these are not in state variable form.) The eigenvalues of the coefficient matrix of this system are

$$\lambda_{1,2,3,4} = \pm\sqrt{\frac{g}{\ell}(-2 \pm \sqrt{2})}$$

Since $g/\ell > 0$, all four of these eigenvalues have only imaginary parts (Fig. 12-8). Thus this is case (3) above and the equilibrium is stable but not asymptotically stable.

Next consider position (b), $\theta_1^* = \pi$, $\theta_2^* = 0$; proceeding as before,

$$\theta_1 = \pi + \eta_1 \ , \quad \dot{\theta}_1 = \eta_2 \ , \quad \theta_2 = \eta_3 \ , \quad \dot{\theta}_2 = \eta_4$$

$$\begin{aligned} &\dot{\eta}_1 = \eta_2 \\ &2\dot{\eta}_2 + \dot{\eta}_4 = 2\frac{g}{\ell}\eta_1 \\ &\dot{\eta}_3 = \eta_4 \\ &\dot{\eta}_4 + \dot{\eta}_2 = \frac{g}{\ell}\eta_3 \end{aligned}$$

$$\lambda_{1,2,3,4} = \pm\sqrt{\frac{g}{\ell}(2 \pm \sqrt{2})}$$

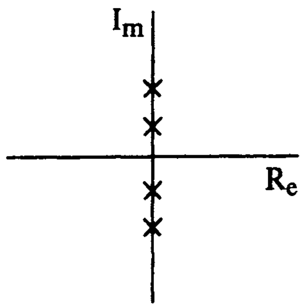


Fig. 12-8

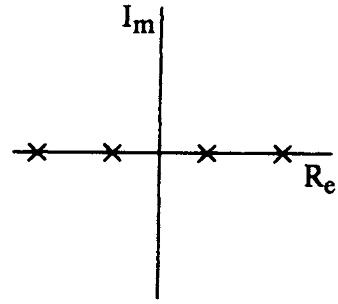


Fig. 12-9

The eigenvalues thus have only real parts and two of them are positive (Fig. 12-9). This is case (2) above and the equilibrium position is unstable.

Equilibrium positions (c) and (d) are also unstable.

Some Simple Examples. Here and in the next section we consider motions in which the reference motion is not an equilibrium position. First, consider the motion of a particle moving vertically near the surface of the earth. The equation of motion is $\ddot{x} = -g$, with x measured upwards from the earth surface. Letting $y = \dot{x}$, the solution with initial conditions $x(0) = x_0$ and $y(0) = y_0$ is

$$x = x_0 + y_0 t - \frac{1}{2} g t^2 , \qquad y = y_0 - gt$$

Suppose the initial conditions are now perturbed so that $x(0) = x_0 + \eta_x$ and $y(0) = y_0 + \eta_y$; then the perturbed motion is given by

$$x' = x_0 + \eta_x + (y_0 + \eta_y)t - \frac{1}{2} g t^2$$

$$y' = y_0 + \eta_y - gt$$

The difference between the two motions is:

$$x' - x = \eta_x + \eta_y t , \qquad y' - y = \eta_y$$

Thus the difference between the reference and the perturbed motion grows with time and the motion is not L-stable.

As a second example, consider the motion defined by the system

$$\dot{x} = -x - y + \frac{ax}{\sqrt{x^2 + y^2}}$$

$$\dot{y} = x - y + \frac{ay}{\sqrt{x^2 + y^2}}$$

Transforming to polar coordinates, the system equations are

$$\dot{r} = a - r\,, \qquad \dot{\theta} = 1$$

With initial conditions $r(0) = r_0$ and $\theta(0) = \theta_0$, the solution is

$$r = (r_0 - a)e^{-t} + a\,, \qquad \theta = \theta_0 + t$$

Now suppose the initial conditions are perturbed, $r(0) = r_0 + \eta_r$ and $\theta(0) = \theta_0 + \eta_\theta$. Then the perturbed motion is

$$r' = (r_0 + \eta_r - a)e^{-t} + a$$

$$\theta' = \theta_0 + \eta_\theta + t$$

The difference between the two is

$$r' - r = \eta_r e^{-t}\,, \qquad \theta' - \theta = \eta_\theta$$

Therefore the perturbation in the motion stays small if η_r and η_θ are small and the system is L-stable. Note that $r' - r \to 0$ as $t \to \infty$ but that $\theta' - \theta$ remains constant. Thus the stability is not asymptotic.

As a third example, consider the motion defined by

$$\dot{x} = -y\sqrt{x^2 + y^2}$$

$$\dot{y} = x\sqrt{x^2 + y^2}$$

Take the initial conditions, without loss of generality, to be

$$x(0) = a\cos\alpha\,, \qquad y(0) = a\sin\alpha$$

Then the motion is

$$x = a\cos(at + \alpha)$$

$$y = a\sin(at + \alpha)$$

The motion therefore describes a circle in the (x, y) plane with radius a and period $2\pi/a$. Now let the initial conditions be perturbed to $a + \eta_a$ and $\alpha + \eta_\alpha$; then the perturbed motion is

$$x' = (a + \eta_a)\cos[(a + \eta_a)t + \alpha + \eta_\alpha]$$

$$y' = (a + \eta_a)\sin[(a + \eta_a)t + \alpha + \eta_\alpha]$$

Because the period has been changed, the system is not L-stable but is P-stable. The situation is similar to that shown on Fig. 12-4.

12.4 Stability of Orbits in a Gravitational Field

Here we consider the important problem of the stability of closed orbits (i.e. elliptical and circular orbits) of a body moving in a central gravitational field. For specificity, the case of a satellite or space craft in a nominally circular earth orbit will be discussed. The perturbation equations, derived in most books on orbital dynamics, are

$$\begin{aligned}
&\ddot{x} - 2n\dot{y} - 3n^2x = 0 \\
&\ddot{y} + 2n\dot{x} = 0 \\
&\ddot{z} + n^2z = 0
\end{aligned} \tag{12.17}$$

where $n = \sqrt{\mu/a^3}$ is called the mean motion and μ and a are the earth's gravitational parameter and the radius of the nominal orbit, respectively. Eqns. (12.17) are called in various places the Hill, the Euler-Hill, or the Clohessy-Wiltshire equations. They have found wide application in orbital dynamics; for example they are used in the analysis of rendezvous between two spacecraft in neighboring circular orbits, docking maneuvers between two spacecraft, and orbital station-keeping.

The solution of Eqns. (12.17) is

$$\begin{aligned}
x(t) &= 2\left(\frac{\dot{y}_0}{n} + 2x_0\right) - \left(2\frac{\dot{y}_0}{n} + 3x_0\right)\cos nt + \frac{\dot{x}_0}{n}\sin nt \\
y(t) &= y_0 - 2\frac{\dot{x}_0}{n} - 3(\dot{y}_0 + 2nx_0)t + 2\frac{\dot{x}_0}{n}\cos nt \\
&\quad + 2\left(2\frac{\dot{y}_0}{n} + 3x_0\right)\sin nt \\
z(t) &= z_0\cos nt + \frac{\dot{z}_0}{n}\sin nt
\end{aligned} \tag{12.18}$$

In these equations, x_0, y_0, z_0, $\dot{x}_0$, $\dot{y}_0$, $\dot{z}_0$ are the perturbations in position and velocity components at time $t = 0$ from a nominal circular orbit, relative to a frame that travels with the orbit (Fig. 12-10), and $x(t)$, $y(t)$, $z(t)$ are the perturbations at some later time t.

Inspection of Eqns. (12.18) shows the following. First, the motion perpendicular to the orbital plane, $z(t)$, is uncoupled from the in-plane motion, and this component of the motion is L-stable but not asymptotically so. Second, because of the term linear in t in the $y(t)$ equation, the in-plane perturbations are not generally bounded and the motion is not L-stable. The radial component, however, is bounded and thus the

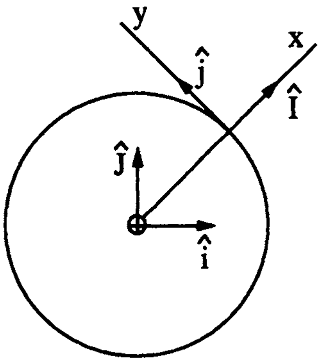


Fig. 12-10

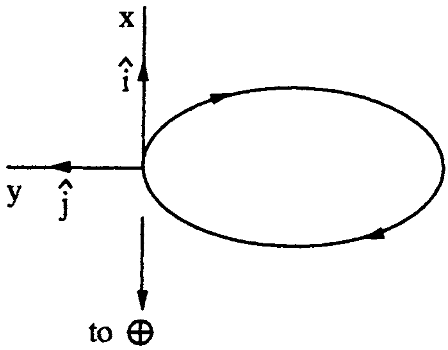


Fig. 12-11

perturbed (elliptical) orbit remains close to the nominal circular one, with an object in the perturbed orbit either pulling ahead or falling back relative to an object in the nominal one. Again the situation is similar to that shown on Fig. 12-4 and the motion is P-stable.

As a simple example, suppose a spacecraft in a circular earth orbit ejects a particle of small mass in the outward radial direction. In this case the initial perturbations are

$$x_0 = y_0 = z_0 = \dot{y}_0 = \dot{z}_0 = 0\,, \quad \dot{x}_0 > 0$$

and Eqns. (12.18) become

$$\begin{aligned} x(t) &= \frac{\dot{x}_0}{n}\sin nt \\ y(t) &= \frac{2\dot{x}_0}{n}\cos nt - \frac{2\dot{x}_0}{n} \\ z(t) &= 0 \end{aligned}$$

These are the equations of an ellipse with semi-major and semi-minor axes of $2\dot{x}_0/n$ and $\dot{x}_0/n$, respectively. Thus, relative to the spacecraft the particle travels in a elliptical orbit (Fig. 12-11) and arrives back at the spacecraft at the time at which the spacecraft has completed one revolution of the earth. In an inertial frame, the particle travels around the earth in an ellipse neighboring the circular orbit with the same period, rendezvousing with the circular orbit after each revolution. This motion is clearly L-stable.

12.5 Liapunov's Direct Method

Autonomous Case. Let $\eta_s(t)$; $s = 1, \cdot\cdot, 2n$ be a perturbation from equilibrium of a dynamical system. Then these functions satisfy equations of the form

$$\dot{\eta}_s = g_s(\eta_1, \cdot\cdot, \eta_{2n}) \; ; \quad s = 1, \cdot\cdot, 2n \tag{12.19}$$

In the autonomous case, these functions do not depend explicitly on t.

We define the following classes of functions:

1. If $V(\eta_1, \cdot\cdot, \eta_{2n})$ is of class C^1 (i.e. continuous with continuous derivatives) in an open region $\Omega \subset I\!E^{2n}$ containing the origin, if $V(0, \cdot\cdot, 0) = 0$, and if $V(\cdot)$ has the same sign everywhere in Ω except at the origin, then $V(\cdot)$ is called *definite* in Ω.

2. If $V(\cdot)$ is positive everywhere except at the origin, it is *positive definite* (Fig. 12-12); if negative everywhere except at the origin, it is *negative definite* (Fig. 12-13).

3. If $V(\cdot)$ has the same sign everywhere where it is not zero, but it can be zero other than at the origin, it is called *semidefinite.* Figure 12-14 shows a positive semidefinite function.

These definitions are generalizations of the idea of positive definite and negative definite quadratic forms.

Now let the η_s in these definitions be solutions of Eqns. (12.19); then

$$\frac{dV}{dt} = \sum_{s=1}^{2n} \frac{\partial V}{\partial \eta_s} \dot{\eta}_s = \sum_{s=1}^{2n} \frac{\partial V}{\partial \eta_s} g_s \tag{12.20}$$

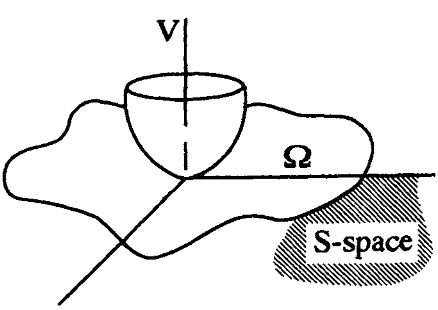


Fig. 12-12

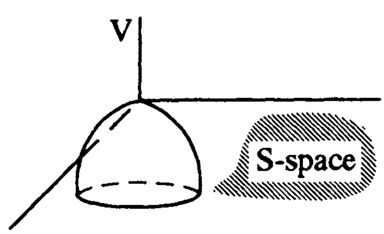


Fig. 12-13

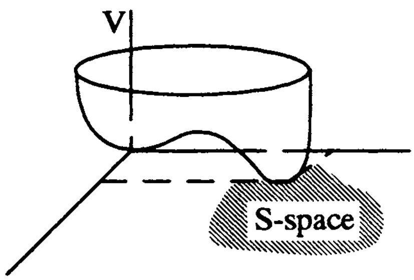


Fig. 12-14

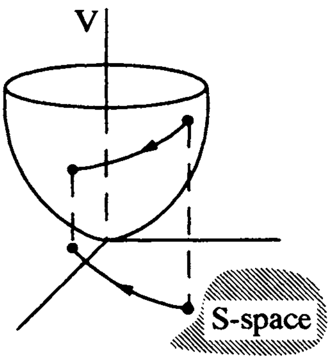


Fig. 12-15

We now define a *Liapunov function* as a function $V(\cdot)$ in Ω definite in sign for which dV/dt is semidefinite and opposite in sign to $V(\cdot)$, or $dV/dt = 0$. The key result is then the following.

Liapunov's Theorem. If a Liapunov function can be constructed for Eqns. (12.19), then the equilibrium position is stable. A geometric proof of the theorem follows from Fig. 12-15. The properties of $V(\cdot)$ ensure that the motions due to small perturbations from equilibrium either tend to zero or remain small.

Application to Dynamics. Consider a natural (holonomic, scleronomic, conservative) system. Lagrange's equations for such a system are[2]

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_s}\right) - \frac{\partial T}{\partial q_s} + \frac{\partial V}{\partial q_s} = 0 \; ; \quad s = 1, \cdot\cdot, n \tag{12.21}$$

Because the system is scleronomic, the transformation equations to generalized coordinates are time independent, and therefore, from Eqn. (6.2) the equilibrium condition is $T = 0$. The equations defining the equilibrium condition are thus:

$$\frac{\partial V}{\partial q_s} = 0 \; ; \quad s = 1, \cdot\cdot, n \tag{12.22}$$

This equation is a necessary condition for an unbounded extremal point of V; that is, V has a stationary value at an equilibrium point. Also, since the system is closed, energy is conserved:

$$E = T + V = h \tag{12.23}$$

Dirichlet's Stability Theorem. An equilibrium solution of the equations of motion for the class of systems defined above is stable if the stationary value of the potential energy is a minimum relative to neighboring points.

To prove this theorem, it suffices to show that E is a Liapunov function. Let $q_1^*, \cdots, q_n^*$ be the equilibrium values, and let $\eta_1, \cdot\cdot, \eta_n$ be small perturbations from these values; that is $q_r = q_r^* + \eta_r$; $r = 1, \cdot\cdot, n$. Then $V = V(\eta_1, \cdot\cdot, \eta_n)$. Choose the datum for V such that $V(0, \cdot\cdot, 0) = 0$. Then, since V has a minimum at $\eta_1 = 0, \cdot\cdot, \eta_n = 0$, V is positive at neighboring points and V is a positive definite function. From Section 6.1, T is always positive definite so that $E = T + V$ is positive definite. Also, $E = h =$ const. implies that $dE/dt = 0$. Consequently, E is a Liapunov function and by Liapunov's Theorem the equilibrium is stable.

Example. Consider again the double pendulum (Fig. 12-6), for which

$$V = 2mg\ell(1 - \cos\theta_1) + mg\ell(1 - \cos\theta_2)$$

Consider the equilibrium position (a), $\theta_1^* = \theta_2^* = 0$. We see that: (i) $V(0,0) = 0$ and (ii) $V(\theta_1, \theta_2) > 0$ for all sufficiently small θ_1 and θ_2. Thus V is positive definite. Since the system is closed, energy is conserved and $dE/dt = 0$. Consequently, E is a Liapunov function and the equilibrium is stable.

Remark. In all but the simplest problems, there is no systematic procedure for finding Liapunov functions and they are generally very difficult to find.

Nonantonomous Case. In this case, one or more of the perturbation equations contains time explicitly:

$$\dot{\eta}_s = g_s(\eta_1, \cdot\cdot, \eta_{2n}, t)\ ; \quad s = 1, \cdot\cdot, 2n \tag{12.24}$$

We now need to introduce two functions $V(\eta_1, \cdot\cdot, \eta_{2n}, t)$ and $W(\eta_1, \cdot\cdot, \eta_{2n})$ such that (i) they vanish at $(\eta_1, \cdot\cdot, \eta_{2n}) = (0, \cdot\cdot, 0)$, (ii) they are single-valued and of class C^1 in Ω, and (iii) $W(\cdot)$ is positive definite. Then:

$$\begin{aligned}
&(1)\quad \text{If } V(\cdot) \geq W(\cdot) \text{ for all } (\eta_1, \cdot\cdot, \eta_{2n}) \in \Omega, \\
&\qquad\ \text{then } V(\cdot) \text{ is } \textit{positive definite.} \\
&(2)\quad \text{If } V(\cdot) \leq W(\cdot) \text{ for all } (\eta_1, \cdot\cdot, \eta_{2n}) \in \Omega, \\
&\qquad\ \text{then } V(\cdot) \text{ is } \textit{negative definite.}
\end{aligned} \tag{12.25}$$

The change in $V(\cdot)$ along a trajectory is now

$$\frac{dV}{dt} = \sum_{s=1}^{2n} \frac{\partial V}{\partial \eta_s} g_s + \frac{\partial V}{\partial t} \tag{12.26}$$

For this case, we call a function $V(\cdot)$ a *Liapunov function* if it is definite in sign in accordance with definitions (12.25) and dV/dt as given in Eqn. (12.26) is semidefinite with opposite sign of $V(\cdot)$.

Liapunov's Theorem (nonantonomous case). If there exists a Liapunov function for Eqns. (12.24), the reference motion is stable.

The proof of this theorem is similar to that for the autonomous case. If the conditions of the theorem are satisfied, the function $V(\cdot)$ will stay "completely inside" the function $W(\cdot)$ (Fig. 12-16), ensuring that motion will tend to zero or remain small. It is clear that this is a much stronger requirement than for the autonomous case, and that Liapunov functions will be even more difficult to find.

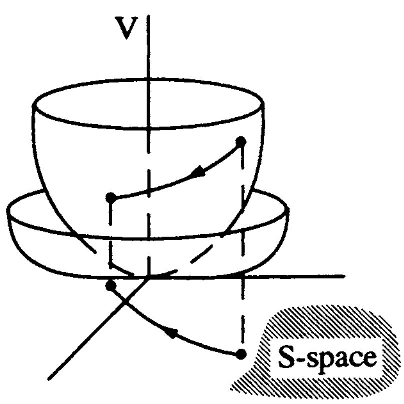


Fig. 12-16

Notes

1 More precisely, $\underline{y} = (y_1, \cdot\cdot, y_{2n})^T$ where T denotes transpose.

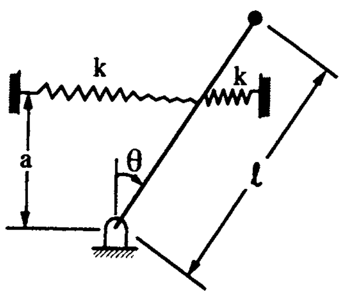
2 Caution: We are using the same symbol, V, for two different functions, Liapunov and potential energy functions.

PROBLEMS

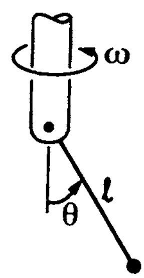
12/1. Investigate the stability of equilibrium position (c) of the double plane pendulum by using linearized equations.

12/2. Investigate the stability of equilibrium position (d) of the double plane pendulum by using linearized equations.

12/3. A heavy, inverted pendulum of mass m is restrained by identical linear springs, as shown. The rigid rod has negligible mass. Examine the stability of small motions about the inverted, vertical position by means of the linearized variational equations.



Problem 12/3



Problem 12/5

12/4. The torque-free motion of a rigid body about a point is given by

$$
\begin{aligned}
I_x\dot{\omega}_x - (I_y - I_z)\omega_y\omega_z &= 0 \,, \\
I_y\dot{\omega}_y - (I_z - I_x)\omega_z\omega_x &= 0 \,, \\
I_z\dot{\omega}_z - (I_x - I_y)\omega_x\omega_y &= 0 \,.
\end{aligned}
$$

Use the linearized variational equations to examine the stability of the steady-state rotation $\omega_z = \Omega = \text{const}$, $\omega_y = \omega_z = 0$. In particular, show that the motion is unstable if I_x is intermediate in magnitude between I_y and I_z.

12/5. A heavy pendulum of mass m rotates with constant angular velocity about the vertical, as shown. The rigid rod has negligible mass. Show that there exist three steady-state motions, for one of which the pendulum angle with the vertical is a non zero constant, and examine the stability of all three steady motions by means of the linearized variational equations.

12/6. Find a Liapunov function for the system of Problem 12/3 thus verifying the results of the linear analysis.

12/7. Investigate the functions

$$I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2$$

and

$$I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2 - I_x\Omega^2$$

as Liapunov functions for the motion of Problem 12/4.

12/8. Investigate the function

$$V = \dot{\theta}^2 + \omega^2\left[\cos^2\theta + 2\alpha(1-\cos\theta)\right]$$

as a Liapunov function for the steady motion with θ not zero of Problem 12/5. If this is not an L-function, can you find one?

12/9. Show that the equilibrium point $\theta = -\frac{\pi}{2}$ of Problem 7/3 is stable by finding a Liapunov function.

12/10. Show, by both the indirect and direct methods, that the equilibrium position of Problem 4/2 is stable.