Chapter 11

Gyroscopic Motion

11.1 Rigid Body Motion with One Point Fixed

Kinetic Energy. Consider the motion of a rigid body such that one of its points, say B, is at rest in an inertial frame (Fig. 11-1). Let $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ be body-fixed principal axes of inertia with origin at B.

The velocity of a typical mass particle of the rigid body is given by Eqn. (1.25) as

$$\underline{v}_i = \underline{v}_B + \underline{v}_{\text{rel}} + \underline{\omega} \times \underline{d}_i$$

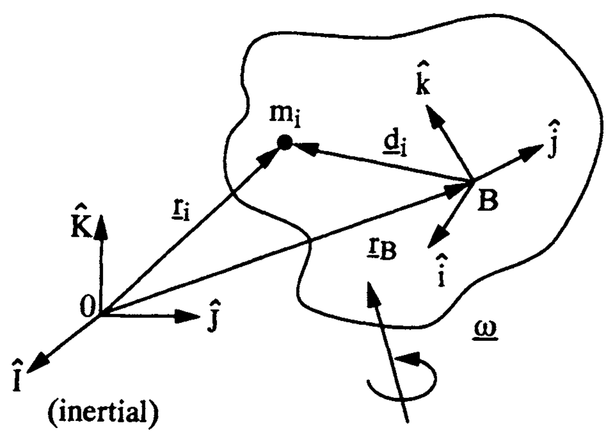


Fig. 11-1

where $\underline{v}_B = \underline{0}$, $\underline{v}_{\text{rel}} = \underline{0}$, and

$$\begin{aligned}\underline{\omega} &= \omega_x\hat{\imath} + \omega_y\hat{\jmath} + \omega_z\hat{k} \\ \underline{d}_i &= x_i\hat{\imath} + y_i\hat{\jmath} + z_i\hat{k}\end{aligned}$$

Thus

$$\begin{aligned}\underline{v}_i &= \left(\omega_x\hat{\imath} + \omega_y\hat{\jmath} + \omega_z\hat{k}\right) \times \left(x_i\hat{\imath} + y_i\hat{\jmath} + z_i\hat{k}\right) \\ &= \left(\omega_y z_i - \omega_z y_i\right)\hat{\imath} + \left(\omega_z x_i - \omega_x z_i\right)\hat{\jmath} + \left(\omega_x y_i - \omega_y x_i\right)\hat{k}\end{aligned} \tag{11.1}$$

Consequently, the kinetic energy of the rigid body is

$$\begin{aligned}T &= \frac{1}{2}\sum_i m_i \underline{v}_i \cdot \underline{v}_i = \frac{1}{2}\sum_i m_i \Big[\left(z_i^2 + y_i^2\right)\omega_x^2 + \left(z_i^2 + x_i^2\right)\omega_y^2 \\ &\quad + \left(y_i^2 + x_i^2\right)\omega_z^2 - 2z_i\, y_i\, \omega_y\, \omega_z - 2x_i\, z_i\, \omega_z\, \omega_x - 2y_i\, x_i\, \omega_x\, \omega_y \Big] \\ &= \frac{1}{2}\left(I_x\,\omega_x^2 + I_y\,\omega_y^2 + I_z\,\omega_z^2\right)\end{aligned} \tag{11.2}$$

where the moments of inertia are

$$\begin{aligned}I_x &= \sum_i m_i(z_i^2 + y_i^2) \\ I_y &= \sum_i m_i(z_i^2 + x_i^2) \\ I_z &= \sum_i m_i(y_i^2 + x_i^2)\end{aligned} \tag{11.3}$$

Note that because $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ are principal axes, all the products of inertia are zero:

$$\begin{aligned}I_{xy} &= \sum_i m_i x_i y_i = 0 \\ I_{yz} &= \sum_i m_i y_i z_i = 0 \\ I_{zx} &= \sum_i m_i z_i x_i = 0\end{aligned}$$

Euler's Angles. The motion of a rigid body without constraints is described by 6 generalized coordinates. Motion with one point fixed implies three holonomic constraints. Without loss of generality, choose the origin of the inertial frame as point B. Then these holonomic constraints are:

$$x = 0\,, \qquad y = 0\,, \qquad z = 0\,. \tag{11.4}$$

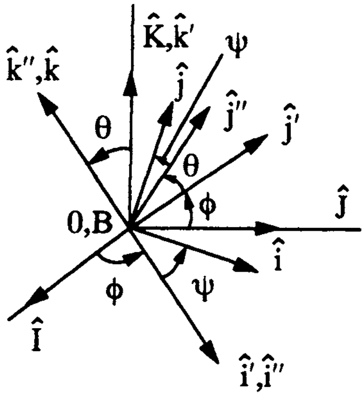


Fig. 11-2

The three generalized coordinates left are orientations or angular measurements. A common choice is the *Euler angles.*[1]

Now consider three reference frames, obtained by rotations relative to the inertial frame $\{\hat{I}, \hat{J}, \hat{K}\}$ (Fig. 11-2):

1. First rotate about $\hat{K} = \hat{k}'$ by ϕ to get $\{\hat{\imath}', \hat{\jmath}', \hat{k}'\}$.
2. Then rotate about $\hat{\imath}' = \hat{\imath}''$ by θ to get $\{\hat{\imath}'', \hat{\jmath}'', \hat{k}''\}$.
3. Finally rotate about $\hat{k}'' = \hat{k}$ by ψ to get $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$.

By a suitable choice of ϕ, θ, ψ, we can get $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ to line up with any arbitrary body-fixed axes. Thus (ϕ, θ, ψ) serve as generalized coordinates.

For the kinetic energy, we need to write ω_x, ω_y, ω_z, the components of $\underline{\omega}$, in body-fixed principal axes, in terms of $\dot{\phi}$, $\dot{\theta}, \dot{\psi}$. From Fig. 11-2, the angular velocity of $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ w.r.t. $\{\hat{I}, \hat{J}, \hat{K}\}$ is

$$\underline{\omega} = \dot{\phi}\hat{k}' + \dot{\theta}\hat{\imath}'' + \dot{\psi}\hat{k} \tag{11.5}$$

where, from Fig. 11-3,

$$\begin{aligned} \hat{\imath}'' &= \cos\psi\hat{\imath} - \sin\psi\hat{\jmath} \\ \hat{\jmath}'' &= \cos\psi\hat{\jmath} + \sin\psi\hat{\imath} \\ \hat{k}' &= \cos\theta\hat{k} + \sin\theta\hat{\jmath}'' \end{aligned} \tag{11.6}$$

Thus

$$\begin{aligned} \underline{\omega} \;=\; & \left(\dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi\right)\hat{\imath} + \left(\dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi\right)\hat{\jmath} \\ & + \left(\dot{\phi}\cos\theta + \dot{\psi}\right)\hat{k} \end{aligned} \tag{11.7}$$

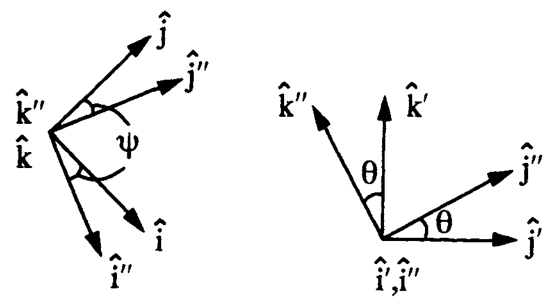


Fig. 11-3

so that

$$\begin{aligned}\omega_x &= \dot\phi \sin\theta \sin\psi + \dot\theta \cos\psi \\ \omega_y &= \dot\phi \sin\theta \cos\psi - \dot\theta \sin\psi \\ \omega_z &= \dot\phi \cos\theta + \dot\psi\end{aligned} \tag{11.8}$$

Substituting Eqns. (11.8) into Eqn. (11.2) then gives the kinetic energy. Instead of doing this in the general case, we will consider the following special case.

11.2 Heavy Symmetrical Top

Formulation. Now suppose the rigid body is axially symmetric and is spinning about its axis of symmetry, Fig. 11-4.

Let frame $\{\hat I, \hat J, \hat K\}$ be inertial, and $\{\hat\imath, \hat\jmath, \hat k\}$ be body-fixed. The angular rates in this case have been given names:

$$\begin{aligned}\dot\phi &= \text{precession} \\ \dot\theta &= \text{nutation} \\ \dot\psi &= \text{spin}\end{aligned}$$

Because of the symmetry, $I_x = I_y$; let

$$I_x = I_y = I\,; \qquad I_z = J \tag{11.9}$$

Then, from Eqns. (11.2), (11.8), and (11.9):

$$\begin{aligned}T &= \frac{1}{2}\left[I\left(\omega_x^2 + \omega_y^2\right) + J\omega_z^2\right] \\ &= \frac{1}{2}\left[I\left(\dot\theta^2 + \dot\phi^2 \sin^2\theta\right) + J\left(\dot\psi + \dot\phi\cos\theta\right)^2\right]\end{aligned} \tag{11.10}$$

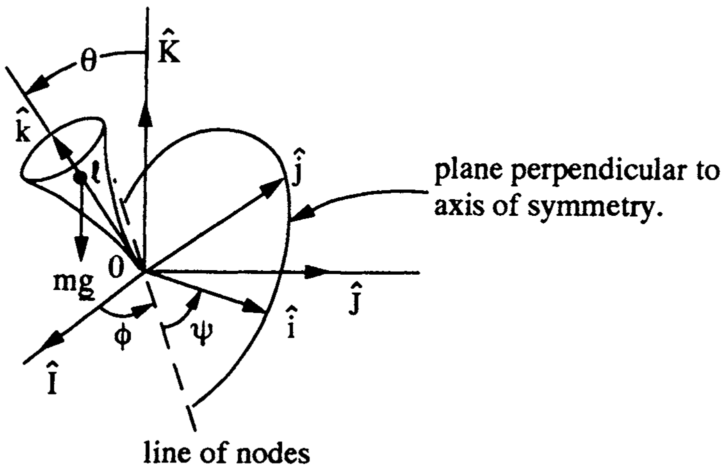


Fig. 11-4

Also,

$$V = mg\ell \cos\theta \tag{11.11}$$

Lagrange's Equations. For a natural system with three generalized coordinates, the proper form of the equations is Eqn. (6.35):

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_s}\right] - \frac{\partial L}{\partial q_s} = 0 \; ; \qquad s = 1, 2, 3 \tag{11.12}$$

Computing partials:

$$\frac{\partial L}{\partial \dot{\theta}} = I\dot{\theta} \; ; \qquad \frac{\partial L}{\partial \dot{\phi}} = I\dot{\phi}\sin^2\theta + J\left(\dot{\psi} + \dot{\phi}\cos\theta\right)\cos\theta$$

$$\frac{\partial L}{\partial \dot{\psi}} = J\left(\dot{\psi} + \dot{\phi}\cos\theta\right)$$

$$\frac{\partial L}{\partial \theta} = I\dot{\phi}^2 \sin\theta\cos\theta - J\left(\dot{\psi} + \dot{\phi}\cos\theta\right)\dot{\phi}\sin\theta + mg\ell\sin\theta$$

$$\frac{\partial L}{\partial \phi} = 0 \; ; \qquad \frac{\partial L}{\partial \psi} = 0$$

Lagrange's equations are then

$$\begin{aligned}
&\frac{d}{dt}\left(I\dot{\theta}\right) - I\dot{\phi}^2\sin\theta\cos\theta + J\left(\dot{\psi} + \dot{\phi}\cos\theta\right)\dot{\phi}\sin\theta \\
&\qquad - \underbrace{mg\ell\sin\theta}_{\substack{= Q_\theta \,=\, \text{moment} \\ \text{due to gravity}}} = 0 \\
&\frac{d}{dt}\left[I\dot{\phi}\sin^2\theta + J\left(\dot{\psi} + \dot{\phi}\cos\theta\right)\cos\theta\right] = 0 \\
&\frac{d}{dt}\left[J\left(\dot{\psi} + \dot{\phi}\cos\theta\right)\right] = 0
\end{aligned} \tag{11.13}$$

We see that ϕ and ψ are ignorable with momentum integrals

$$\begin{aligned}
&J\left(\dot{\psi} + \dot{\phi}\cos\theta\right) = p_\psi = P_2 = \text{constant} \\
&I\dot{\phi}\sin^2\theta + P_2\cos\theta = p_\phi = P_1 = \text{constant}
\end{aligned} \tag{11.14}$$

Solving these for $\dot{\phi}$ and $\dot{\psi}$ and substituting into the first of Eqns. (11.13) gives a second order differential equation in θ:

$$I\ddot{\theta} - \frac{P_1 - P_2\cos\theta}{I\sin^3\theta}(P_1\cos\theta - P_2) - mg\ell\sin\theta = 0 \tag{11.15}$$

with, from Eqns. (11.14),

$$\dot{\phi} = \frac{P_1 - P_2\cos\theta}{I\sin^2\theta} \qquad \dot{\psi} = \frac{P_2}{J} - \left(\frac{P_1 - P_2\cos\theta}{I\sin^2\theta}\right)\cos\theta \tag{11.16}$$

Energy Integral. Since the system is closed (catastatic and potential),

$$T + V = h = \text{constant}$$

$$\frac{1}{2}\left\{I\left[\dot{\theta}^2 + \left(\frac{P_1 - P_2\cos\theta}{I\sin^2\theta}\right)^2\sin^2\theta\right] + \frac{P_2^2}{J}\right\} + mg\ell\cos\theta = h \tag{11.17}$$

This is a first integral of Eqn. (11.15); we now have 3 integrals of the motion. The energy integral, Eqn. (11.17), may be written in quadrature as

$$t = \int f(\theta)d\theta + \text{constant}$$

which can be inverted in principle to give $\theta = f_\theta(t)$. Substitution into Eqns. (11.16) then allows integration to get $\phi = f_\phi(t)$ and $\psi = f_\psi(t)$. The solution has been reduced to quadratures.

We now analyze the behavior of the top via "qualitative integration", that is without numerically evaluating the integrals.

Qualitative Analysis. Let

$$\begin{aligned} &u = \cos\theta\,, \qquad a = \frac{2mg\ell}{I}\,, \qquad \alpha = \frac{2h}{I} - \frac{P_2^2}{IJ} \\ &\beta = \frac{P_1}{I}\,, \qquad \gamma = \frac{P_2}{I} \end{aligned} \tag{11.18}$$

Then the energy integral, Eqn. (11.17), becomes

$$\dot{u}^2 = f(u) = (1-u^2)(\alpha - au) - (\beta - \gamma u)^2 \tag{11.19}$$

First we note that for real $\dot{u}$ we must have $f(u) \geq 0$ and that $f(u) = 0$ gives $\dot{u} = 0 \Longrightarrow u = \text{constant} \Longrightarrow \theta = \text{constant}$. The function $f(u)$ is a cubic equation with properties:

(a) $f(u) \to +\infty$ as $u \to +\infty$

$$\left\{ \begin{aligned} \lim_{u\to+\infty} f(u) &= \ell\left[(-u^2)(-au) - (\gamma u)^2\right] \\ &= \ell\left[au^3 - \gamma^2u^2\right] = a\ell\left[u^3\right] \\ &= +\infty \quad \text{because } a > 0 \end{aligned} \right\}$$

(b) $f(u) \to -\infty$ as $u \to -\infty$

(c) $f(\pm 1) = -(\beta \mp \gamma)^2 \leq 0$

(d) $f(u)$ has a zero between $+1$ and $+\infty$.

(e) If $f(u)$ has three zeros, the other two must lie in $-1 \leq u \leq 1$.

Consequently, $f(u)$ looks as shown on Fig. 11-5 ($-1 \leq u \leq +1$ because $u = \cos\theta$), where u_1 and u_2 are the two zeros of $f(u)$, $-1 \leq u \leq 1$, if any exist.

First consider the special case $u_1 = u_2 = u_0$, Fig. 11-6. Now, $\ddot{\theta} = 0$; combining the first of Eqns. (11.13) with Eqns. (11.14) and (11.18) gives

$$\dot{\phi}^2 \cos\theta_0 - \gamma\dot{\phi} + \frac{a}{2} = 0$$

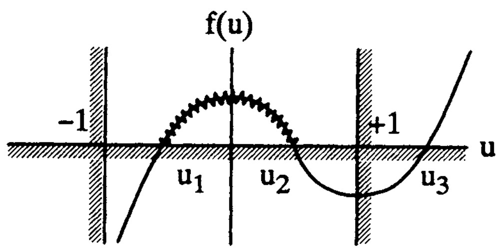


Fig. 11-5

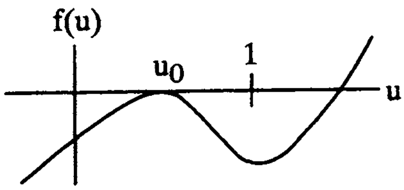


Fig. 11-6

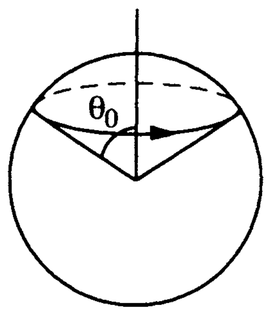


Fig. 11-7

with solution

$$\dot{\phi}_{1,2} = \frac{\gamma}{2\cos\theta_0}\left[1 \pm \left(1 - \frac{2a\cos\theta_0}{\gamma^2}\right)^{1/2}\right] \tag{11.20}$$

Provided

$$\gamma^2 \geq 2a\cos\theta_0$$

that is

$$\left(\dot{\psi} + \dot{\phi}\cos\theta_0\right)^2 \geq \frac{4Img\ell}{J^2} \tag{11.21}$$

there will be two real roots, corresponding to a fast and a slow precession. The inequality will hold when the spin $\dot{\psi}$ is sufficiently high. To visualize the motion, consider the path traced out by body axis z on a sphere (Fig. 11-7). It traces out a circle at cone angle θ_0 at rate $\dot{\phi}_1$ or $\dot{\phi}_2$. This is called steady, or regular, precession.

Another special case is $u_0 = 1(\theta = 0)$ for which the z axis stays vertical. This can be satisfied in either of two ways as shown on Fig. 11-8 (only one is stable). This is called the *sleeping* top. It takes special

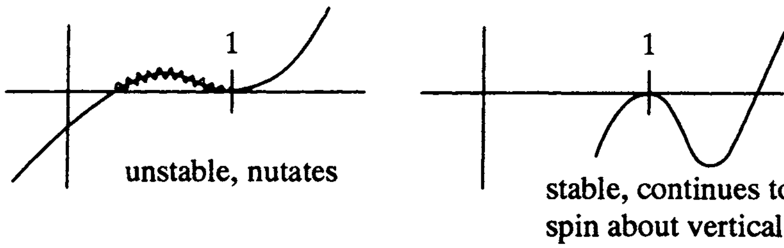


Fig. 11-8

initial conditions $\theta(0)$, $\psi(0)$, $\phi(0)$, $\dot{\theta}(0)$, $\dot{\psi}(0)$, $\dot{\phi}(0)$ to produce these special cases.

Next consider the special case of initial conditions:

$$\theta(0) = \theta_2 , \qquad \dot{\theta}(0) = 0 , \qquad \dot{\phi}(0) = 0.$$

That is, the top is released with no precession and no nutation at some angle θ_2. Substituing these initial conditions into Eqns. (11.14) and using Eqns. (11.18):

$$\begin{aligned} J\dot{\psi} &= P_2 = \gamma I \\ P_2 u_2 &= P_1 = \beta I \end{aligned} \tag{11.22}$$

Thus at $t = 0$ Eqn. (11.19) becomes

$$0 = \left(1 - u_2^2\right)(\alpha - a u_2)$$

so that $\alpha = a u_2$, where $u_2 = \cos\theta_2$. Then for any time,

$$\begin{aligned} f(u) &= (1 - u^2)a(u_2 - u) - \gamma^2 (u_2 - u)^2 \\ &= (u_2 - u)\left[(1 - u^2)a - \gamma^2 (u_2 - u)\right] \end{aligned} \tag{11.23}$$

Clearly, $u = u_2$ is one zero as expected; the other two are given by

$$(1 - u^2)a - \gamma^2 (u_2 - u) = 0$$

which implies that

$$u_{1,3} = \frac{\gamma^2}{2a} \pm \left(\frac{\gamma^4}{4a^2} - \frac{\gamma^2}{2} u_2 + 1\right)^{\frac{1}{2}} \tag{11.24}$$

As mentioned previously, only the lower one is physically possible (see Fig. 11-5):

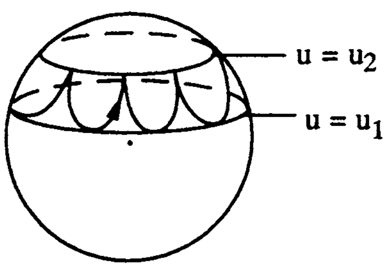
$$u_1 = \cos\theta_1 = \frac{\gamma^2}{2a} - \left(\frac{\gamma^4}{4a^2} - \frac{\gamma^2}{a}\cos\theta_2 + 1\right)^{\frac{1}{2}} \tag{11.25}$$

Therefore the z axis falls from θ_2 to θ_1 and then oscillates between the two values. From the second of Eqns. (11.14), $\dot{\phi}$ is

$$\dot{\phi} = \frac{\gamma(u_2 - u)}{1 - u^2} \tag{11.26}$$

Hence both $\dot{\phi}$ and $\dot{u}$, and thus $\dot{\theta}$ as well, are zero simultaneously at $u = u_2$; geometrically, such a point is cusp. The motion is thus as follows: After release, the top falls under gravity but then begins to precess and nutate (Fig. 11-9). Essentially, the decrease in potential energy is accounted for by an increase in kinetic energy of the same amount. When $u = u_2$ is again reached, the initial state is duplicated.

More generally, if the initial conditions are $\dot{\phi}(0) \neq 0$, we get either of the two cases shown on Fig. 11-10.



$\dot{\theta}(0) = \dot{\phi}(0) = 0$

Fig. 11-9

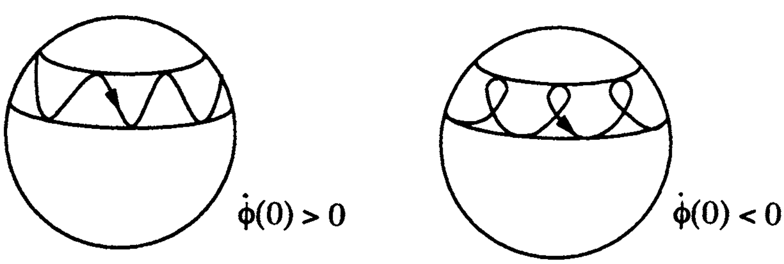


Fig. 11-10

11.3 Some Applications

Precession of the Equinoxes. Because the earth had an initial spin on its polar axis when formed and because it is an oblate spheroid (slightly flattened at the poles), it acts like a top (Fig. 11-11). The torque is due to gravitational attraction, primarily by the sun and moon, and would be zero if the earth was spherical. This torque is extremely weak and gives a processional period of 26,000 years; in 80 years the spin axis precesses 1°.

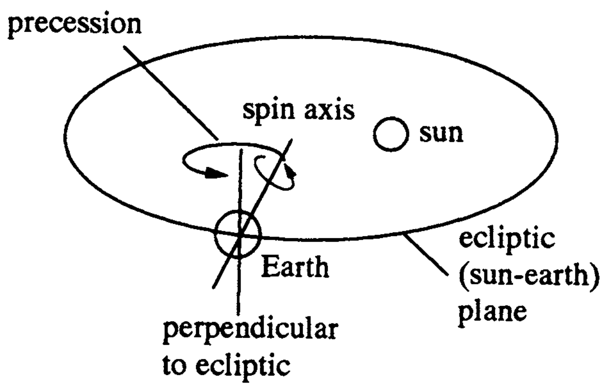


Fig. 11-11

Gyroscope. Consider now a spinning, heavy body with no gravity torque ($\ell = 0$) and constrained such that

$$\theta = \frac{\pi}{2}\,, \qquad \dot{\theta} = 0\,, \qquad \dot{\psi} = \text{constant} \tag{11.27}$$

Lagrange's equations are

$$\begin{aligned} &\frac{d}{dt}\left[I\dot{\phi}\sin^2\theta + J\left(\dot{\psi} + \dot{\phi}\cos\theta\right)\cos\theta\right] - Q_\phi = 0 \\ &\frac{d}{dt}\left[J\left(\dot{\psi} + \dot{\phi}\cos\theta\right)\right] - Q_\psi = 0 \\ &\frac{d}{dt}\left(I\dot{\theta}\right) - I\dot{\phi}^2\sin\theta\cos\theta + J\dot{\phi}\left(\dot{\psi} - \dot{\phi}\cos\theta\right)\sin\theta - Q_\theta = 0 \end{aligned} \tag{11.28}$$

where the Q_i's are the components of the torque exerted by the bearings required to keep the motion as specified. Carrying out the differentiation, and using Eqns. (11.27),

$$Q_\phi = 0\,, \qquad Q_\psi = 0\,, \qquad Q_\theta = J\dot{\phi}\dot{\psi} \tag{11.29}$$

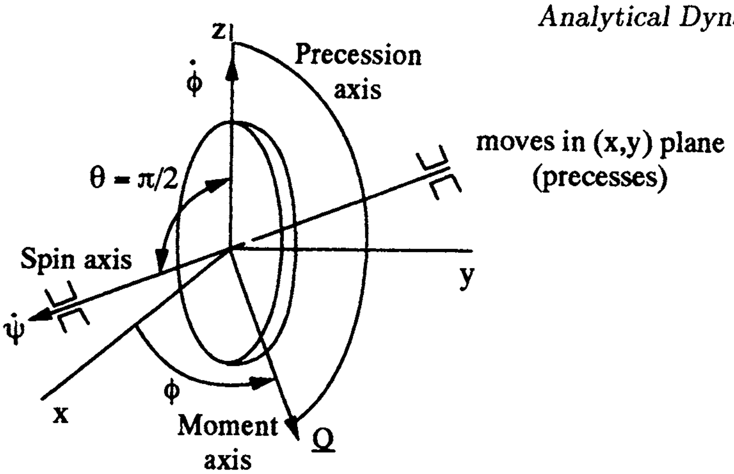


Fig. 11-12

Therefore there is a torque required in the line of nodes (Fig. 11-12), perpendicular to both the $\dot{\phi}$ and $\dot{\psi}$ axes, to keep the motion as specified.

This can be used to *detect* motion. The body is set spinning about its axis of symmetry. Then motion in a perpendicular direction can be detected by measuring moments in the bearings in the third orthogonal direction. This is one application of the gyroscope. Three such devices in perpendicular directions will detect any angular motion.

Another use of the gyroscope is as an angular reference in an *inertial* navigation system. The gyroscope is mounted in gimbals such that it nominally experiences no torque (Fig. 11-13). As the vehicle moves, the gyroscope remains fixed in orientation relative to an inertial reference frame. Therefore, measuring the orientation of the vehicle relative to that of gyroscope gives the orientation of the vehicle relative to an inertial reference frame.

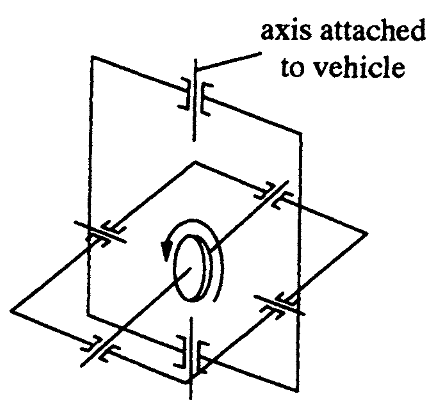


Fig. 11-13

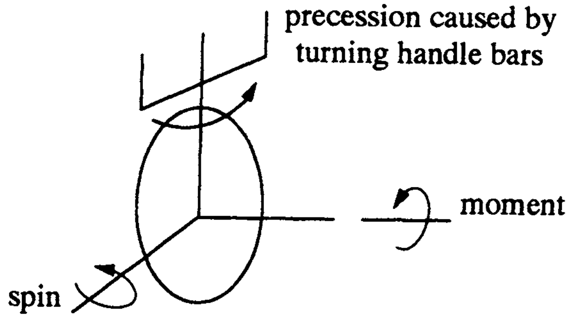


Fig. 11-14

It is very important to minimize friction in the bearings and drag on the spinning disk – these produce moments that make the gyro precess and nutate, known as drift. The latest technology is a "ring-laser" gyro which has very low friction and in which angles are measured by lasers.

Gyrocompass. This is a gyroscope fixed to the earth in such a way that the rotation of the earth causes the gyroscope to precess with a period of one day. This causes bearing torques to act in such a way that the gyroscope axis always lines up with the direction of precession, or the northernly direction.

Other Applications. Gyroscopic motion also partly explains why one can stay up on a bicycle when it's moving (Fig. 11-14) and why a football travels with a constant orientation along its path when spun and thrown[2].

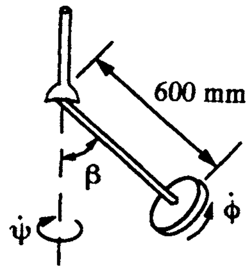
Notes

1 Other choices are the Rodriques parameters or quaternions.

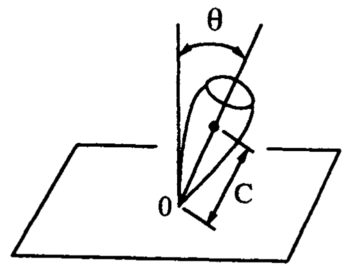
2 See Ardema, *Newton-Euler Dynamics*.

PROBLEMS

11/1. A disk of mass 2 kg and diameter 150 mm is attached to a rod AB of negligible mass to a ball-and-socket joint at A. The disk precesses at a steady rate about the vertical axis of $\dot{\psi} = 36$ rpm and the rod makes an angle of $\beta = 60°$ with the vertical. Determine the spin rate of the disk about rod AB.



Problem 11/1

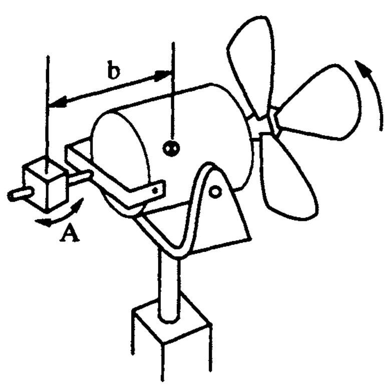


Problem 11/3

11/2. Same as Problem 11/1 except that $\beta = 30°$.

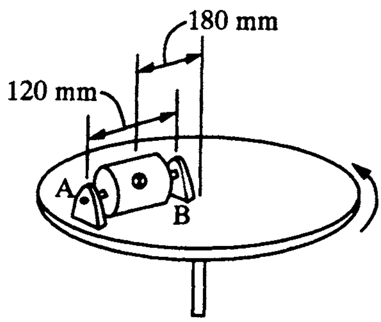
11/3. The figure shows a top weighting 3 oz. The radii of gyration of the top are 0.84 in. and 1.80 in. about the axis of symmetry and about a perpendicular axis passing through the support point 0, respectively. The length $C = 1.5$ in., the steady spin rate of the top about it's axis is 1800 rpm, and $\theta = 30°$. Determine the two possible rates of precession.

11/4. A fan is made to rotate about the vertical axis by using block A to create a moment about a horizontal axis. The parts of the fan that spin when it is turned on have a combined mass of 2.2 kg with a radius of gyration of 60 mm about the spin axis. The block A may be adjusted. With the fan turned off, the unit is balanced when $b = 180$ mm. The fan spins at a rate of 1725 rpm with the fan turned on. Find the value of b that will produce a steady precession about the vertical of 0.2 rad/s.



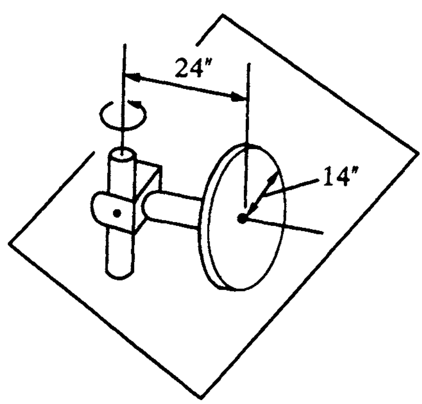
Problem 11/4

11/5. The motor shown has a total mass of 10 Kg and is attached to a rotating disk. The rotating components of the motor have a combined mass of 2.5 Kg and a radius of gyration of 35 mm. The motor rotates with a constant angular speed of 1725 rpm in a counter clockwise direction when viewed from A to B, and the turntable revolves about a vertical axis at a constant rate of 48 rpm in the direction shown. Determine the forces in the bearings A and B.



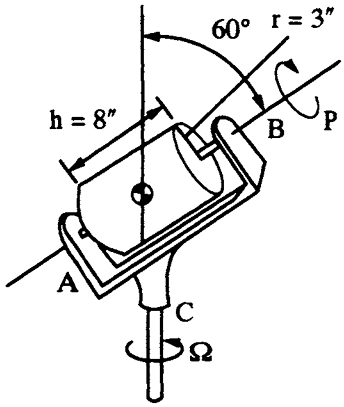
Problem 11/5

11/6. A rigid homogeneous disk of weight 96.6 lb. rolls on a horizontal plane on a circle of radius 2 ft. The steady rate of rotation about the vertical axis is 48 rpm. Determine the normal force between the wheel and the horizontal surface. Neglect the weight of all components except the disk.



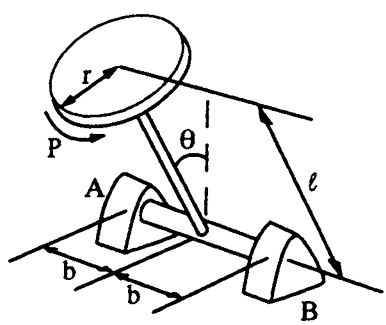
Problem 11/6

11/7. The 64.4 lb homogeneous cylinder is mounted in bearings at A and B to a bracket which rotates about a vertical axis. If the cylinder spins at steady rate $p = 50$ rad/s and the bracket at 30 rad/s, compute the moment that the assembly exerts on the shaft at C. Neglect the mass of everything except the cylinder.



Problem 11/7

11/8. A homogeneous thin disk of mass m and radius r spins on its shaft at a steady rate P. This shaft is rigidly connected to a horizontal shaft that rotates in bearings at A and B. If the assembly is released from rest at the vertical position ($\theta = 0$, $\dot{\theta} = 0$), determine the forces in the bearing at A and B as the horizontal position ($\theta = \pi/2$) is passed. Neglect the masses of all components except that of the disk.



Problem 11/8