Chapter 1

Review of Newtonian Dynamics

1.1 Basic Concepts

Assumptions. Classical mechanics rests on three basic assumptions:

1. The physical world is a three dimensional Euclidean space. This implies that the Pythagorean theorem, vector addition by parallelograms, and all elementary geometry and trigonometry are valid.

2. There exist inertial (Galilean) reference frames in this space. An inertial frame is one in which Newton's three laws hold to a sufficient degree of accuracy. We generally will take reference frames fixed relative to the surface of the earth to be inertial.

3. The quantities mass and time are invariant, that is, they are measured as the same by all observers.

4. Physical objects are particles or collections of particles constituting rigid bodies.

Assumptions (1) – (3) were regarded as laws of physics at one time; now they are regarded as engineering approximations.[1] Assumption (4) is clearly an approximation; all known materials deform under forces, but this deformation is frequently negligible.

Newton's Laws. Let $\sum \underline{F}$ be the resultant (vector sum) of all the forces acting on a mass particle of mass m. Then Newton's Second Law states that:[2]

$$\sum \underline{F} = m\underline{a} \tag{1.1}$$

where $\underline{a} = d^2\underline{r}/dt^2 = \ddot{\underline{r}}$ is the acceleration of the mass particle, and where $\underline{r}$ is the position vector of the mass particle in an inertial frame of reference (Fig. 1-1) and $d(\ )/dt = (\dot{\ })$ is the time derivative in that frame. That is, force is proportional to acceleration with proportionality constant m.

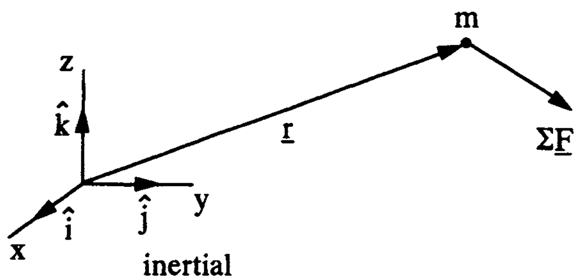


Fig. 1-1

Newton's Third Law states that given any two particles p_1 and p_2 with masses m_1 and m_2, the force exerted by p_1 on p_2, say $\underline{F}_{21}$, is equal and opposite to that exerted by p_2 on p_1, $\underline{F}_{12}$, and these forces act on a line adjoining the two particles (Fig. 1-2):

$$\underline{F}_{12} = |\underline{F}_{12}|\,\hat{e} = -\underline{F}_{21} \tag{1.2}$$

where $\hat{e}$ is a unit vector in direction $\underline{F}_{12}$.

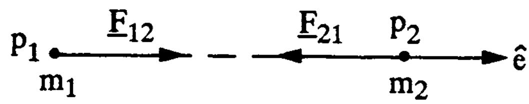


Fig. 1-2

Newton's First Law states that if $\sum \underline{F} = \underline{0}$ then $\underline{v}(t) = \underline{\text{constant}}$. This "Law" is therefore a consequence of the Second Law.

Newton stated these laws for a single particle, as we have just done; L. Euler and others generalized them to a rigid body, that is a collection of particles whose relative positions are fixed.

Definitions. The subject of mechanics is conveniently divided into branches as follows:

1. Statics is that branch of mechanics concerned with the special case $\underline{v}(t) = \underline{0}$. This implies that $\underline{a}(t) = \underline{0}$ and consequently $\sum \underline{F} = \underline{0}$.

2. Dynamics is that branch of mechanics concerned with $\underline{v}(t) \neq \underline{\text{constant}}$.

3. Kinematics is that branch of dynamics concerned with motion independent of the forces that produce the motion.

4. Kinetics is that branch of dynamics concerned with the connection between forces and motion, as defined by Newton's three laws.

Basic Problems in Kinetics. It is clear that there are two basic problems:

1. Given the forces, find the motion (that is, the position, velocity, and acceleration as a function of time). This is sometimes called the "forward" or "dynamics" problem.

2. Given the motion, find the forces that produced it (actually, one can usually only find the resultant force). Statics is a special case of this. This is sometimes called the "backward" or "controls" problem.

In practice, mixed problems frequently arise; for example, given the motion and some of the forces, what is the resultant of the remaining force(s)?

Reasons for Reviewing Newtonian Dynamics. In the rest of this chapter, we will review elementary Newtonian dynamics, for the following reasons:

1. Some of this material will be needed later.

2. It allows a chance in a familiar setting to get used to the approach and notation used throughout this book.

3. It gives insight that leads to other approachs to dynamics.

4. It provides a benchmark with which to measure the worth of these new approachs.

Many of the following results will be presented without proof.

1.2 Kinematics and Newtonian Particle Dynamics

Motion of a Point. Consider a point P moving along a curve C relative to a reference frame $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$. Denote the position vector of P at time t by $\underline{r}(t)$. Then the velocity of P is defined as (Fig. 1-3):

$$\underline{v}(t) = \frac{d\underline{r}}{dt} = \dot{\underline{r}} = \lim_{\Delta t \to 0} \frac{\underline{r}(t+\Delta t) - \underline{r}(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta \underline{r}}{\Delta t} \tag{1.3}$$

Similarly the acceleration of the point is defined as:

$$\underline{a}(t) = \frac{d\underline{v}}{dt} = \dot{\underline{v}} = \ddot{\underline{r}} = \lim_{\Delta t \to 0} \frac{\underline{v}(t+\Delta t) - \underline{v}(t)}{\Delta t} \tag{1.4}$$

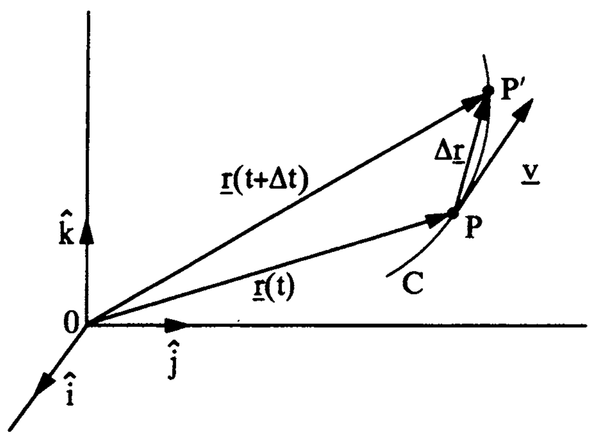


Fig. 1-3

Note that $\underline{v}(t)$ is tangent to the curve C. The magnitude of the velocity vector, $v(t) = |\underline{v}(t)|$, is called the speed of the point.

Rectangular Components. To obtain scalar equations of motion, the vectors of interest are written in components. In rectangular components (Fig. 1-4), the position vector is given by:

$$\underline{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k} \tag{1.5}$$

From Eqns. (1.3) and (1.4) $\underline{v}$ and $\underline{a}$ are:

$$\underline{v} = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath} + \dot{z}\hat{k} \tag{1.6}$$

$$\underline{a} = \ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath} + \ddot{z}\hat{k} \tag{1.7}$$

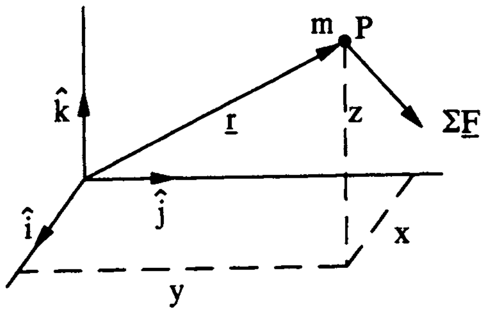


Fig. 1-4

We call (x, y, z) the rectangular components of position (or the rectangular coordinates) of point P. Similarly, $(\dot{x}, \dot{y}, \dot{z})$ and $(\ddot{x}, \ddot{y}, \ddot{z})$ are the rectangular components of velocity and acceleration, respectively. The distance of P from the origin and the speed of P are given by:

$$r = |\underline{r}| = \sqrt{x^2 + y^2 + z^2}\,, \qquad v = |\underline{v}| = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} \tag{1.8}$$

Expressing the resultant force on the mass m at point P in rectangular components

$$\sum \underline{F} = \sum F_x \hat{\imath} + \sum F_y \hat{\jmath} + \sum F_z \hat{k} \tag{1.9}$$

and combining this with Eqns. (1.1) and (1.7) gives three scalar equations of motion:

$$\sum F_x = m\ddot{x}\,, \qquad \sum F_y = m\ddot{y}\,, \qquad \sum F_z = m\ddot{z} \tag{1.10}$$

This is a sixth order system of ordinary differential equations.

Any vector in three dimensions can be written as a linear combination of any three linearly independent vectors, called basis vectors. In this book, all basis vectors will be triads of mutually-orthogonal, right-handed unit vectors and will be denoted by "hats".

Normal – Tangential Components. It is possible, and frequently desirable, to express $\underline{r}$, $\underline{v}$, and $\underline{a}$ in components along directions other than $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$. For planar motion (take this to be in the (x, y) plane), normal-tangential components are frequently useful. Introduce unit vectors tangent and normal to $\underline{v}$ as shown in Fig. 1-5.

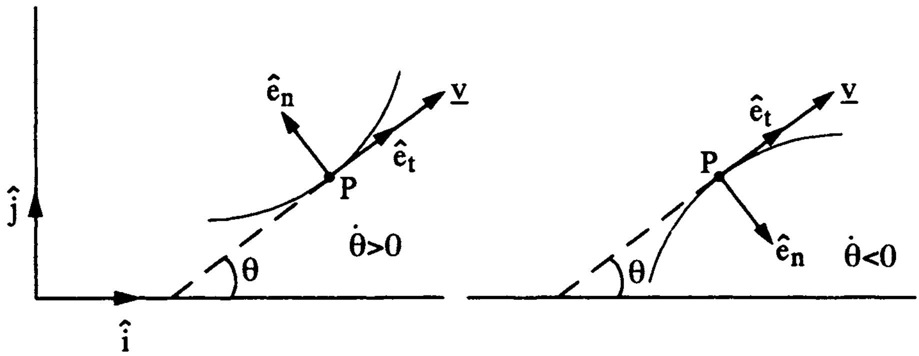


Fig. 1-5

The velocity and acceleration vectors expressed in these components are:

$$\underline{v} = v\hat{e}_t \tag{1.11}$$

$$\underline{a} = \frac{d\underline{v}}{dt} = \dot{v}\hat{e}_t + v\frac{d\hat{e}_t}{dt} \tag{1.12}$$

It is clear that in general $\hat{e}_t$ will vary with time and thus $d\hat{e}_t/dt \neq \underline{0}$. We must consider the two cases shown on Fig. 1-5 separately. First, for $\dot{\theta} > 0$ (Fig. 1-6):

$$\hat{e}_t = \cos\theta\hat{i} + \sin\theta\hat{j}$$

$$\hat{e}_n = -\sin\theta\hat{i} + \cos\theta\hat{j}$$

$$\frac{d\hat{e}_t}{dt} = -\dot{\theta}\sin\theta\hat{i} + \dot{\theta}\cos\theta\hat{j} = \dot{\theta}\hat{e}_n$$

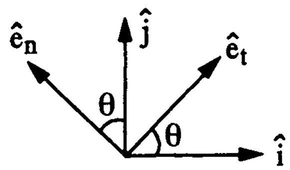


Fig. 1-6

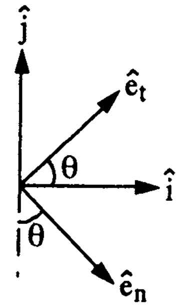


Fig. 1-7

Next for $\dot{\theta} < 0$ (Fig. 1-7):

$$\hat{e}_t = \cos\theta\hat{\imath} + \sin\theta\hat{\jmath}$$

$$\hat{e}_n = \sin\theta\hat{\imath} - \cos\theta\hat{\jmath}$$

$$\frac{d\hat{e}_t}{dt} = -\dot{\theta}\sin\theta\hat{\imath} + \dot{\theta}\cos\theta\hat{\jmath} = -\dot{\theta}\hat{e}_n$$

Thus for both cases:

$$\frac{d\hat{e}_t}{dt} = |\dot{\theta}|\hat{e}_n \tag{1.13}$$

so that, from Eqn. (1.12),

$$\underline{a} = \dot{v}\hat{e}_t + v|\dot{\theta}|\hat{e}_n \tag{1.14}$$

We call $(\dot{v},\ v|\dot{\theta}|)$ the tangential and normal components of acceleration.

Now let $\rho = \dfrac{v}{|\dot{\theta}|} \geq 0$. Suppose the motion is on a circle of radius R (Fig. 1-8); then

$$S = R\theta \Longrightarrow \dot{S} = R\dot{\theta} = v \Longrightarrow R = \frac{v}{\dot{\theta}}$$

Thus we call in general ρ the *radius of curvature.* Hence we may write

$$\underline{a} = \dot{v}\hat{e}_t + \frac{v^2}{\rho}\hat{e}_n \tag{1.15}$$

Note that $\hat{e}_n$ is undefined and $\rho = \infty$ for $\dot{\theta} = 0$, i.e. for rectilinear motion or at a point of inflection (Fig. 1-9). If the resultant force acting on a particle is expressed in normal-tangential components, $\sum \underline{F} = \sum F_t\hat{e}_t + \sum F_n\hat{e}_n$, then Eqns. (1.1) and (1.15) give the scalar equations of motion:

$$\sum F_t = m\dot{v}\ , \qquad \sum F_n = m\frac{v^2}{\rho} \tag{1.16}$$

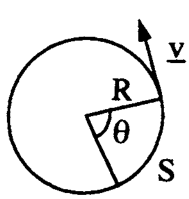


Fig. 1-8

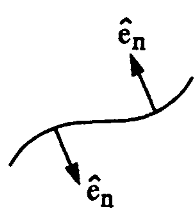


Fig. 1-9

Cylindrical and Spherical Coordinates and Components. In three-dimensional (3-D) motion it is often advantageous to resolve the velocity and acceleration into cylindrical or spherical components; only the velocity components will be given here. The cylindrical coordinates (r, ϕ, z) are shown on Fig. 1-10. From the geometry, the cylindrical and rectangular coordinates are related by

$$\begin{aligned} x &= r\cos\phi \\ y &= r\sin\phi \\ z &= z \end{aligned} \tag{1.17}$$

The velocity expressed in cylindrical components is

$$\underline{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta + \dot{z}\hat{k} \tag{1.18}$$

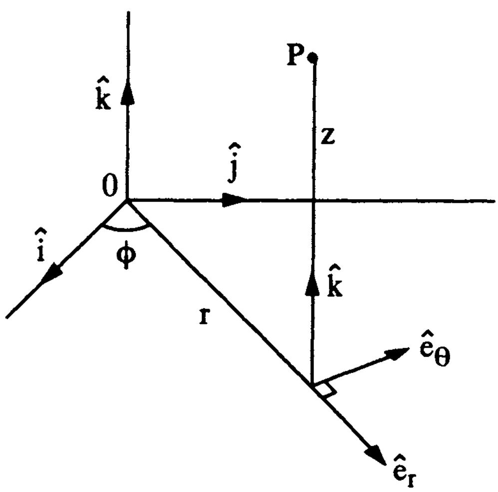


Fig. 1-10

Note that if $z =$ constant, (r, θ) are just the familiar plane polar coordinates.

The spherical coordinates (r, θ, ϕ) are shown on Fig. 1-11. The relation to rectangular coordinates is given by

$$\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned} \tag{1.19}$$

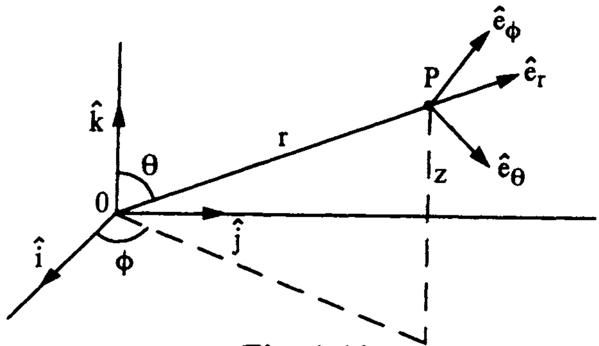


Fig. 1-11

and the velocity in spherical components is

$$\underline{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta + r\dot{\phi}\sin\theta\hat{e}_\phi \tag{1.20}$$

Spherical coordinates and components are particularly advantageous for central force motion (Chapter 10).

Relative Velocity. It is sometimes necessary to relate the motion of a point as measured in one reference frame to the motion of the same point as measured in another frame moving with respect to (w.r.t.) the first one. First consider two reference frames moving with respect to each other such that one axis, say z, is always aligned (Fig. 1-12). Define the *angular velocity* and *angular acceleration* of frame $\{\hat{i}, \hat{j}\}$ w.r.t. frame $\{\hat{I}, \hat{J}\}$ by:

$$\underline{\omega} \;=\; \dot{\theta}\hat{k} = \dot{\theta}\hat{K} \tag{1.21}$$

$$\underline{\alpha} \;=\; \underline{\dot{\omega}} = \ddot{\theta}\hat{k} = \ddot{\theta}\hat{K} \tag{1.22}$$

where we have used the fact that $\hat{k} = \hat{K}$ is a constant vector.

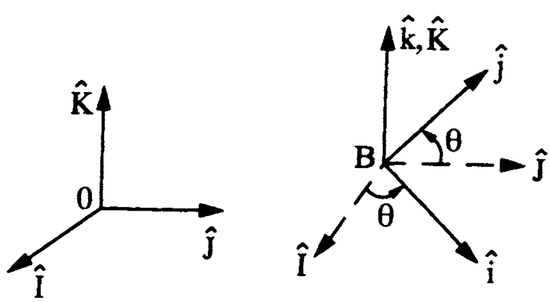


Fig. 1-12

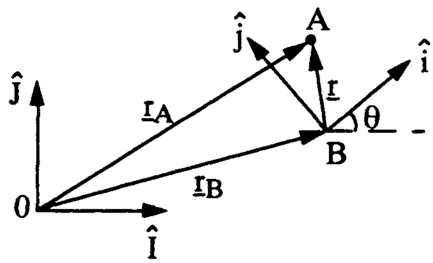


Fig. 1-13

Figure 1-13 shows a point A moving in a plane w.r.t. to two frames which are also moving w.r.t. each other.

Let

$$\begin{aligned}\frac{D}{Dt} &= \text{time derivative w.r.t.}\{\hat{I},\hat{J}\}\\ \frac{d}{dt} &= \text{time derivative w.r.t.}\{\hat{\imath},\hat{\jmath}\}\end{aligned}$$

For a scalar Q, $DQ/Dt = dQ/dt$, but for a vector $\underline{Q}$, $D\underline{Q}/Dt \neq d\underline{Q}/dt$, in general. The relation between the two is given by the *basic kinematic equation*

$$\frac{D\underline{Q}}{Dt} = \frac{d\underline{Q}}{dt} + \underline{\omega} \times \underline{Q} \tag{1.23}$$

which holds for any vector $\underline{Q}$. We are now ready to derive the relative velocity equation. From Fig. 1-13:

$$\underline{r}_A = \underline{r}_B + \underline{r} \tag{1.24}$$

Differentiating and applying Eqn. (1.23):

$$\frac{D\underline{r}_A}{Dt} = \frac{D\underline{r}_B}{Dt} + \frac{D\underline{r}}{Dt}$$

$$\begin{aligned}\underline{v}_A &= \underline{v}_B + \frac{d\underline{r}}{dt} + \underline{\omega} \times \underline{r}\\ \underline{v}_A &= \underline{v}_B + \underline{v}_r + \underline{\omega} \times \underline{r} \qquad (1.25)\end{aligned}$$

where

$$\begin{aligned}\underline{v}_A &= D\underline{r}_A/Dt = \underline{\text{velocity}} \text{ of } A \text{ w.r.t. } \{\hat{I},\hat{J}\}\\ \underline{v}_B &= D\underline{r}_B/Dt = \underline{\text{velocity}} \text{ of } B \text{ w.r.t. } \{\hat{I},\hat{J}\}\\ \underline{v}_r &= d\underline{r}/dt = \underline{\text{velocity}} \text{ of } A \text{ w.r.t. } \{\hat{\imath},\hat{\jmath}\}\end{aligned}$$

These results also apply to general 3-D motion provided that the angular velocity is suitably defined. This is most conveniently done using Euler's Theorem. This theorem states that any displacement of one reference frame relative to another may be replaced by a simple rotation about some line. The motion of the one frame w.r.t. the other may then be thought of as a sequence of such rotations. At any instant, $\underline{\omega}$ is defined as the vector whose direction is the axis of rotation and whose magnitude is the rotation rate. With this definition of $\underline{\omega}$, Eqns. (1.23) and (1.25) are valid for 3-D motion. For a full discussion of 3-D kinematics see Ardema, *Newton-Euler Dynamics*.

Example. Car B is rounding a curve of radius R with speed v_B (Fig. 1-14). Car A is traveling toward car B at speed v_A and is distance x from car B at the instant shown. We want the velocity of car A as seen by car B. The cars are modelled as points.

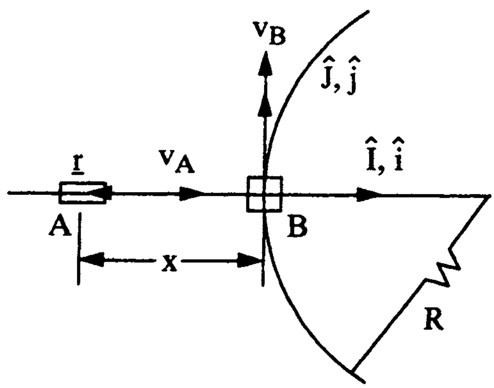


Fig. 1-14

Introduce reference frames:

$\{\hat{I}, \hat{J}\}$ fixed in ground (the data is given in this frame)

$\{\hat{\imath}, \hat{\jmath}\}$ fixed in car B (the answer is required in this frame)

Applying Eqn. (1.25):

$$\underline{v}_A = \underline{v}_B + \underline{v}_r + \underline{\omega} \times \underline{r} \qquad \underline{v}_A = v_A\hat{I}$$

$$\underline{v}_B = v_B\hat{J}$$

$$\underline{v}_r = \underline{v}_A - \underline{v}_B - \underline{\omega} \times \underline{r} \qquad \underline{r} = -x\hat{I}$$

$$= v_A\hat{I} - v_B\hat{J} - \left(-\frac{v_B}{R}\hat{K}\right) \times (-x\hat{I}) \qquad \underline{\omega} = -\frac{v_B}{R}\hat{K}$$

$$\underline{v}_r = v_A\hat{I} - \left(v_B + \frac{v_B x}{R}\right)\hat{J}$$

1.3 Work and Energy

Definitions. Suppose a force $\underline{F}$ acts on a particle of mass m as it moves along curve C (Fig. 1-15). Define the *work done* by $\underline{F}$ during the displacement of m from $\underline{r}_0$ to $\underline{r}_1$ along C by

$$U_{0,1} = \int_{\underline{r}_0}^{\underline{r}_1} \underline{F} \cdot d\underline{r} \tag{1.26}$$

Since $\underline{v} = \dfrac{d\underline{r}}{dt}$, this may be written as

$$U_{0,1} = \int_{t_0}^{t_1} \underline{F} \cdot \underline{v}\, dt = \int_{t_0}^{t_1} P\, dt \tag{1.27}$$

where $P = \underline{F} \cdot \underline{v}$ is called the power.

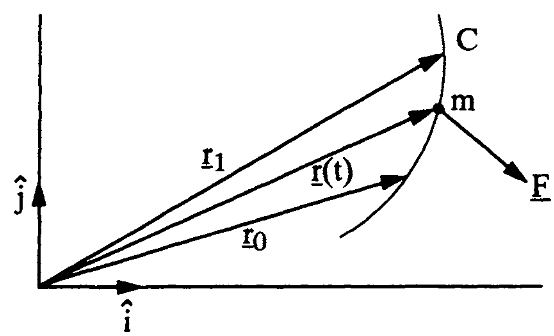


Fig. 1-15

Now suppose $\underline{F}$ is the resultant of all forces and $\{\hat{\imath}, \hat{\jmath}\}$ is an inertial frame; then, Newton's Second Law, Eqn. (1.1), holds:

$$m\,\frac{d\underline{v}}{dt} = \underline{F}$$

Taking the scalar product of both sides with $\underline{v}$ and inserting the result in Eqn. (1.27):

$$m\,\frac{d\underline{v}}{dt} \cdot \underline{v} = \underline{F} \cdot \underline{v}$$

$$\begin{aligned} U_{0,1} &= \int_{t_0}^{t_1} m \frac{d\underline{v}}{dt} \cdot \underline{v}\, dt = m \int_{t_0}^{t_1} \underline{v} \cdot d\underline{v} = \frac{1}{2} m \left(v_1^2 - v_0^2 \right) \\ &= T_1 - T_0 = \Delta T_{0,1} \end{aligned} \tag{1.28}$$

where the *kinetic energy* of the particle is defined as:

$$T = \frac{1}{2} m v^2 \tag{1.29}$$

In words, Eqn. (1.28) states that the change in kinetic energy from position 0 to position 1 is equal to the work done by the resultant force from 0 to 1.

Potential Energy. In rectangular coordinates,

$$\begin{aligned} &\underline{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k} \; ; \\ &d\underline{r} = dx\hat{\imath} + dy\hat{\jmath} + dz\hat{k} \; ; \\ &\underline{F} = F_x\hat{\imath} + F_y\hat{\jmath} + F_z\hat{k} \end{aligned}$$

and if $\underline{F}$ is a function only of position $\underline{r}$, Eqn. (1.26) gives

$$U_{0,1} = \int_{\underline{r}_0}^{\underline{r}_1} \underline{F} \cdot d\underline{r} = \int_{x_0}^{x_1} F_x dx + \int_{y_0}^{y_1} F_y dy + \int_{z_0}^{z_1} F_z dz \tag{1.30}$$

Generally, this integration will depend on path C, and not just the end points.

Recall that the *gradient* of a scalar function of a vector argument $V(\underline{r})$ in rectangular coordinates is

$$\text{grad } V(\underline{r}) = \frac{\partial V}{\partial x}\hat{\imath} + \frac{\partial V}{\partial y}\hat{\jmath} + \frac{\partial V}{\partial z}\hat{k} \tag{1.31}$$

Suppose that $\underline{F}(\underline{r})$ is such that there exists a function $V(\underline{r})$ such that

$$\underline{F}(\underline{r}) = -\text{grad } V(\underline{r}) = F_x\hat{\imath} + F_y\hat{\jmath} + F_z\hat{k} \tag{1.32}$$

Then, comparing Eqns. (1.31) and (1.32), and writing $V = V(x, y, z)$,

$$F_x = -\frac{\partial V}{\partial x}, \qquad F_y = -\frac{\partial V}{\partial y}, \qquad F_z = -\frac{\partial V}{\partial z} \tag{1.33}$$

so that

$$\underline{F} \cdot d\underline{r} = -\frac{\partial V}{\partial x} dx - \frac{\partial V}{\partial y} dy - \frac{\partial V}{\partial z} dz = -dV \tag{1.34}$$

Therefore, from Eqn. (1.26),

$$U_{0,1} = \int_{V_0}^{V_1} (-dV) = -(V_1 - V_0) = -\Delta V_{0,1} \tag{1.35}$$

This shows that now the work done by $\underline{F}$ depends only on the endpoints and *not* on the path C.

$V(\underline{r})$ is called a *potential energy function* and $\underline{F}(\underline{r})$ with this property is called a *conservative force.*

Gravitation. Consider two masses with the only force acting on them being their mutual gravitation (Fig. 1-16). If m_e (the earth, for example) $\gg m$ (an earth satellite, for example), we may take m_e as fixed in an inertial frame. If the two bodies are spherically symmetric they can be regarded as particles for the purpose of determing the gravitational force.

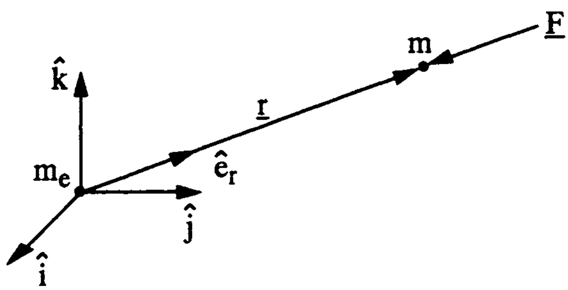


Fig. 1-16

Newton's law of gravitation gives the force acting on mass m as

$$\underline{F} = -\frac{K m_e m}{r^2} \hat{e}_r \tag{1.36}$$

where $K = 6.673 \times 10^{-11}$ m^3/(Kg$\cdot$ sec^2) is the universal gravitational constant.

Because

$$\underline{r} = r\hat{e}_r$$

$$\underline{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$r = (x^2 + y^2 + z^2)^{1/2}$$

we have

$$\underline{F} = -\frac{K m_e m}{r^3} (x\hat{i} + y\hat{j} + z\hat{k}) \tag{1.37}$$

so that

$$F_x = -Km_em\frac{x}{(x^2+y^2+z^2)^{3/2}}$$
$$F_y = -Km_em\frac{y}{(x^2+y^2+z^2)^{3/2}} \tag{1.38}$$
$$F_z = -Km_em\frac{z}{(x^2+y^2+z^2)^{3/2}}$$

Thus the gravitational force is conservative with potential energy function given by

$$V = -\frac{Km_em}{(x^2+y^2+z^2)^{1/2}} \tag{1.39}$$

This is verified by observing that Eqns. (1.33) are satisfied for this function.

In central force motion, as mentioned earlier, it is usually best to use spherical coordinates. Writing $V = V(r, \theta, \phi)$, the gradient of V in spherical components is

$$\text{grad } V(\underline{r}) = \frac{\partial V}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{e}_\phi \tag{1.40}$$

Thus the gravitational potential function is

$$V = -\frac{Km_em}{r} \tag{1.41}$$

which could have been obtained directly from Eqn. (1.39).

For motion over short distances on or near the surface of the earth it is usually sufficient to take the gravitational force as a constant in both magnitude and direction (Fig. 1-17). The force acting on the particle is

$$\underline{F} = F_x\hat{\imath} = -mg\hat{\imath}$$

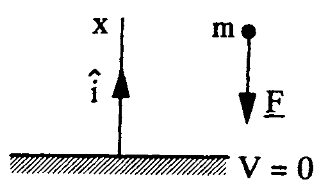


Fig. 1-17

Therefore the gravitational potential energy function is

$$V(x) = mgx \tag{1.42}$$

because $-\dfrac{\partial V}{\partial x} = -mg = F_x$.

Energy Equation. Suppose a number of forces act on m, some conservative and some not. Then

$$\underline{F}_i^c = -\text{grad } V_i \tag{1.43}$$

for each conservative force. The resultant force is

$$\underline{F} = \sum_{i=1}^{n_c} \underline{F}_i^c + \sum_{j=1}^{n_{nc}} \underline{F}_j^{nc} = \sum_i (-\text{grad } V_i) + \sum_j \underline{F}_j^{nc}$$

The work done is

$$U_{0,1} = \int_{\underline{r}_0}^{\underline{r}_1} \underline{F} \cdot d\underline{r} = -\sum_i \Big[V_i(\underline{r}_1) - V_i(\underline{r}_0) \Big] + \int_{\underline{r}_0}^{\underline{r}_1} \sum_j \underline{F}_j^{nc} \cdot d\underline{r}$$

Using $U_{0,1} = \Delta T_{0,1}$, this becomes

$$\Delta T_{0,1} = -\Delta V_{0,1} + U_{0,1}^{nc} \tag{1.44}$$

where V is the sum of all potential energies and U^{nc} is the work done by all nonconservative forces.

Let the *total mechanical energy* be defined by:

$$E = T + V \tag{1.45}$$

Then Eqn. (1.44) may be written

$$\Delta E_{0,1} = U_{0,1}^{nc} \tag{1.46}$$

and, in particular if all forces are conservative and accounted for in V,

$$\Delta E_{0,1} = 0 \tag{1.47}$$

that is, energy is conserved.

Remarks:

1. U is defined over an interval of motion but T, V, and E are defined at an instant.

2. U, T, V, and E are all scalars. Therefore, the energy equation gives only one piece of information.

3. The energy equation is a once-integrated form of Newton's Second Law; it is a relation among speeds, not accelerations.

4. The energy equation is most useful when a combination of the following factors is present: the problem is of low dimension, forces are not needed to be determined, and energy is conserved.

5. The energy equation involves only *changes* in T and V between two positions; thus adding a constant to either one does not change the equation.

1.4 Eulerian Rigid Body Dynamics

Kinetics of Particle System. First consider a collection of particles (not necessarily rigid), Fig. 1-18. Let $\{\hat{\imath}, \hat{\jmath}\}$ be an inertial reference frame and:

$$\underline{F}_i^e \;=\; \text{sum of all external } \underline{\text{forces}} \text{ on particle } i.$$
$$\underline{F}_{ij} \;=\; \text{(internal) } \underline{\text{force}} \text{ exerted by particle } j \text{ on particle } i.$$

The *center of mass* is a position, labeled G, whose position vector is given by

$$\bar{\underline{r}} = \frac{1}{m} \sum_i m_i \underline{r}_i \tag{1.48}$$

where $m = \sum_i m_i$ is the total mass.

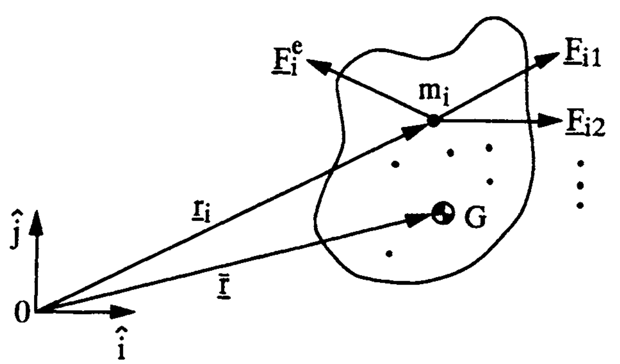


Fig. 1-18

Newton's Second Law for particle i is:

$$\underline{F}_i^e + \sum_j \underline{F}_{ij} = m_i \frac{d^2\underline{r}_i}{dt^2} = m_i\underline{a}_i$$

Sum these equations for all particles and recall from Newton's Third Law that $\underline{F}_{ij} = -\underline{F}_{ji}$ for all i, j:

$$\sum_i \underline{F}_i^e + \sum_i \sum_j \underline{F}_{ij} = \sum_i m_i \frac{d^2\underline{r}_i}{dt^2}$$

$$\sum_i \underline{F}_i^e + \underline{0} = \frac{d^2}{dt^2}\left(\sum_i m_i\underline{r}_i\right) = \frac{d^2}{dt^2}(m\underline{\overline{r}})$$

$$\sum_i \underline{F}_i^e = m \frac{d^2\underline{\overline{r}}}{dt^2} = m\underline{\overline{a}} \tag{1.49}$$

Therefore, for a system of constant mass, including a rigid body, the sum of all *external* forces equals the total mass times acceleration of the center of mass. This means that if the same force $\underline{F}$ is applied to two dissimilar rigid bodies, each having the same mass, the accelerations of their centers of mass will be the same (Fig. 1-19).

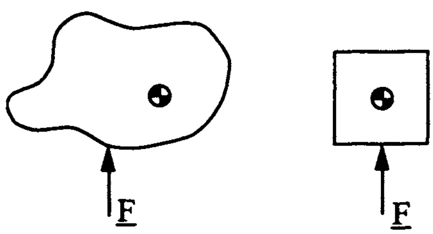


Fig. 1-19

Rigid Body. A *rigid body* is a collection of particles such that there exists a reference frame in which all particles have fixed positions in this reference frame. The *angular velocity of a rigid body* relative to a given reference frame is just the angular velocity of any body fixed reference frame w.r.t. the other frame.[3]

Planar Rigid Body Kinetics. Here we consider the 2-D motion of a rigid body, which we take to be in the $\{\hat{\imath}, \hat{\jmath}\}$ plane. Let $\{\hat{\imath}', \hat{\jmath}'\}$ be a body fixed frame and let the position of a particle in the rigid body be given by $\underline{d}_i = x_i\hat{\imath}' + y_i\hat{\jmath}'$ (Fig. 1-20).[4] The quantities $\underline{d}_i$, x_i, and y_i are all

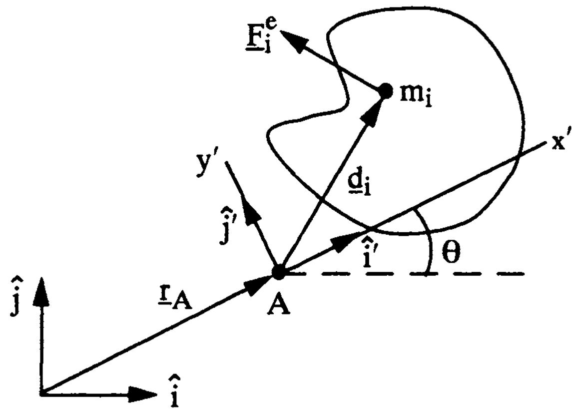


Fig. 1-20

constants. Also let the position of A, the origin of $\{\hat{\imath}', \hat{\jmath}'\}$, w.r.t. the inertial frame $\{\hat{\imath}, \hat{\jmath}\}$ be given by $\underline{r}_A = x\hat{\imath} + y\hat{\jmath}$. It is clear that the location of all the particles of the rigid body are known if the values of x, y and θ are known. We say that the body has three degrees of freedom.

The first two degrees of freedom are accounted for by Eqn. (1.49); in rectangular components:

$$\sum_i F^e_{x_i} = m\ddot{x}\ ; \qquad \sum_i F^e_{y_i} = m\ddot{y} \tag{1.50}$$

The third degree of freedom comes from relating the rate of change of angular momentum to the sum of the *external* force moments. The result is in general quite complicated.

A special case is

$$\sum_i M^e_{A_i}\hat{k} = I_A\alpha\hat{k} \Longrightarrow \sum_i M^e_{A_i} = I_A\alpha \tag{1.51}$$

Here, the sum of the moments of all external forces about the body fixed point A is (Fig. 1-21)

$$\sum_i \underline{M}^e_{A_i} = \sum_i \underline{d}_i \times \underline{F}^e_i \tag{1.52}$$

and

$$I_A = \sum_i \left(x_i'^2 + y_i'^2\right) m_i = \int_m (x'^2 + y'^2)dm \tag{1.53}$$

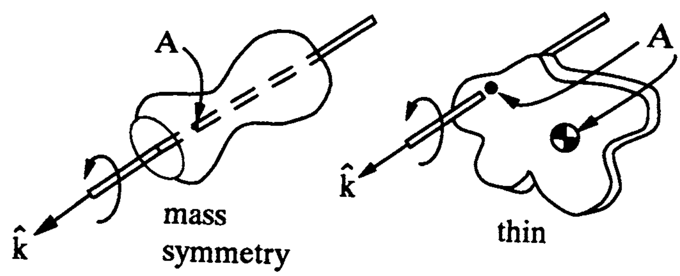


Fig. 1-21

is the mass moment of inertia of the body about the axis through A parallel to $\hat{k}$, and $\alpha = \dot{\omega} = \ddot{\theta}$ is the angular acceleration of the body. The first representation of I_A in Eqn. (1.53) considers the rigid body to be a finite collection of mass particles and the second is the limit as the number of particles tends to infinity, that is the body is regarded as a continuum.

Equation (1.51) is only valid if [5](see Fig. 1-21):

1. Point A is either G, the center of mass, or moves with constant velocity in the inertial frame; and
2. Axis $\hat{k}$ is an axis of rotational mass symmetry or the body is "thin".

Concepts of 3-D kinetics will be introduced as needed in future Chapters.

Work and Energy for Rigid Body. As before, Eqn. (1.44) applies, repeated here:

$$\Delta T_{0,1} + \Delta V_{0,1} = U_{0,1}^{np} \tag{1.54}$$

For most of the approaches to dynamics developed later in this book, an essential step in deriving the equations of motion of a dynamic system is the determination of the system's kinetic energy. If a body is considered as a collection of mass particles, its kinetic energy is, by definition,

$$T = \frac{1}{2}\sum_r m_r v_r^2 \tag{1.55}$$

Use of this formula is seldom convenient, however, and in what follows we present several alternative methods for obtaining T for rigid bodies.

First, for 2-D motion T can be obtained from

$$T = \frac{1}{2}m\bar{v}^2 + \frac{1}{2}\bar{I}\omega^2 \tag{1.56}$$

where $\overline{v}$ is the speed of the center of mass and $\overline{I}$ is the moment of inertia about an axis passing through the center of mass and parallel to the axis of rotation. As an alternative for 2-D problems

$$T = \frac{1}{2} I_A \omega^2 \tag{1.57}$$

where I_A is the moment of inertia about an axis passing through a body-fixed point that is also fixed in an inertial frame.

A result that is sometimes useful is Koenig's theorem. Consider a rigid body moving w.r.t. an inertial frame $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$, as shown on Fig. 1-22. Introduce a frame $\{\hat{\imath}', \hat{\jmath}', \hat{k}'\}$ with origin at the body's center of mass that moves in such a way that it's axes always remain parallel to those of the inertial frame (thus this frame is not body fixed). Let the coordinates of mass particle r (with mass m_r) be (x_r, y_r, z_r) and (ζ_r, η_r, ν_r) in the inertial and the other frame, respectively. Then

$$x_r = \overline{x} + \zeta_r \,, \qquad y_r = \overline{y} + \eta_r \,, \qquad z_r = \overline{z} + \nu_r$$

Koenig's theorem states that the kinetic energy of the body is given by

$$T = \frac{1}{2} m \left(\dot{\overline{x}}^2 + \dot{\overline{y}}^2 + \dot{\overline{z}}^2 \right) + \frac{1}{2} \sum_r m_r \left(\dot{\zeta}_r^2 + \dot{\eta}_r^2 + \dot{\nu}_r^2 \right) \tag{1.58}$$

where $m = \sum_r m_r$ is the mass of the body. Note that, like Eqn. (1.56), this equation divides T into a translational and a rotational part.

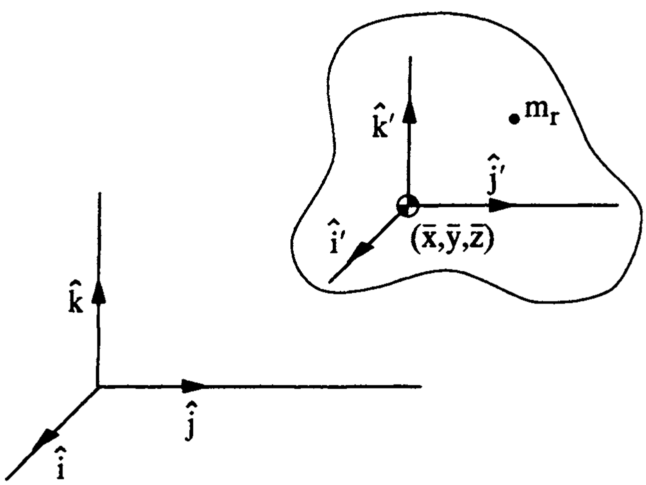


Fig. 1-22

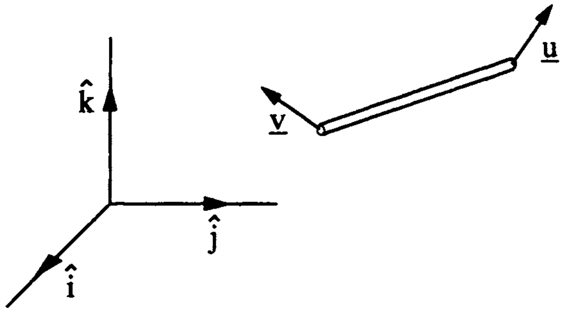


Fig. 1-23

A method of obtaining T for a rigid body for the special case in which one point of the body remains fixed in an inertial frame is given in Section 11.1.

One final result is useful for a special case. Suppose a rigid body may be idealized as a homogeneous, thin, straight rod and suppose that at some instant the velocities of the ends of the rod are $\underline{u}$ and $\underline{v}$ (Fig. 1-23). Then T of the rod is

$$T = \frac{1}{6}m(\underline{u} \cdot \underline{u} + \underline{u} \cdot \underline{v} + \underline{v} \cdot \underline{v}) \tag{1.59}$$

System of Rigid Bodies. Consider a system of constant mass, not necessarily rigid, but consisting of a number of rigid bodies. Then for the system Eqn. (1.54) applies where now T is the sum of all the kinetic energies of all the rigid bodies, V is the total potential energy of the system, and U^{nc} is the work done by all the nonconservative forces.

1.5 Examples

Simple Pendulum. A bob of mass m is suspended by an inextensible, weightless cord and moves in the (x, y) plane (Fig. 1-24). We obtain the equation of motion by three methods:

(a) $\sum \underline{F} = m\underline{a}$ in rectangular components (Eqns. 1.10):

$$\sum F_x = m\ddot{x} \quad \Longrightarrow \quad -T\sin\theta = m\ddot{x}$$

$$\sum F_y = m\ddot{y} \quad \Longrightarrow \quad T\cos\theta - mg = m\ddot{y}$$

eliminate T to get: $\quad -\dfrac{\cos\theta}{\sin\theta}\ddot{x} - g = \ddot{y}$

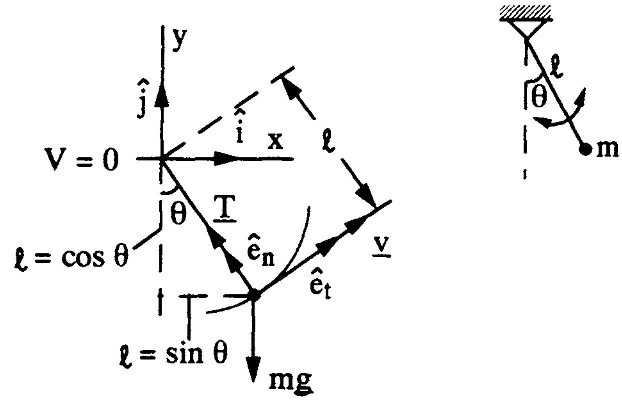


Fig. 1-24

but

$$x = \ell \sin\theta \quad \Longrightarrow \quad \ddot{x} = \ell \cos\ddot{\theta} - \ell \sin\theta\dot{\theta}^2$$

$$y = -\ell \cos\theta \quad \Longrightarrow \quad \ddot{y} = \ell \sin\theta\ddot{\theta} + \ell \cos\theta\dot{\theta}^2$$

Therefore

$$\ddot{\theta} + \frac{g}{\ell}\sin\theta = 0 \tag{1.60}$$

(b) $\sum \underline{F} = m\underline{a}$ in normal-tangential components (Eqn. 1.16):

$$\sum F_t = m\dot{v} \quad \Longrightarrow \quad -mg\sin\theta = m\ell\ddot{\theta}$$

$$\sum F_n = m\frac{v^2}{\rho} \quad \Longrightarrow \quad T - mg\cos\theta = m\tfrac{v^2}{\ell}$$

The first equation provides the equation of motion:

$$\ddot{\theta} + \frac{g}{\ell}\sin\theta = 0$$

and the second gives the force T.

(c) The work-energy relation (Eqn. 1.46): Since the chord tension $\underline{T}$ is perpendicular to $\underline{v}$, $U = \int \underline{T} \cdot \underline{v}dt = 0$ and $\underline{T}$ does no work. The only force doing work is weight, $\underline{mg}$, and this force is conservative; therefore energy is conserved and Eqn. (1.47) applies:

$$E = T + V = \frac{1}{2}m\underbrace{v^2}_{\ell^2\dot{\theta}^2} - mg\ell\cos\theta = \text{constant}$$

$$\dot{E} = 2\frac{1}{2}m\ell^2\dot{\theta}\ddot{\theta} + mg\ell\sin\theta\dot{\theta} = 0$$

$$\ddot{\theta} + \frac{g}{\ell}\sin\theta = 0$$

Clearly, method (a) requires the most work and (c) the least. Note that method (c) does not give force T, which may be of interest, and gives a once integrated form of the equation of motion.

Robot Link. A rigid body moves in the (x, y) plane such that one of its points, say B, remains fixed in an inertial frame. An external moment $\underline{M} = M\hat{k}$ is applied at B (Fig. 1-25). We obtain the equation of motion by two methods.

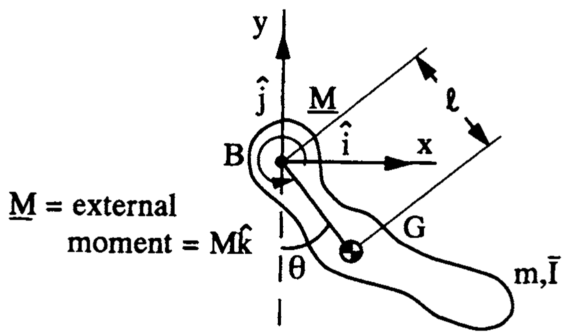


Fig. 1-25

(a) Equation (1.51) relative to point B:

$$\sum M_B = I_B\alpha \implies M = (\bar{I} + m\ell^2)\ddot{\theta}$$

$$[I_B = \bar{I} + m\ell^2 \text{ by the } \textit{parallel axis theorem}]$$

$$\ddot{\theta} - \frac{1}{(\bar{I} + m\ell^2)}M = 0$$

(b) Equations (1.50) and (1.51) relative to point A:

$$F_x = m\ddot{\bar{x}}\,, \quad F_y = m\ddot{\bar{y}}\,, \quad M - F_y\ell\sin\theta - F_x\ell\cos\theta = \bar{I}\ddot{\theta}$$

$$\bar{x} = \ell \sin\theta\,, \qquad \bar{y} = -\ell\cos\theta$$
$$M - m\ell^2\ddot{\theta} = \bar{I}\ddot{\theta}$$
$$\ddot{\theta} - \frac{1}{(\bar{I} + m\ell^2)} M = 0 \tag{1.61}$$

Physical Pendulum. This may be treated as a special case of the robot link with the gravitational force providing the moment (Fig. 1-26). The equation of motion is obtained by three methods.

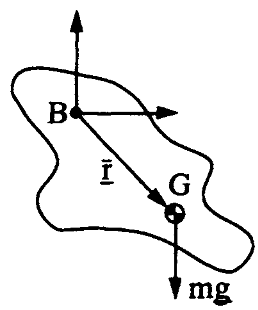


Fig. 1-26

(a) Equation (1.61):

$$\underline{M} = \underline{\bar{r}} \times m\underline{g}$$
$$M = -mg\ell \sin\theta$$
$$\ddot{\theta} + \frac{mg\ell}{(\bar{I} + m\ell^2)} \sin\theta = 0$$

(b) Conservation of energy, Eqn. (1.47), with T computed from Eqn. (1.57):

$$E = T + V = \frac{1}{2} I_B \omega^2 - mg\ell\cos\theta = \text{constant}$$
$$\dot{E} = 2\frac{1}{2}(\bar{I} + m\ell^2)\dot{\theta}\ddot{\theta} + mg\ell\sin\theta\dot{\theta} = 0$$
$$\ddot{\theta} + \frac{mg\ell}{(\bar{I} + m\ell^2)} \sin\theta = 0$$

(c) Conservation of energy with T computed from Eqn. (1.56):

$$E = T + V = \frac{1}{2}\bar{I}\underbrace{\omega^2}_{\dot{\theta}^2} + \frac{1}{2}m\underbrace{\bar{v}^2}_{\ell^2\dot{\theta}^2} - mg\ell\cos\theta = \text{constant}$$

$$E = \tfrac{1}{2}(\bar{I} + m\ell^2)\dot{\theta}^2 - mg\ell\cos\theta = \text{constant}$$

The simple pendulum, Eqn. (1.60), is the special case of $\bar{I} = 0$.

Car Accelerating Up a Hill. A car of mass m has acceleration $\underline{\bar{a}}$ and velocity $\underline{\bar{v}}$ up an incline of angle θ (Fig. 1-27). The wind resistance is $\underline{D}$ and each of the four wheels has a moment of inertia $\bar{I}$ and a radius r. Friction is sufficient to prevent wheel slipping. We want to find the power, P_e, delivered by the engine to the wheels.

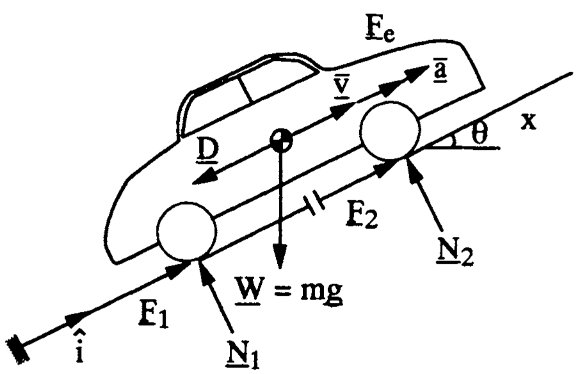


Fig. 1-27

Since the wheels roll without slipping, there is no velocity at the point of contact with the road. Consequently, from Eqn. (1.27), $\underline{F}_1$ and $\underline{F}_2$ do no work. Also, $\underline{N}_1$ and $\underline{N}_2$ do no work because they are normal to any possible velocity, even if there is slipping. The kinetic and potential energies are found from Eqns. (1.56) and (1.42) as (see Fig. 1-28):

$$\begin{aligned} T &= \frac{1}{2}m\bar{v}^2 + 4\frac{1}{2}\bar{I}\left(\frac{\bar{v}}{r}\right)^2 \\ V &= mg\sin\theta x \end{aligned}$$

The work done by force D is

$$U^D = \int \underline{D}\cdot\underline{\bar{v}}dt = -\int D\bar{v}dt$$

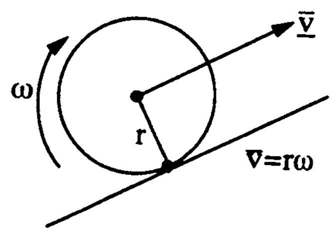


Fig. 1-28

Now differentiate the energy equation, Eqn. (1.44), and substitute the above relations:

$$U_{0,1}^{nc} = \Delta T_{0,1} + \Delta V_{0,1}$$

$$\int_{t_0}^{t_1} \underline{F}^{nc} \cdot \underline{v} dt = T_1 - T_0 + V_1 - V_0$$

$$\int_{t_0}^{t} \underline{F}^{nc} \cdot \underline{v} dt = T - T_0 + V - V_0$$

$$F^{nc}\overline{v} = \dot{T} + \dot{V}$$

$$F_e\overline{v} - D\overline{v} = 2\frac{1}{2}m\overline{v}\,\overline{a} + 4\frac{1}{2}\overline{I}\frac{2\overline{v}\,\overline{a}}{r^2} + mg\overline{v}\sin\theta$$

$$P_e = F_e\overline{v} = m\overline{v}\,\overline{a} + 4\frac{\overline{I}\overline{v}\,\overline{a}}{r^2} + mg\sin\theta\overline{v} + D\overline{v}$$

This shows that the power produced by the engine is used in four separate ways: (1) to accelerate the car, (2) to spin-up the wheels, (3) to gain altitude, and (4) to overcome air resistance.

1.6 Motivation for Analytical Dynamics

Lessons from Newtonian Dynamics. This chapter has revealed the following:

1. There are differences in forces. Forces may be classified as external or internal (the latter cancel out in a system of particles). They may be conservative or nonconservative (the former can be accounted for by potential energy functions). And finally, some forces do work while others do not.

2. Components along the coordinate axes are not always the best parameters to describe the motion; for example, θ is the best parameter for the simple pendulum.

3. The energy method is simple when appropriate. The advantages are: (i) Unit vectors, free-body diagrams, and coordinate systems are not needed, (ii) It gives an integral of the motion; that is, it gives a relation of speeds, not accelerations. The disadvantages are: (i) It gives only one equation, and thus possibly doesn't give the information desired, (ii) It is cumbersome when forces do work and are nonconservative.

These observations motivate the search for new approachs to dynamics.

Purposes of Analytical Dynamics. The main goals of the rest of this book are to:

1. Obtain scalar equations of motion invariant to coordinate transformation in the minimum number of variables.

2. Eliminate constraint forces and treat conservative forces via potential energy functions.

3. Obtain solutions of the equations of motion.

Notes

1 About 100 years ago, experiments showed that in certain situations Newton's Laws were significantly inaccurate. Relativity theory was developed to account for these discrepancies, and is now the most accurate description of dynamics. In relativity theory, the physical world is a four dimensional, non-Eulclidean space, there are no inertial reference frames, and the values of mass and time are different for different observers.

2 Vectors will be underlined here and throughout most of the book; the exception is unit vectors, which will get "hats".

3 It may be shown (Ardema, *Newton-Euler Dynamics*) that all body-fixed frames have the same angular velocity w.r.t. any other frame.

4 Note that the origin of the body-fixed frame need not be "in" the body.

5 If these conditions are not met, then the equation contains products as well as moments of inertia. See Ardema, *Newton-Euler Dynamics*, for a complete discussion of Eulerian dynamics of a rigid body.

PROBLEMS

1/1. A point moves at a constant speed of 5 ft/s along a path given by $y = 10e^{-2x}$, where x and y are in ft. Find the acceleration of the point when $x = 2$ ft.

1/2. A point moves at constant speed v along a curve defined by $r = A\theta$, where A is a constant. Find the normal and tangential components of acceleration.

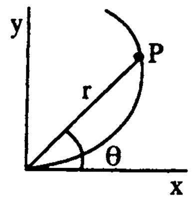


Fig. 1/2

Problem 1/2

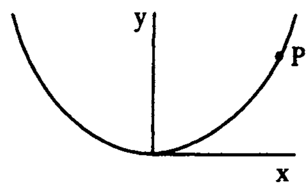
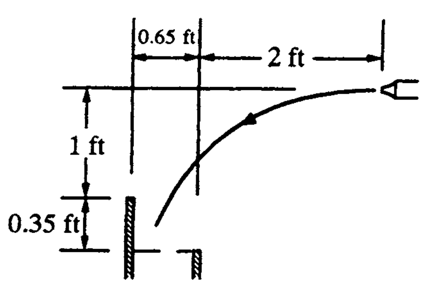


Fig. 1/3

Problem 1/3

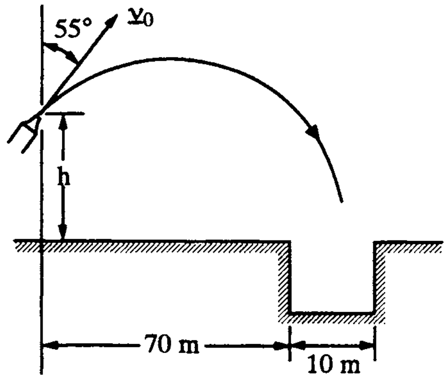
1/3. A point travels along a parabola $y = kx^2$, k a constant, such that the horizontal component of velocity, $\dot{x}$, remains a constant. Determine the acceleration of the point as a function of position.

1/4. Grain is being discharged from a nozzle into a vertical shute with an initial horizontal velocity $\underline{v}_0$. Determine the range of values of v_0 for which the grain will enter the shute.



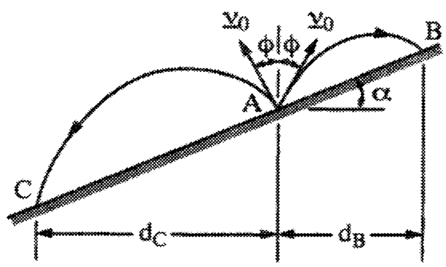
Problem 1/4

1/5. A nozzle is located at a height h above the ground and discharges water at a speed $v_0 = 25$ m/s at an angle of 55° with the vertical. Determine the range of values of h for which the water enters the trench in the ground.

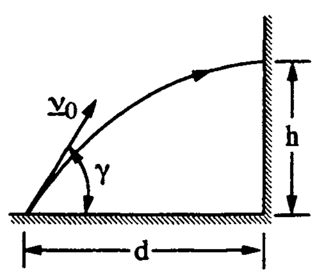


Problem 1/5

1/6. A rotating water sprinkler is positioned at point A on a lawn inclined at an angle $\alpha = 10°$ relative to the horizontal. The water is discharged with a speed of $v_0 = 8$ ft/s at an angle of $\phi = 40°$ to the vertical. Determine the horizontal distances d_C and d_B where the water lands.



Problem 1/6



Problem 1/7

1/7. A ball is thrown with velocity $\underline{v}_0$ against a vertical wall a distance d away. Determine the maximum height h at which the ball can strike the wall and the corresponding angle γ, in terms of v_0, d, and g.

1/8. In Problem 1/7, is the ball ascending or descending when it strikes the wall? What minimum speed v_0 is needed to strike the wall at all?

1/9. A condition of "weightlessness" may be obtained by an airplane flying a curved path in the vertical plane as shown. If the plane's speed is $v = 800$ km/h, what must be the rate of rotation of the airplane $\dot{\gamma}$ to obtain this condition at the top of its loop?

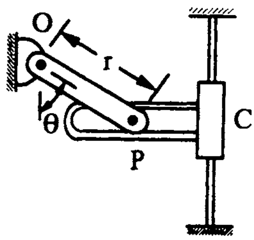
1/10. The speed of a car is increasing at a constant rate from 60 mi/h to 75 mi/h over a distance of 600 ft along a curve of 800 ft radius. What is the magnitude of the total acceleration of the car after it has traveled 400 ft along the turn?

1/11. Consider the situation of Problem 1/5. Determine the radius of curvature of the stream both as it leaves the nozzle and at its maximum height.

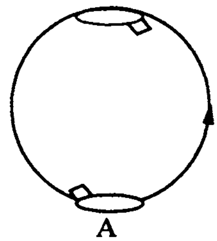
1/12. Consider again the situation of Problem 1/5. It was observed that the radius of curvature of the stream of water as it left the nozzle was 35 ft. Find the speed v_0 with which the water left the nozzle, and the radius of curvature of the stream when it reaches its maximum height, for $\theta = 36.87°$.

1/13. The velocity of a point at a certain instant is $\underline{v} = 3\hat{\imath} + 4\hat{\jmath}$ ft/s, and the radius of curvature of its path is 6.25 ft. The speed of the point is decreasing at the rate of 2 ft/s^2. Express the velocity and acceleration of the point in tangential-normal components.

1/14. Link OP rotates about O, and pin P slides in the slot attached to collar C. Determine the velocity and acceleration of collar C as a function of θ for the following cases:
(i) $\dot{\theta} = \omega$ and $\ddot{\theta} = 0$,
(ii) $\dot{\theta} = 0$ and $\ddot{\theta} = \alpha$.



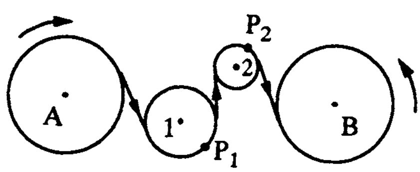
Problem 1/14



Problem 1/15

1/15. At the bottom A of a vertical inside loop, the magnitude of the total acceleration of the airplane is 3g. If the airspeed is 800 mph and is increasing at the rate of 20 mph per second, determine the radius of curvature of the path at A.

1/16. Tape is being transferred from drum A to drum B via two pulleys. The radius of pulley 1 is 1.0 in and that of pulley 2 is 0.5 in. At P_1, a point on pulley 1, the normal component of acceleration is 4 in/s^2 and at P_2, a point on pulley 2, the tangential component of acceleration is 3 in/s^2. At this instant, compute the speed of the tape, the magnitude of the total acceleration at P_1, and the magnitude of the total acceleration at P_2.



Problem 1/16

1/17. The shape of the stationary cam is that of a limacon, defined by $r = b - c\cos\theta$, $b > c$. Determine the magnitude of the total acceleration as a function of θ if the slotted arm rotates with a constant angular rate $\omega = \dot{\theta}$ in the counter clockwise direction.

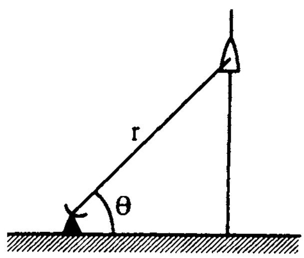
1/18. A radar used to track rocket launches is capable of measuring r, $\dot{r}$, $\dot{\theta}$, and $\ddot{\theta}$. The radar is in the vertical plane of the rocket's flight path. At a certain time, the measurements of a rocket are

$r = 35,000$ m, $\dot{r} = 1600$ m/s, $\dot{\theta} = 0$, and $\ddot{\theta} = -0.0072$ rad/s^2. What direction is the rocket heading relative to the radar at this time? What is the radius of curvature of its path?

1/19. A collar A slides on a thin rod OB such that $r = 60t^2 - 20t^3$, with r in meters and t in seconds. The rod rotates according to $\theta = 2t^2$, with θ in radius. Determine the velocity and total acceleration of the collar when $t = 1s$, using radial-transverse components.

1/20. Consider the same situation as in Problem 1/19, but with $r = 1.25t^2 - 0.9t^3$ and $\theta = \frac{1}{2}\pi(4t - 3t^3)$. Answer the same questions.

1/21. A vertically ascending rocket is tracked by radar as shown. When $\theta = 60°$, measurements give $r = 30,000$ ft, $\ddot{r} = 70$ ft/s^2, and $\dot{\theta} = 0.02$ rad/s. Determine the magnitudes of the velocity and the acceleration of the rocket at this instant.

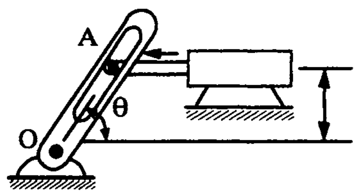


Problem 1/21

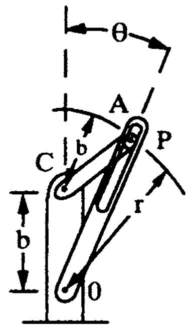
1/22. The path of fluid particles in a certain centrifugal pump is closely approximated by $r = r_0 e^{n\theta}$ where r_0 and n are constants. If the pump turns at a constant rate $\dot{\theta} = \omega$, determine the expression for the magnitude of the acceleration of a fluid particle when $r = R$.

1/23. The pin A at the end of the piston of the hydraulic cylinder has a constant speed 3 m/s in the direction shown. For the instant when $\theta = 60°$, determine $\dot{r}, \ddot{r}, \dot{\theta}$ and $\ddot{\theta}$, where $r = \overline{OA}$.

1/24. Slotted arm OA oscillates about O and drives crank P via the pin at P. For an interval of time, $\dot{\theta} = \omega =$ constant. During this time, determine the magnitude of the acceleration of P as a function of θ. Also, show that the magnitudes of the velocity and acceleration of P are constant during this time interval.

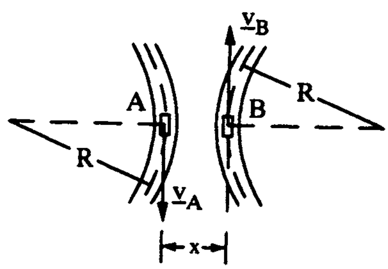


Problem 1/23



Problem 1/24

1/25. Two cars, labeled A and B, are traveling on curves with constant equal speeds of 72 km/hr. The curves both have radius $R = 100$m and their point of closest approach is $x = 30$m. Find the velocity of B relative to the occupants of A at the point of closest approach.

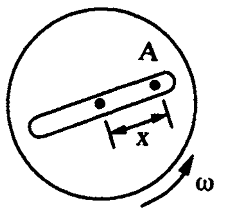


Problem 1/25

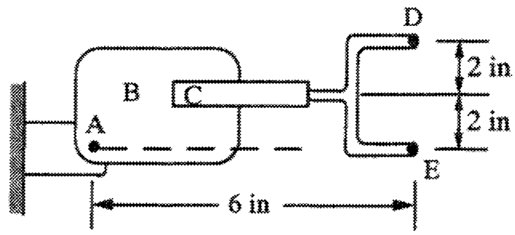
1/26. For the same conditions of Problem 1/25, find the acceleration of B relative to A.

1/27. For the same conditions of Problem 1/25, find the acceleration of B relative to A if A is speeding up at the rate of 3 m/s^2 and B is slowing down at the rate of 6 m/s^2.

1/28. At a certain instant, the disk is rotating with an angular speed of $\omega = 15$ rad/s and the speed is increasing at a rate of 20 rad/s^2. The slider moves in the slot in the disk at the constant rate $\dot{x} = 120$ in/s and at the same instant is at the center of the disk. Obtain the acceleration and velocity of the slider at this instant.



Problem 1/28

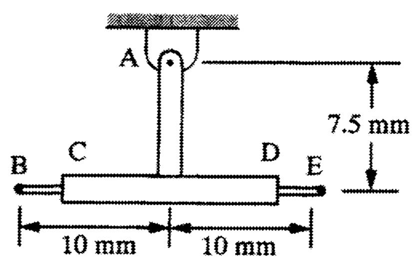


Problem 1/29

1/29. Shown is an automated welding device. Plate B rotates about point A, and the welding bracket with tips D and E moves in a cylinder C attached to B. At a certain instant, bracket DE is moving to the right with respect to plate B at a constant rate of 3 in/s and B is rotating counter clockwise about A at a constant rate of 1.6 rad/s. Determine the velocity and acceleration of tip E at that instant.

1/30. For the same situation as in Problem 1/29, determine the velocity and acceleration of tip D.

1/31. Bracket ACD is rotating clockwise about A at the constant rate of 2.4 rad/s. When in the position shown, rod BE is moving to the right relative to the bracket at the constant rate of 15 mm/s. Find the velocity and acceleration of point B.



Problem 1/31

1/32. Same as Problem 1/31, except that the rotation of the bracket is speeding up at a rate of 0.3 rad/s^2.

1/33. Same as Problem 1/31, except that the rod is slowing down at the rate of 2 mm/s^2.

1/34. Find the velocity and acceleration of point E for the situation in Problem 1/31.

1/35. Prove Eqn. (1.56).

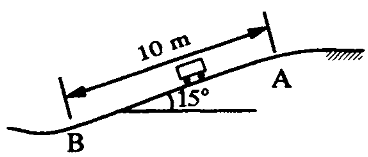
1/36. Prove Eqn. (1.57).

1/37. Prove Koenig's theorem, Eqn. (1.58).

1/38. Show that Eqn. (1.58) reduces to Eqn. (1.57) for the case of 2-D motion.

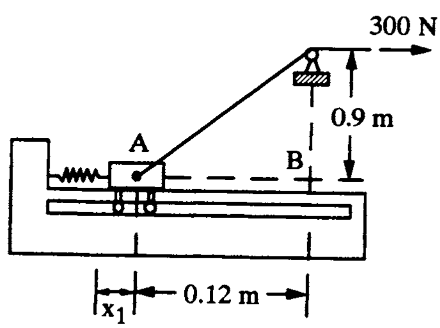
1/39. Prove Eqn. (1.59).

1/40. A 50 kg cart slides down an incline from A to B as shown. What is the speed of the cart at the bottom at B if it starts at the top at A with a speed of 4 m/s? The coefficient of kinetic friction is 0.30.



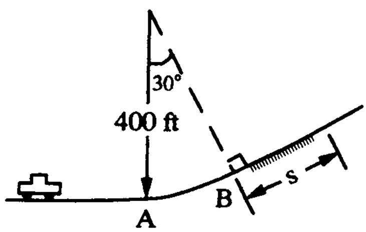
Problem 1/40

1/41. A 50 kg block slides without friction as shown. There is a constant force of 300 N in the cable and the spring attached to the block has stiffness 80 N/m. If the block is released from rest at a position A in which the spring is stretched by amount $x_1 = 0.233$ m, what is the speed when the block reaches position B.

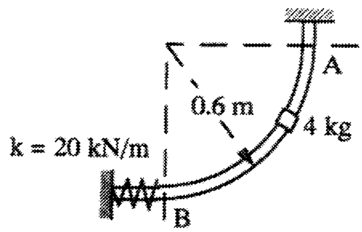


Problem 1/41

1/42. A 4000 lb car travels up a hill as shown. The car starts from rest at A and the engine exerts a constant force in the direction of travel of 1000 lb until position B is reached, at which time the engine is shut off. How far does the car roll up the hill before stopping? Neglect all friction and air resistance.



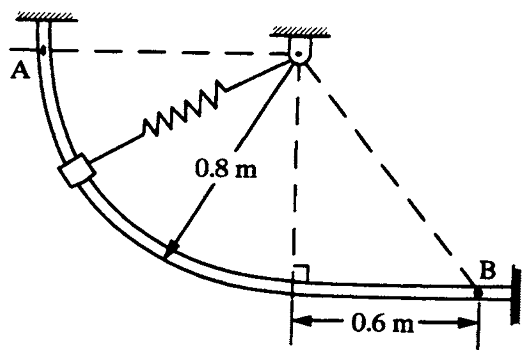
Problem 1/42



Problem 1/43

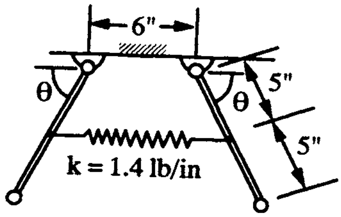
1/43. The small 4 kg collar is released from rest at A and slides down the circular rod in the vertical plane. Find the speed of the collar as it reaches the bottom at B and the maximum compression of the spring. Neglect friction.

1/44. The small 3 kg collar is released from rest at A and slides in the vertical plane to B. The attached spring has stiffness 200 N/m and an unstretched length of 0.4 m. What is the speed of the collar at B? Neglect friction.



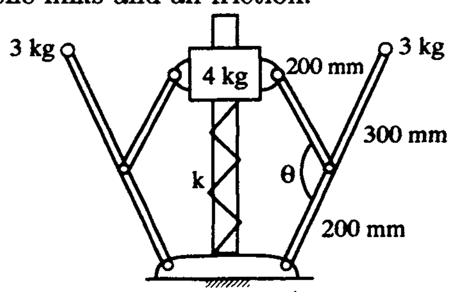
Problem 1/44

1/45. The identical links are released simultaneously from rest at $\theta = 30°$ and rotate in the vertical plane. Find the speed of each 2 lb mass when $\theta = 90°$. The spring is unstretched when $\theta = 90°$. Ignore the mass of the links and model the masses as particles.



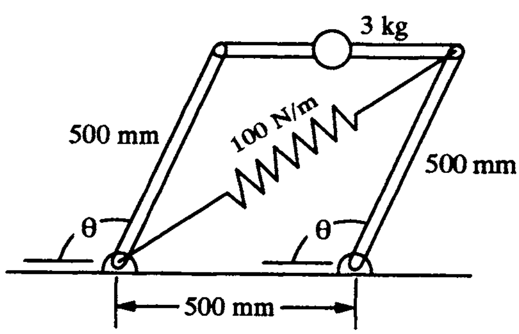
Problem 1/45

1/46. The device shown is released from rest with $\theta = 180°$ and moves in the vertical plane. The spring has stiffness 900 N/m and is just touching the underside of the collar when $\theta = 180°$. Determine the angle θ when the spring reaches maximum compression. Neglect the masses of the links and all friction.



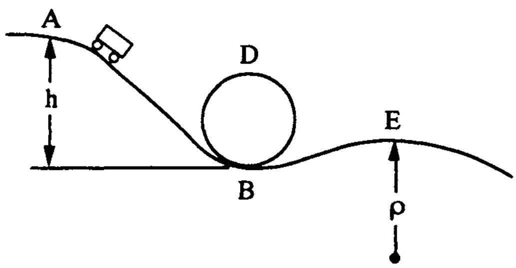
Problem 1/46

1/47. Shown is a frame of negligible weight and friction that rotates in the vertical plane and carries a 3 kg mass. The spring is unstretched when $\theta = 90°$. If the frame is released from rest at $\theta = 90°$, determine the speed of the mass when $\theta = 135°$ is passed.



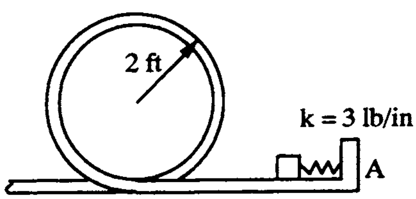
Problem 1/47

1/48. A roller coaster car starts from rest at A, rolls down the track to B, transits a circular loop of 40 ft diameter, and then moves over the hump at E. If $h = 60$ ft, determine (a) the force exerted by his or her seat on a 160 lb rider at both B and D, and (b) the minimum value of the radius of curvature of E if the car is not to leave the track at that point. Neglect all friction and air resistance.

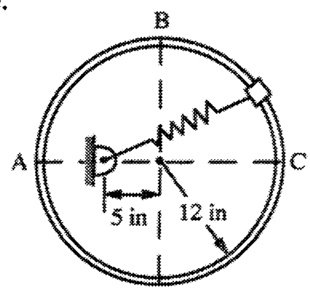


Problem 1/48

1/49. The 0.5 lb pellet is pushed against the spring at A and released from rest. Determine the smallest deflection of the spring for which the pellet will remain in contact with the circular loop at all times. Neglect friction and air resistance.



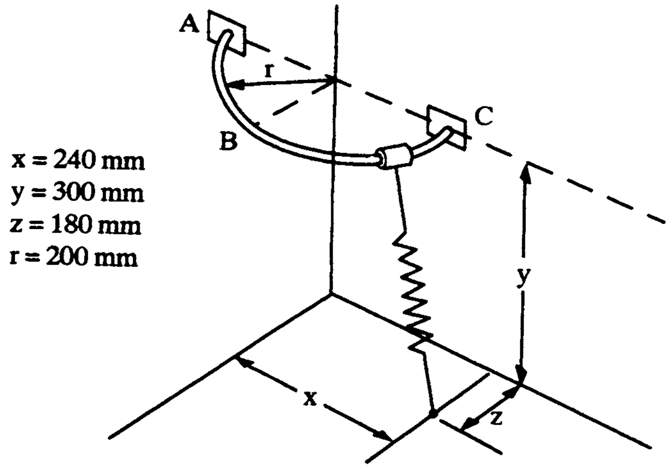
Problem 1/49



Problem 1/50

1/50. A 3 lb collar is attached to a spring and slides without friction on a circular hoop in a horizontal plane. The spring constant is 1.5 lb/in and is undeformed when the collar is at A. If the collar is released at C with speed 6 ft/s, find the speeds of the collar as it passes through points B and A.

1/51. A 600 g collar slides without friction on a horizontal semicircular rod ABC of radius 200 mm and is attached to a spring of spring constant 135 N/m and undeformed length 250 mm. If the collar is



Problem 1/51

released from rest at A, what are the speeds of the collar at B and C?

1/52. Prove that a force is conservative if and only if the following relations are satisfied:

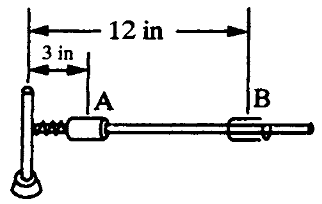
$$\frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x}, \qquad \frac{\partial F_y}{\partial z} = \frac{\partial F_z}{\partial y}, \qquad \frac{\partial F_z}{\partial x} = \frac{\partial F_x}{\partial z}$$

1/53. Show that the force

$$\underline{F} = (x\hat{\imath} + y\hat{\jmath} + z\hat{k})(x^2 + y^2 + z^2)$$

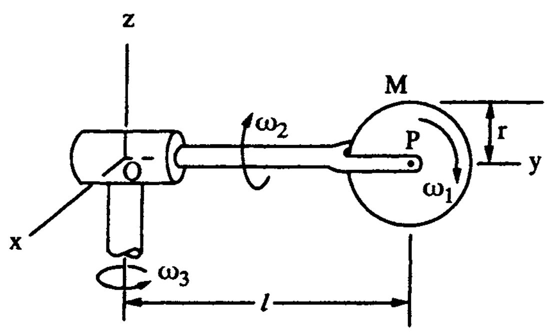
is conservative by applying the results of Problem 1/52. Also find the potential energy function $V(x, y, z)$ associated with $\underline{F}$.

1/54. A 1/2 lb collar slides without friction on a horizontal rod which rotates about a vertical shaft. The collar is initially held in position A against a spring of spring constant 2.5 lb/ft and unstretched length 9 in. As the rod is rotating at angular speed 12 rad/s the cord is cut, releasing the collar to slide along the rod. The spring is attached to the collar and the rod. Find the angular speed of the rod and the radial and transverse components of the velocity of the collar as the rod passes postion B. Also find the maximum distance from the vertical shaft that the collar will reach.



Problem 1/54

1/55. Determine the kinetic energy of the uniform circular disk of mass M at the instant shown.



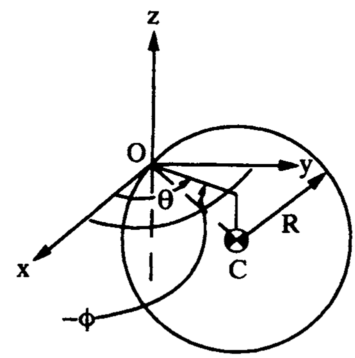
Problem 1/55

1/56. Find the kinetic energy of a homogeneous solid disk of mass m and radius r that rolls without slipping along a straight line. The center of the disk moves with constant velocity v.

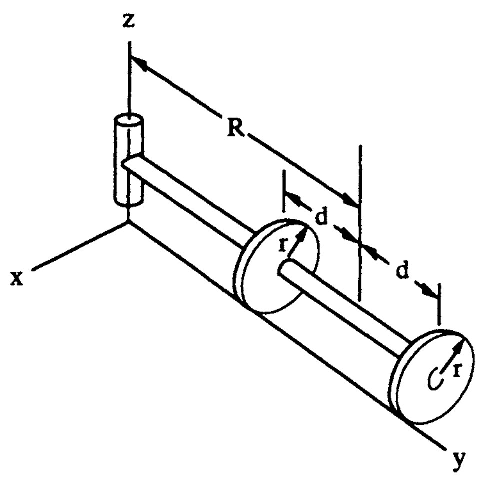
1/57. A homogeneous solid sphere of mass M and radius R is fixed at a point O on its surface by a ball joint. Find the kinetic energy of the sphere for general motion.

1/58. Two uniform circular disks, each of mass M and radius r, are mounted on the same shaft as shown. The shaft turns about the z-axis, while the two disks roll on the xy-plane without slipping. Prove that the ratio of the kinetic energies of the two disks is

$$\frac{6(R+d)^2+r^2}{6(R-d)^2+r^2}$$



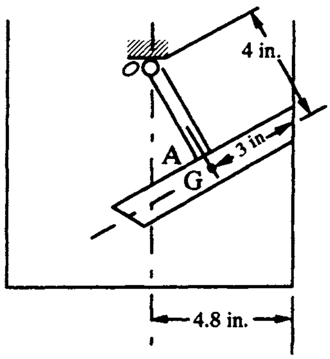
Problem 1/57



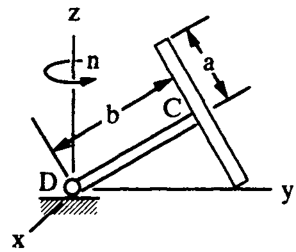
Problem 1/58

1/59. A disk with arm OA is attached to a socket joint at O. The moment of inertia of the disk and arm about axis OA is I and the total mass is M, with the center of gravity at G. The disk rolls inside a cylinder whose radius is 4.8 in. Find the kinetic energy of the disk when the line of contact turns around the cylinder at 10 cycles per second.

1/60. A uniform circular disk of radius a and mass M is mounted on a weightless shaft CD of length b. The shaft is normal to the disk at its center C. The disk rolls on the xy-plane without slipping, with point D remaining at the origin. Determine the kinetic energy of the disk if shaft CD rotates about the z-axis with constant angular speed n.

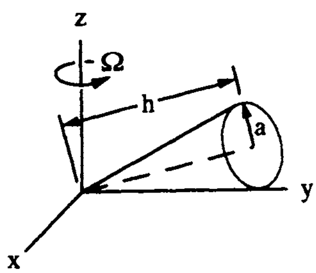


Problem 1/59

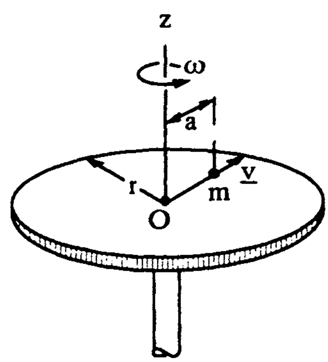


Problem 1/60

1/61. A homogeneous solid right circular cone rolls on a plane without slipping. The line of contact turns at constant angular speed Ω about the z-axis. Find the kinetic energy of the cone.



Problem 1/61

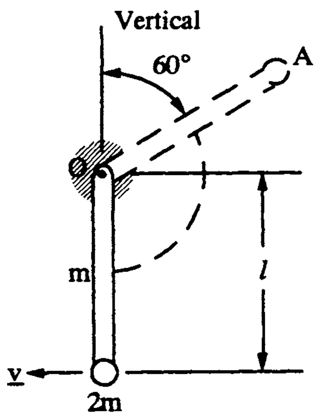


Problem 1/62

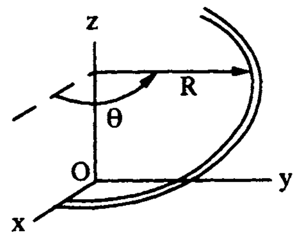
1/62. A particle of mass m slides along one radius of a circular platform of mass M. At the instant shown, the platform has an angular velocity ω and the particle has a velocity v relative to the platform. Determine the kinetic energy and the angular momentum of the system about point O.

1/63. A pendulum consists of a uniform rod of mass m and a bob of mass $2m$. The pendulum is released from rest at position A as shown. What is the kinetic energy of the system at the lowest position? What is the velocity of the bob at the lowest position?

1/64. A particle of mass m is attracted toward the origin by a force with magnitude $(mK)/r^2$ where K is a constant and r is the distance



Problem 1/63



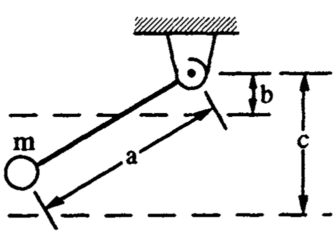
Problem 1/64

between the particle and the center of attraction. The particle is constrained to move in a frictionless tube which lies along the space curve given by

$$\left.\begin{aligned} z &= 5\theta \\ R &= 1 + \frac{1}{2}\theta \end{aligned}\right\} \quad \text{in cylindrical coordinates}$$

If the particle was at rest when $z = 10$, what is the velocity of the particle at $z = 0$?

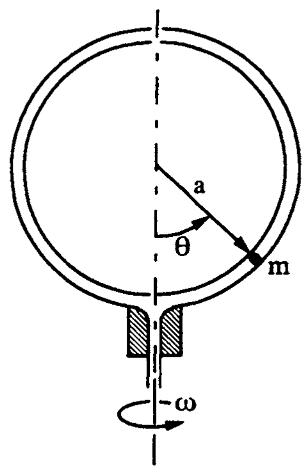
1/65. A spherical pendulum of mass m and length a oscillates between levels b and c, located below the support. Find the expression for the total energy of the system in terms of a, b, c, m, and g, taking the horizontal plane passing through the support as the zero potential level.



Problem 1/65

1/66. A spherical pendulum, consisting of a massless rod and a bob of mass m, is initially held at rest in the horizontal plane. A horizontal velocity v_0 is imparted to the bob normal to the rod. In the resulting motion, what is the angle between the rod and the horizontal plane when the bob is at its lowest position?

1/67. A particle of mass m is placed inside a frictionless tube of negligible mass. The tube is bent into a circular ring with the lowest point left open as shown. The ring is given an initial angular velocity ω about the vertical axis passing through the diameter containing the opening, and simultaneously the particle is released from rest (relative to the tube) at $\theta = \pi/2$. In the subsequent motion, will the particle drop through the opening?



Problem 1/67