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ORIGINAL RESEARCH

Studies of Norm-Based Locality Measures of Two-Dimensional Hilbert Curves

H. K. Dai[1] · **H. C. Su**[2]

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Abstract

A discrete space-filling curve provides a one-dimensional indexing or traversal of a multi-dimensional grid space. Sample applications of space-filling curves include multi-dimensional indexing methods, data structures and algorithms, parallel computing, and image compression. Common measures for the applicability of space-filling curve families are locality and clustering. Locality preservation reflects proximity between grid points, that is, close-by grid points are mapped to close-by indices or vice versa. We present analytical and empirical studies on the locality properties of the two-dimensional Hilbert curve family. The underlying locality measure, based on the p-normed metric d_p, is the maximum ratio of $d_p(v,u)^m$ to $d_p(\tilde{v},\tilde{u})$ over all corresponding point-pairs (v,u) and $(\tilde{v},\tilde{u})$ in the m-dimensional grid space and one-dimensional index space, respectively. Our analytical results close the gaps between the current best lower and upper bounds with exact formulas for $p \in \{1,2\}$, and extend to all reals $p \geq 2$. We also verify the results with computer programs over various grid-orders and p-values. Our empirical results will shed some light on determining the exact formulas for the locality measure for all reals $p \in (1,2)$.

Keywords Index structures · Space-filling curves · Hilbert curves · z-order curves · Locality

Preliminaries

Discrete space-filling curves have a wide range of applications in databases, parallel computation, algorithms, in which linearization techniques of multi-dimensional arrays or computational grids are needed. Sample applications include heuristics for combinatorial algorithms and data structures: traveling salesperson algorithm [30] and nearest-neighbor finding [9], multi-dimensional space-filling indexing methods [3, 7, 16, 23], image compression [25], dynamic unstructured mesh partitioning [21], and linearization and traversal of sensor networks [5, 34]. Some recent diverse applications of space-filling curves extend to statistical sampling [18] and bioinformatics [22]. For a comprehensive historical development of classical space-filling curves, see [4, 32].

For a positive integer n, denote $[n] = \{1, 2, \ldots, n\}$. For a positive integer m, and m-dimensional (discrete) space-filling curve of length n^m is a bijective mapping $C : [n^m] \rightarrow [n]^m$, which provides a linear indexing/traversal or total ordering of the grid points in $[n]^m$. For a positive integer k, an m-dimensional grid is of order k if it has side-length $n = 2^k$; a space-filling curve has order k if its codomain is a grid of order k. A mathematical construction of a sequence of multi-dimensional space-filling curves of successive orders usually follows a recursive framework on the dimensionality and order, with which a few classical families arise, such as Gray-coded curves, Hilbert curves, Peano curves, and z-order curves (see, for examples, [2, 27]).

A mathematical formulation of discrete Hilbert curves based on generators and permutations (on a corner-labeling hypercube) in [2] shows that the descriptional complexity and structural analysis of multi-dimensional Hilbert curves can be reduced to a combinatorial analysis of their

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✉ H. K. Dai
dai@cs.okstate.edu

H. C. Su
suh@astate.edu

1 Computer Science Department, Oklahoma State University, Stillwater, OK 74078, USA

2 Department of Computer Science, Arkansas State University, State University, Jonesboro, AR 72467, USA

generators. One of the salient characteristics of space-filling curves is their "self-similarity". Denote by H_k^m and Z_k^m an m-dimensional Hilbert and z-order, respectively, space-filling curve of order k. Figure 1 illustrates the recursive geometric generations of H_k^m and Z_k^m for $m = 2$, and $k = 1, 2$, and $m = 3$, and $k = 1$.

We gauge the applicability of a family of space-filling curves based on: (1) their common structural characteristics that measure locality and clustering, (2) descriptional simplicity that facilitates their construction and combinatorial analysis in arbitrary dimensions, and (3) computational complexity in the grid space-index space transformation. Locality preservation measures proximity between the grid points of $[n]^m$, that is, close-by points in $[n]^m$ are mapped to close-by indices/numbers in $[n^m]$, or vice versa. Clustering performance evaluates the distribution of continuous runs of grid points (clusters) over identically shaped subspaces of $[n]^m$, which can be characterized by the average number of clusters and the average inter-cluster distance (in $[n^m]$) within a subspace.

Empirical and analytical studies of clustering performances of various low-dimensional space-filling curves have been reported in the literature (see [8, 11, 13, 19, 20, 27, 31] for details). Generally, the Hilbert and z-order curve families exhibit good performance in this respect.

Jagadish [20] derives exact formulas for the mean numbers of clusters over all rectangular 2×2 and 3×3 subgrids of a two-dimensional H_k^2-structural grid space. Moon, Jagadish, Faloutsos, and Saltz [27] prove that in a sufficiently large m-dimensional H_k^m-structural grid space, the mean number of clusters over all rectilinear polyhedral queries with surface area $S_{m,k}$ approaches $\frac{1}{2}\frac{S_{m,k}}{m}$ as k approaches ∞. They also extend the work in [20] to obtain the exact formula for the mean number of clusters over all rectangular $2^q \times 2^q$ subgrids of a two-dimensional H_k^2-structural grid space.

Xu and Tirthapura [36] generalize the above asymptotic mean number of clusters over all rectilinear polyhedral queries with common surface area from m-dimensional Hilbert curves to arbitrary continuous space-filling curves (with which contiguously indexed grid points are at a rectilinear distance of 1). Note that rectangular queries with common volume yield the optimal asymptotic mean number of clusters for a continuous space-filling curve.

For an m-dimensional H_k^m-structural grid space with $m = 3$, there are 1536 structurally different three-dimensional Hilbert curves [2]. Based on a canonical version of an H_k^3-curve, Dai and Su [14] develop the exact formula for the mean-clustering statistics for the mean number of clusters over all rectangular $2^q \times 2^q \times 2^q$ subgrids of the canonical H_k^3-curve — which extends the two-dimensional exact result in [27].

For clustering performance based on inter-cluster statistics, Dai and Su [11] obtain the exact formulas for the following three statistics for two-dimensional H_k^2 and Z_k^2: (1) the summation of all inter-cluster distances over all $2^q \times 2^q$ query subgrids, (2) the universe mean inter-cluster distance over all inter-cluster gaps from all $2^q \times 2^q$ subgrids, and (3) the mean total inter-cluster distance over all $2^q \times 2^q$ subgrids. Based on the analytical results, the asymptotic comparisons indicate that, for a two-dimensional grid space, the z-order curve family performs better than the Hilbert curve family with respect to the statistics.

Alber and Niedermeier [2] give a simple mathematical mechanism to describe and analyze the combinatorial properties of Hilbert curves in arbitrary dimensions. The structure-theoretic viewpoint provides a framework for combinatorial studies and mechanized analysis of multi-dimensional Hilbert indexings via reduction to a structural analysis of basic generating elements and permutations operating on a corner-labeling hypercube. Lawder and King [24] implement effective methods for range and partial-match query execution for multi-dimensional Hilbert indexing schemes.

The studies above show that the Hilbert and z-order curve families manifest good data clustering properties according to some quality clustering measures, robust mathematical formalism, and viable indexing techniques for querying multi-dimensional data, when compared with other curve families.

Locality Measures and Related Work

The locality preservation of space-filling curve families is crucial for the efficiency of their supported indexing schemes on computational grids, and data structures and algorithmic applications for combinatorial optimization; for examples, spatial correlation in multi-dimensional indexings, compression in image processing, and communication optimization in mesh-connected parallel computing. Rigorous analyses of locality depends on the availability of robust and practical measures: good bounds (lower and upper) on the locality measure translate into good bounds on the declustering (locality loss) in one space in the presence of locality in the other space.

A few locality measures have been proposed and analyzed for space-filling curves in the literature for their diverse applications. Denote by d and d_p the Euclidean metric and p-normed metric (rectilinear metric ($p = 1$) and maximum metric ($p = \infty$)), respectively. Let $\mathcal{C}$ denote a family of m-dimensional curves of successive orders.

For quantifying the proximity preservation of close-by grid points in the m-dimensional space $[n]^m$, Pérez, Kamata, and Kawaguchi [29] employ an average locality measure:

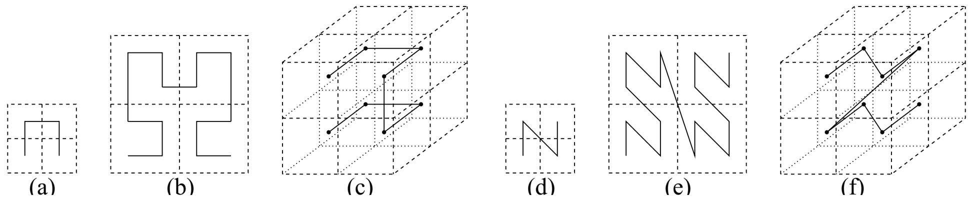


Fig. 1 Recursive self-similar generations of Hilbert and z-order curves of higher order (respectively, H_k^m and Z_k^m) by interconnecting symmetric subcurves, via reflection and/or rotation, of lower order (respectively, H_{k-1}^m and Z_{k-1}^m) along an order-1 subcurve (respectively, H_1^m and Z_1^m): **a** H_1^2; **b** H_2^2; **c** H_1^3; **d** Z_1^2; **e** Z_2^2; **f** Z_1^3

$$L_{\mathrm{PKK}}(C) = \sum_{i,j \in [n^m] | i<j} \frac{|i-j|}{d(C(i), C(j))} \text{ for } C \in \mathcal{C},$$

and provide a hierarchical construction for a two-dimensional $\mathcal{C}$ with good but suboptimal locality with respect to this measure.

Mitchison and Durbin [26] use a more restrictive locality measure parameterized by q:

$$L_{\mathrm{MD},q}(C) = \sum_{i,j \in [n^m] | i<j \text{ and } d(C(i),C(j))=1} |i-j|^q \text{ for } C \in \mathcal{C}$$

to study optimal two-dimensional mappings for $q \in [0, 1]$. For the case $q = 1$, the optimal mapping with respect to $L_{\mathrm{MD},1}$ is very different from that in [29]. For the case $q < 1$, they prove a lower bound for arbitrary two-dimensional curve C:

$$L_{\mathrm{MD},q}(C) \geq \frac{1}{1+2q} n^{1+2q} + O(n^{2q}),$$

and provide an explicit construction for two-dimensional $\mathcal{C}$ with good but suboptimal locality. They conjecture that the space-filling curves with optimal locality (with respect to $L_{\mathrm{MD},q}$ with $q < 1$) must exhibit a "fractal" character.

Dai and Su [12] consider a locality measure similar to $L_{\mathrm{MD},1}$ conditional on a 1-normed distance of δ between points in $[n]^m$:

$$L_\delta(C) = \sum_{i,j \in [n^m] | i<j \text{ and } d_1(C(i),C(j))=\delta} |i-j| \text{ for } C \in \mathcal{C}.$$

They derive exact formulas for L_δ for the Hilbert curve family $\{H_k^m \mid k = 1, 2, \ldots\}$ and z-order curve family $\{Z_k^m \mid k = 1, 2, \ldots\}$ for $m = 2$ and arbitrary δ that is an integral power of 2, and $m = 3$ and $\delta = 1$:

$$L_\delta(H_k^2) = \begin{cases} \frac{17}{2\cdot7} \cdot 2^{3k} - \frac{5}{2\cdot3} \cdot 2^{2k} - \frac{2^3}{3\cdot7} & \text{if } \delta = 1 \\ \frac{17}{2\cdot7} \cdot 2^{3k+2\log\delta} - \frac{2^3\cdot3\cdot5^2\cdot7(k-\log\delta)+5\cdot7\cdot383}{2^4\cdot3^3\cdot5\cdot7} \cdot 2^{2k+3\log\delta} & \\ \quad + \frac{2\cdot3\cdot5(k-\log\delta)-1}{2^2\cdot3^3} \cdot 2^{2k+\log\delta} - \frac{2^2\cdot41}{3^3\cdot5\cdot7} \cdot 2^{5\log\delta} & \\ \quad - \frac{2}{3^3} \cdot 2^{3\log\delta} - \frac{2}{3\cdot5} \cdot 2^{\log\delta} & \text{otherwise,} \end{cases}$$

$$L_\delta(Z_k^2) = \begin{cases} 2^{3k} - 2^k & \text{if } \delta = 1 \\ 2^{3k+2\log\delta} - (\frac{2}{3^2}(k-\log\delta) + \frac{1949}{2^5\cdot3^3\cdot7})2^{2k+3\log\delta} & \\ \quad + (\frac{2}{3^2}(k-\log\delta) + \frac{7}{2^2\cdot3^3})2^{2k+\log\delta} + \frac{19}{2^2\cdot3\cdot7} \cdot 2^{2k} & \\ \quad - \frac{2^2}{7} \cdot 2^{k+4\log\delta} - \frac{3}{7} \cdot 2^{k+\log\delta} + \frac{2\cdot5}{3^3\cdot7} \cdot 2^{5\log\delta} & \\ \quad - \frac{2^2}{3^3} \cdot 2^{3\log\delta} + \frac{2}{3\cdot7} \cdot 2^{2\log\delta} & \text{otherwise,} \end{cases}$$

$$L_1(H_k^3) = \frac{67}{2\cdot31} \cdot 2^{5k} - \frac{11}{2\cdot7} \cdot 2^{3k} - \frac{2^6}{7\cdot31}, \text{ and}$$

$$L_1(Z_k^3) = 2^{5k} - 2^{2k}.$$

With respect to the locality measure L_δ and for sufficiently large k and $\delta \ll 2^k$, the z-order curve family performs better than the Hilbert curve family for $m = 2$ and over the δ-spectrum of integral powers of 2. When $\delta = 2^k$, the domination reverses. The superiority of the z-order curve family persists but declines for $m = 3$ with unit 1-normed distance for L_δ.

Xu and Tirthapura [35] consider a variant of the all-pairs locality measure L_δ via the notion of nearest-neighbor stretch of a single-source grid point — conditional on the unit 1-normed metric d_1; that is, for an m-dimensional space-filling curve C and a grid point v indexed by C, denote the nearest-neighbor of v in $[n]^m$, $N_1(v,C) = \{u \in [n]^m \mid d_1(u,v) = 1\}$, and:

$$\text{average nearest-neighbor stretch } (v,C) = \frac{\sum_{u \in N_1(v,C)} |C^{-1}(v) - C^{-1}(u)|}{|N_1(v,C)|}, \text{ and}$$

$$\text{maximum nearest-neighbor stretch } (v,C) = \max_{u \in N_1(v,C)} |C^{-1}(v) - C^{-1}(u)|.$$

The average-quantifications of these two nearest-neighbor stretches for C result in: average-average nearest-neighbor stretch $D^{\text{avg}}(C)$ and average-maximum nearest-neighbor stretch $D^{\text{max}}(C)$ for C. They obtain a lower bound for $D^{\text{avg}}(C)$ for arbitrary m-dimensional curve C with grid space $[n]^m$:

$$(D^{\text{max}}(C) \geq)\ D^{\text{avg}}(C) \geq \frac{2}{3m}(n^{m-1} - n^{-m-1}),$$

and show that, for an m-dimensional row-major space-filling curve S with grid space $[n]^m$,

$$D^{\text{avg}}(S) \sim \frac{1}{m} n^{m-1} \text{ and } D^{\text{max}}(S) = n^{m-1}.$$

Voorhies [33] defines a heuristic locality measure, tailored to computer graphics applications, and the corresponding empirical study indicates that the Hilbert space-filling curve family outperforms other curve families.

For measuring the proximity preservation of close-by points in the indexing space $[n^m]$, Gotsman and Lindenbaum [17] consider the following measures:

$$L_{\text{GL,min}}(C) = \min_{i,j \in [n^m] | i<j} \frac{d(C(i), C(j))^m}{|i-j|}, \text{ and}$$

$$L_{\text{GL,max}}(C) = \max_{i,j \in [n^m] | i<j} \frac{d(C(i), C(j))^m}{|i-j|}, \text{ for } C \in \mathcal{C}.$$

They show that for arbitrary m-dimensional curve C,

$$L_{\text{GL,min}}(C) = O(n^{1-m}), \text{ and}$$

$$L_{\text{GL,max}}(C) > (2^m - 1)(1 - \frac{1}{n})^m.$$

For the m-dimensional Hilbert curve family $\{H_k^m \mid k = 1, 2, \ldots\}$, they prove that:

$$L_{\text{GL,max}}(H_k^m) \leq 2^m (m+3)^{\frac{m}{2}}.$$

Alber and Niedermeier [1, 2] generalize $L_{\text{GL,max}}$ to L_p by employing the p-normed metric d_p for real norm-parameter $p \geq 1$ in place of the Euclidean metric d, which is the locality measure studied in our work (and the preliminary versions in [12, 15]). We summarize below: (1) the representative lower- and upper-bound results and exact formulas for the locality measure L_p of the two-dimensional Hilbert curve family H_k^2 for various norm-parameter p-values and grid-order k-values, and (2) the contribution of our studies:

1. For $p = 1$: Niedermeier, Reinhardt, and Sanders [28] give a lower bound for $L_1(H_k^2)$: for all $k \geq 1$,

   $$L_1(H_k^2) \geq \frac{(3 \cdot 2^{k-1} - 2)^2}{4^{k-1}},$$

   and Chochia, Cole, and Heywood [10] provide a matching upper bound for $L_1(H_k^2)$ for all $k \geq 2$. We will prove the exact formula for $L_1(H_k^2)$ for all $k \geq 2$ (preliminary version in [12]).
2. For $p = 2$: Gotsman and Lindenbaum [17] derive a lower and upper bounds for $L_2(H_k^2)$: for all $k \geq 6$,

   $$\frac{(2^{k-1} - 1)^2}{\frac{2}{3} \cdot 4^{k-2} + \frac{1}{3}} \leq L_2(H_k^2) \leq 6\frac{2}{3},$$

   and Alber and Niedermeier [2] improves the upper bound for $L_2(H_k^2)$: for all $k \geq 1$,

   $$L_2(H_k^2) \leq 6\frac{1}{2}.$$

   We will prove that the lower bound above [17] is the exact formula for $L_2(H_k^2)$ (preliminary version in [12]): for all $k \geq 5$,

   $$L_2(H_k^2) = 6 \cdot \frac{2^{2k-3} - 2^{k-1} + 2^{-1}}{2^{2k-3} + 1}.$$

   Bauman [6] obtains a matching lower and upper bounds for $L_2(H_k^2)$ for $k = \infty$:

   $$L_2(H_\infty^2) = 6.$$

3. For $2 < p \leq \infty$: Due to the monotonicity of the underlying p-normed metric: for every grid-point pair (v, u), the p-normed metric $d_p(v, u)$ is strictly decreasing in $p \in [1, \infty)$, we will prove the same exact formula for $L_p(H_k^2)$ as for the case when $p = 2$ (preliminary version in [12]):

$$L_p(H_k^2) = 6 \cdot \frac{2^{2k-3} - 2^{k-1} + 2^{-1}}{2^{2k-3} + 1} \text{ for all reals } p \geq 2.$$

When $p = \infty$, Alber [1] and Alber and Niedermeier [2] establish a lower and upper bounds for $L_\infty(H_k^2)$, respectively:

$$6(1 - O(2^{-k})) \leq L_\infty(H_k^2) \leq 6\frac{2}{5}.$$

We present analytical and empirical studies on the locality measure L_p for the two-dimensional Hilbert curve family over the entire spectrum of possible norm-parameter values. Our proofs of the exact formulas of $L_p(H_k^2)$ for $p \in \{1, 2\}$ follow a uniform approach: identifying all the representative grid-point pairs, which realize the $L_p(H_k^2)$-value, for each $p \in \{1, 2\}$. The analytical results close the gap between the current best lower and upper bounds with exact formulas for $p \in \{1, 2\}$, and extend to all reals $p \geq 2$.

While the three most obviously important norm-parameter p-values: $\{1, 2, \infty\}$ (rectilinear, Euclidean, and maximum metrics, respectively) are intimately related to intuitive concepts, in some cases the structure of applications of the Hilbert curves may suggest a different choice of p-value in the real unit interval (1, 2) as the most natural setting for the underlying locality measure. While not addressing the candidate exact formulas for $L_p(H_k^2)$ for $p \in (1, 2)$ (partial result in [15]), we present an empirical study on $L_p(H_k^2)$ for all norm-parameters $p \in [1, 2]$, which complements the incomplete analytical study and shows that: (1) The analytical results are consistent with program verification over various norm-parameter p-values and sufficiently large grid-order k-values, (2) As p increases over the real unit interval [1, 2], the locations of candidate representative grid-point pairs agree with the intuitive interpolation effect over the two delimiting p-values, and (3) Our empirical study will shed some light on determining the exact formulas for the locality measure for all reals $p \in (1, 2)$.

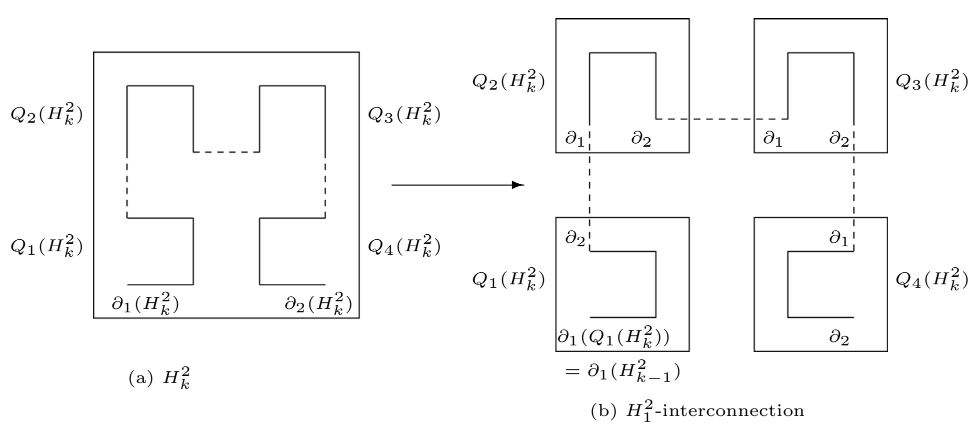
With diverse applications of the two-dimensional Hilbert curve family H_k^2, a practical implication of our results on the locality measure $L_p(H_k^2)$ over all real norm-parameters $p \in \{1\} \cup [2, \infty)$ is that the exact formulas provide precise bounds on measuring the loss in data locality in the one-dimensional index space, while spatial correlation exists in the two-dimensional grid space, or vice versa.

Analytical Studies of $L_p(H_k^2)$ with $p \geq 1$

For two-dimensional Hilbert curves, the recursive self-similar structural property decomposes H_k^2 into four identical H_{k-1}^2-subcurves via reflection and/or rotation, which are amalgamated together by an H_1^2-curve — inducing unique orientations of the four H_{k-1}^2-subcurves relative to that of the H_1^2-curve for only the case of a two-dimensional H_k^2. Following the linear order along this H_1^2-curve, we denote the four H_{k-1}^2-subcurves (quadrants) as $Q_1(H_k^2)$, $Q_2(H_k^2)$, $Q_3(H_k^2)$, and $Q_4(H_k^2)$.

We extend the notations to identify all H_l^2-subcurves of a structured H_k^2 for all $l \in [k]$ inductively on the grid-order. Let $Q_i(H_k^2)$ denote the ith H_{k-1}^2-subcurve (along the amalgamating H_1^2-curve) for all $i \in [2^2]$. Then for the ith H_{l-1}^2-subcurve, $Q_i(H_l^2)$, of H_l^2, where $2 < l \leq k$ and $i \in [2^2]$, let $Q_j(Q_i(H_l^2))$ denote the jth H_{l-2}^2-subcurve of $Q_i(H_l^2)$ for all $j \in [2^2]$. We write $Q_i^{q+1}(H_l^2)$ for $Q_i(Q_i^q(H_l^2))$ for all $l \in [k]$ and all positive integers $q < l$. The notation $Q_i^l(H_l^2)$ identifies the ith grid point in the H_1^2-subcurve $Q_i^{l-1}(H_l^2)$.

Fig. 2 Generation of H_k^2 from a H_1^2-interconnection of four H_{k-1}^2-subcurves with their labeled entries and exits



For a two-dimensional Hilbert curve H_k^2 indexing the grid $[2^k]^2$, with a canonical orientation shown in Fig. 2a, denote by $\partial_1(H_k^2)$ and $\partial_2(H_k^2)$ the entry and the exit, respectively, grid points in $[2^k]^2$ (with respect to the canonical orientation). Figure 2 depicts the decomposition of H_k^2 and the ∂_1- and ∂_2 -labels of four H_{k-1}^2-subcurves.

For a two-dimensional Hilbert curve H_k^2 in a Cartesian x-y coordinate system, and for a grid point v indexed by H_k^2, we denote by $\mathrm{x}(v)$ and $\mathrm{y}(v)$ the x- and y-coordinate of v, respectively, and by $(\mathrm{x}(v), \mathrm{y}(v))$ the grid point v in the coordinate system. For an H_l^2-subcurve C of H_k^2, where $l \in [k]$, notice that its entry $\partial_1(C)$ and exit $\partial_2(C)$ differ in exactly one coordinate: x- or y-coordinate, say $z \in \{x, y\}$. We say that the subcurve C is z^+-oriented (respectively, z^--oriented) if the z-coordinate of $\partial_1(C)$ is less than (respectively, greater than) that of $\partial_2(C)$. Note that: (1) the x- and y-coordinates of $\partial_1(H_k^2)$ and $\partial_2(H_k^2)$ uniquely determine those of $\partial_1(H_l^2)$ and $\partial_2(H_l^2)$ for all $l \in [k]$, and (2) the two subcurves $Q_2(H_k^2)$ and $Q_3(H_k^2)$ inherit the orientation from their supercurve H_k^2.

For a space-filling curve C indexing an m-dimensional grid space, the notation "$v \in C$" refers to "the grid point v indexed by C", and $C^{-1}(v)$ gives the index of v in the one-dimensional index space. We denote, for m-dimensional grid-point pair $v = (v_1, v_2, \dots, v_m)$ and $u = (u_1, u_2, \dots, u_m)$, and for positive real norm-parameter p,

$$d_p(v, u) = (\sum_{i=1}^{m} |v_i - u_i|^p)^{\frac{1}{p}}.$$

Note that, for $0 < p < 1$, the formula of d_p fails to be a norm since it defines an absolutely homogeneous function but is not subadditive. The locality measure in our studies is, for all reals $p \geq 1$,

$$L_p(C) = \max_{\text{indices } i,j \in [n^m]} \frac{d_p(C(i), C(j))^m}{d_p(i,j)} \left(= \max_{\text{indices } i,j \in [n^m]} \frac{d_p(C(i), C(j))^m}{|i-j|}\right) = \max_{v,u \in C} \frac{d_p(v, u)^m}{|C^{-1}(v) - C^{-1}(u)|}.$$

When $m = 2$, the following denotations represent the above locality measure with respect to a grid-point pair and a subcurve pair. We write $\mathcal{L}_{C,p}(v, u) = \frac{d_p(v,u)^2}{\delta_C(v,u)}$, where $\delta_C(v, u)$ denotes the index-difference $|C^{-1}(v) - C^{-1}(u)|$, and generalize the notations $L_p(C)$ and $\mathcal{L}_{C,p}$ for a subcurve C (of a two-dimensional space-filling curve) in an obvious manner. For two subcurves C_1 and C_2 of a two-dimensional space-filling curve C, denote:

$$\mathcal{L}_{C,p}(C_1, C_2) = \max_{(v,u) \in C_1 \times C_2} \mathcal{L}_{C,p}(v, u).$$

We define order relations among grid-point pairs and subcurve pairs with respect to the locality measure $\mathcal{L}_{C,p}$ as follows. For subcurves C_1, C_2, C_1', and C_2' of C, a grid-point pair $(v_1, v_2) \in C_1 \times C_2$ is reducible to a grid-point pair $(v_1', v_2') \in C_1' \times C_2'$ if $\mathcal{L}_{C,p}(v_1, v_2) \leq \mathcal{L}_{C,p}(v_1', v_2')$ — denoted by $(v_1, v_2) \preceq (v_1', v_2')$, and subcurve pair $C_1 \times C_2$ is reducible to subcurve pair $C_1' \times C_2'$ if for every $(v_1, v_2) \in C_1 \times C_2$, there exists $(v_1', v_2') \in C_1' \times C_2'$ such that (v_1, v_2) is reducible to (v_1', v_2') — denoted by $C_1 \times C_2 \preceq C_1' \times C_2'$. We define the strict reducibility, denoted by $\prec$, for grid-point pairs and subcurve pairs via the strict inequality of $\mathcal{L}_{C,p}$-values in an obvious manner.

For two grid-point pairs (v, u) and (v', u') indexed by C, denote:

$$s_{C,p}(v', u', v, u) = d_p(v', u')^2 \delta_C(v, u) - d_p(v, u)^2 \delta_C(v', u').$$

Grid-point pairs can be ordered with respect to the measure $\mathcal{L}_{C,p}$ via the algebraic sign of $s_{C,p}$-values. We summarize the reducibility conditions via $s_{C,p}$-values in Lemma 1, whose proof simply follows from the definitions.

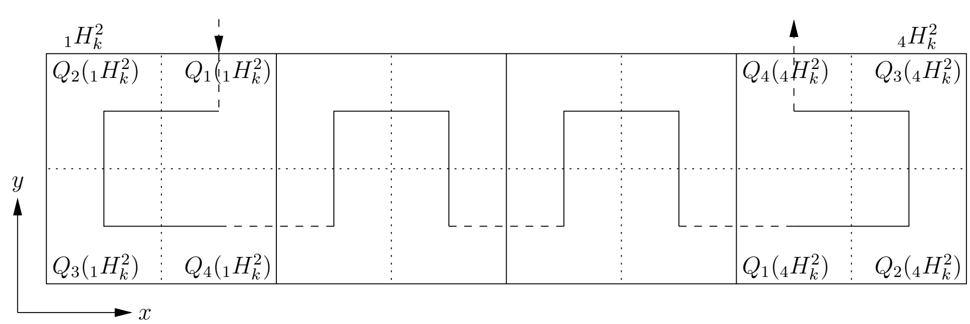


Fig. 3 Four linearly-contiguous H_k^2-subcurves in a canonical Cartesian coordinate system

Lemma 1 *For two arbitrary grid-point pairs,* (v, u) *and* (v', u')*, indexed by a space-filling curve* C *of a two-dimensional grid-space, and all real norm-parameters* $p \geq 1$, $\mathcal{L}_{C,p}(v,u) \leq \mathcal{L}_{C,p}(v',u')$ *(equivalently,* $(v,u) \preceq (v',u')$*) if and only if* $s_{C,p}(v',u',v,u)$ $(= d_p(v',u')^2\delta_C(v,u) - d_p(v,u)^2\delta_C(v',u')) \geq 0$; *the equivalence remains true also for strict inequalities and strict reducibility.*

A pair of grid points v and u indexed by C is a representative for C with respect to L_p if $\mathcal{L}_{C,p}(v,u) = L_p(C)$, or, equivalently, for all $v', u' \in C$, $(v',u') \preceq (v,u)$. Many of our main results encompass identifications of candidate representative grid-point pairs for C, which often involve sequences of reductions via successive considerations of two grid-point pairs and the comparisons of their $\mathcal{L}_{C,p}$-values. Our studies of $L_p(H_k^2)$ cover all real norm-parameters $p \geq 1$. The geometric characteristics of the underlying p-norm that is rectilinear or Euclidean metric of $p = 1$ or $p = 2$, respectively, help distinguish candidate representative grid-point pairs and verify tedious reductions. However, for all reals $p \in (1, 2)$, the lack of geometric clarity for interpreting $\mathcal{L}_{C,p}$- and hence L_p-values adversely increases the complexity: (1) of identifying candidate representative grid-point pairs, and (2) in comparing $\mathcal{L}_{H_k^2,p}$-values for reductions due to the complex interplay of the norm-parameter p-value and grid-order k-value.

Exact Formulas for $L_p(H_k^2)$ with $p \geq 2$

To obtain exact formulas for $L_p(H_k^2)$ for all reals $p \geq 2$, it suffices to consider identifying all representative pairs that yield, for $p = 2$, $\mathcal{L}_{H_k^2,2}(v,u) = L_2(H_k^2)$, due to the monotonicity of the underlying p-normed metric. In "Exact Formulas for $L_p(H_k^2)$ with $p > 2$" section, Lemma 9 and Theorem 3 reduce the consideration of $L_2(H_k^2)$ for the case of $p > 2$ to $p = 2$.

A more refined combinatorial analysis based on the upper-bound argument in [17] reveals in Theorem 2 below that the representative grid-point pair resides in a subcurve C composed of four linearly-contiguous Hilbert subcurves. In "L_2-Locality of Four Linearly Contiguous Hilbert Subcurves" and "Exact Formula for $L_2(H_k^2)$" sections, we derive the exact formula for $L_2(C)$, which is used to deduce that for $L_2(H_k^2)$.

L_2-Locality of Four Linearly Contiguous Hilbert Subcurves

For a two-dimensional Hilbert curve H_l^2 with $l \geq 4$, there exists a subcurve C that is composed of four linearly-contiguous H_k^2-subcurves with $k = l - 3$. Figure 3 depicts the arrangement in a canonical Cartesian coordinate system. Denote the leftmost and rightmost (first and fourth in the traversal order) H_k^2-subcurves by ${}_1H_k^2$ (y^--oriented) and ${}_4H_k^2$ (y^+-oriented), respectively.

In this subsection, we assume the canonical coordinate system as shown in Fig. 3 such that the lower-left corner grid point of ${}_1H_k^2$ is the origin (1, 1) of the coordinate system. In the following analysis, we identify a pair of grid points $v' \in {}_1H_k^2$ and $u' \in {}_4H_k^2$ such that $\mathcal{L}_{C,2}(v',u') = \mathcal{L}_{C,2}({}_1H_k^2, {}_4H_k^2)$; we show explicitly that such a grid-point pair must necessarily be the lower-left and lower-right corners of C. In "Exact Formula for $L_2(H_k^2)$" section, we prove that (v', u') (or its symmetry) serves as the representative pair for the entire H_k^2 with respect to L_2.

To locate a candidate representative grid-point pair $v \in {}_1H_k^2$ and $u \in {}_4H_k^2$, Lemmas 2–4 show that the possibility "$v \in Q_3({}_1H_k^2)$ and $u \in Q_3({}_4H_k^2)$" is reduced to, with respect to u, seeking v in successive Q_3-subcurves of ${}_1H_k^2$.

Lemma 2 *For all positive integers* $k \geq 2$, *and all grid-point pairs* $v \in Q_3({}_1H_k^2) - Q_3(Q_3({}_1H_k^2))$ *and* $u \in Q_3({}_4H_k^2)$, *there exists* $v' \in Q_3(Q_3({}_1H_k^2))$ *such that* $(v,u) \prec (v',u)$ *via the comparison*: $\mathcal{L}_{C,2}(v,u) < \mathcal{L}_{C,2}(v',u)$.

Proof Note that the partition of $Q_3({}_1H_k^2) - Q_3(Q_3({}_1H_k^2))$ $= Q_1(Q_3({}_1H_k^2)) \cup Q_2(Q_3({}_1H_k^2)) \cup Q_4(Q_3({}_1H_k^2))$ suggests the consideration of the following three cases, in which the geometric interpretation of the underlying 2-normed (Euclidean) distance helps identify and verify sequences of reductions in maximizing $\mathcal{L}_{C,2}$-values.

Case 1: $v \in Q_2(Q_3({}_1H_k^2))$. Consider $v' \in Q_3(Q_3({}_1H_k^2))$ with $\mathrm{x}(v') = \mathrm{x}(v)$, then we have $d_2(v',u)^2 > d_2(v,u)^2$ and $\delta_C(v',u) < \delta_C(v,u)$, which yield that $s_{C,2}(v',u,v,u) > 0$ in Lemma 1; we have $\mathcal{L}_{C,2}(v,u) < \mathcal{L}_{C,2}(v',u)$.

Case 2: $v \in Q_1(Q_3({}_1H_k^2))$. Consider $v'' \in Q_2(Q_3({}_1H_k^2))$ with $\mathrm{y}(v'') = \mathrm{y}(v)$, then, as in Case 1, we have $s_{C,2}(v'',u,v,u) > 0$ and $\mathcal{L}_{C,2}(v,u) < \mathcal{L}_{C,2}(v'',u)$. From Case 1, there exists $v' \in Q_3(Q_3({}_1H_k^2))$ such that $\mathcal{L}_{C,2}(v,u) < \mathcal{L}_{C,2}(v'',u) < \mathcal{L}_{C,2}(v',u)$.

Case 3: $v \in Q_4(Q_3({}_1H_k^2))$. Consider $v' \in Q_3(Q_3({}_1H_k^2))$ with $\mathrm{x}(v') = 1$ and $\mathrm{y}(v') = \mathrm{y}(v)$, and we show that $s_{C,2}(v',u,v,u) > 0$ as follows.

1. We expand $s_{C,2}(v',u,v,u)$ in terms of x- and y-coordinates of relevant grid points:

$$
\begin{aligned}
&s_{C,2}(v', u, v, u) \\
&= d_2(v', u)^2 \delta_C(v, u) - d_2(v, u)^2 \delta_C(v', u) \\
&= ((\mathrm{x}(u) - \mathrm{x}(v'))^2 + (\mathrm{y}(u) - \mathrm{y}(v'))^2) \\
&\quad \cdot (\delta_C(v, \partial_2({}_1H_k^2)) + 2 \cdot 2^{2k} + \delta_C(u, \partial_1({}_4H_k^2)) + 1) \\
&\quad - ((\mathrm{x}(u) - \mathrm{x}(v))^2 + (\mathrm{y}(u) - \mathrm{y}(v))^2) \\
&\quad \cdot (\delta_C(v', \partial_2({}_1H_k^2)) + 2 \cdot 2^{2k} + \delta_C(u, \partial_1({}_4H_k^2)) + 1) \\
&= ((\mathrm{x}(u) - 1)^2)(\delta_C(v, \partial_2({}_1H_k^2)) + 2 \cdot 2^{2k} + \delta_C(u, \partial_1({}_4H_k^2)) + 1) \\
&\quad + (\mathrm{y}(u) - \mathrm{y}(v))^2(\delta_C(v, \partial_2({}_1H_k^2)) + 2 \cdot 2^{2k} + \delta_C(u, \partial_1({}_4H_k^2)) + 1) \\
&\quad (\text{note that } \mathrm{x}(v') = 1 \text{ and } \mathrm{y}(v') = \mathrm{y}(v)) \\
&\quad - ((\mathrm{x}(u) - \mathrm{x}(v))^2)(\delta_C(v', \partial_2({}_1H_k^2)) + 2 \cdot 2^{2k} + \delta_C(u, \partial_1({}_4H_k^2)) + 1) \\
&\quad - (\mathrm{y}(u) - \mathrm{y}(v))^2(\delta_C(v', \partial_2({}_1H_k^2)) + 2 \cdot 2^{2k} + \delta_C(u, \partial_1({}_4H_k^2)) + 1) \\
&= \mathrm{x}(u)^2(\delta_C(v, \partial_2({}_1H_k^2)) - \delta_C(v', \partial_2({}_1H_k^2))) \\
&\quad + (-2\mathrm{x}(u) + 1 + 2\mathrm{x}(u)\mathrm{x}(v) - \mathrm{x}(v)^2)(\delta_C(v, \partial_2({}_1H_k^2)) + \delta_C(u, \partial_1({}_4H_k^2))) \\
&\quad + (2\mathrm{x}(u)\mathrm{x}(v) - \mathrm{x}(v)^2)(\delta_C(v', \partial_2({}_1H_k^2)) - \delta_C(v, \partial_2({}_1H_k^2))) \\
&\quad + (\mathrm{y}(u) - \mathrm{y}(v))^2 \delta_C(v, \partial_2({}_1H_k^2)) - (\mathrm{y}(u) - \mathrm{y}(v))^2 \delta_C(v', \partial_2({}_1H_k^2)) \\
&= \mathrm{x}(u)^2(\delta_C(v, \partial_2({}_1H_k^2)) - \delta_C(v', \partial_2({}_1H_k^2))) \\
&\quad + (2\mathrm{x}(u) - \mathrm{x}(v) - 1)(\mathrm{x}(v) - 1)(2 \cdot 2^{2k} + 1) \\
&\quad + (2\mathrm{x}(u) - \mathrm{x}(v) - 1)(\mathrm{x}(v) - 1)(\delta_C(v, \partial_2({}_1H_k^2)) + \delta_C(u, \partial_1({}_4H_k^2))) \\
&\quad + (2\mathrm{x}(u)\mathrm{x}(v) - \mathrm{x}(v)^2)(\delta_C(v', \partial_2({}_1H_k^2)) - \delta_C(v, \partial_2({}_1H_k^2))) \\
&\quad + (\mathrm{y}(u) - \mathrm{y}(v))^2(\delta_C(v, \partial_2({}_1H_k^2)) - \delta_C(v', \partial_2({}_1H_k^2))).
\end{aligned}
$$

2. We bound all the x- and y-coordinate, and index-differences of relevant grid points by noting that $u \in Q_3({}_4H_k^2)$, and $v \in Q_4(Q_3({}_1H_k^2))$ and its corresponding $v' \in Q_3(Q_3({}_1H_k^2))$:

$$
\begin{aligned}
&4 \cdot 2^k \geq \mathrm{x}(u) \geq \frac{7}{2} \cdot 2^k + 1, \quad 2^k \geq \mathrm{y}(u) \geq \frac{1}{2} \cdot 2^k + 1; \\
&\frac{1}{2} \cdot 2^k \geq \mathrm{x}(v) \geq \frac{1}{4} \cdot 2^k + 1, \quad \frac{1}{4} \cdot 2^k \geq \mathrm{y}(v) \geq 1; \\
&\frac{5}{16} \cdot 2^{2k} > \delta_C(v, \partial_2({}_1H_k^2)) \geq \frac{1}{4} \cdot 2^{2k}, \\
&\frac{6}{16} \cdot 2^{2k} > \delta_C(v', \partial_2({}_1H_k^2)) \geq \frac{5}{16} \cdot 2^{2k}.
\end{aligned}
$$

3. The lower and upper bounds in item 2 above yield the following bounds for the five terms appearing in $d_2(v', u)^2 \delta_C(v, u) - d_2(v, u)^2 \delta_C(v', u)$ in item 1:

   (a) $\mathrm{x}(u)^2(\delta_C(v, \partial_2({}_1H_k^2)) - \delta_C(v', \partial_2({}_1H_k^2))) \geq -2 \cdot 2^{4k}$,
   (b) $(2\mathrm{x}(u) - \mathrm{x}(v) - 1)(\mathrm{x}(v) - 1)(2 \cdot 2^{2k} + 1) \geq (7 \cdot 2^k - \frac{1}{4} \cdot 2^k + 1)(\frac{1}{4} \cdot 2^k)(2 \cdot 2^{2k} + 1) > \frac{27}{8} \cdot 2^{4k}$,
   (c) $(2\mathrm{x}(u) - \mathrm{x}(v) - 1)(\mathrm{x}(v) - 1)(\delta_C(v, \partial_2({}_1H_k^2)) + \delta_C(u, \partial_1({}_4H_k^2))) \geq (7 \cdot 2^k - \frac{1}{4} \cdot 2^k + 1)(\frac{1}{4} \cdot 2^k)(\frac{3}{4} \cdot 2^{2k}) > 0$,
   (d) $(2\mathrm{x}(u)\mathrm{x}(v) - \mathrm{x}(v)^2)(\delta_C(v', \partial_2({}_1H_k^2)) - \delta_C(v, \partial_2({}_1H_k^2))) > 0$, and
   (e) $(\mathrm{y}(u) - \mathrm{y}(v))^2(\delta_C(v, \partial_2({}_1H_k^2)) - \delta_C(v', \partial_2({}_1H_k^2))) \geq (2^k - 1)^2(-\frac{2}{16} \cdot 2^{2k}) > -\frac{1}{8} \cdot 2^{4k}$.

   These five terms together show that the grid point $v' \in Q_3(Q_3({}_1H_k^2))$ with $\mathrm{x}(v') = 1$ and $\mathrm{y}(v') = \mathrm{y}(v)$ satisfies that:

$$s_{C,2}(v', u, v, u) = d_2(v', u)^2 \delta_C(v, u) - d_2(v, u)^2 \delta_C(v', u) > 0,$$

   hence $\mathcal{L}_{C,2}(v, u) < \mathcal{L}_{C,2}(v', u)$.

Combining the three cases, the lemma is proved. □

Lemma 3 *For all positive integers k and h with $1 \leq h < k$, and all grid-point pairs $v \in Q_3^h({}_1H_k^2) - Q_3^{h+1}({}_1H_k^2)$ and $u \in Q_3({}_4H_k^2)$, there exists $v' \in Q_3^{h+1}({}_1H_k^2)$ such that $(v, u) \prec (v', u)$ via the comparison: $\mathcal{L}_{C,2}(v, u) < \mathcal{L}_{C,2}(v', u)$.*

Proof Similar to the proof of Lemma 2. By focusing on $Q_3^{h-1}({}_1H_k^2)$, we rephrase the statement of the lemma as: for all integers k and h with $1 \leq h < k$, and all $v \in Q_3(Q_3^{h-1}({}_1H_k^2)) - Q_3(Q_3(Q_3^{h-1}({}_1H_k^2)))$ and $u \in Q_3({}_4H_k^2)$, there exists $v' \in Q_3(Q_3(Q_3^{h-1}({}_1H_k^2)))$ such that $(v, u) \prec (v', u)$ via the comparison: $\mathcal{L}_{C,2}(v, u) < \mathcal{L}_{C,2}(v', u)$.

We proceed analogously as in the proof of Lemma 2. Noting that:

$$Q_3(Q_3^{h-1}({}_1H_k^2)) - Q_3(Q_3(Q_3^{h-1}({}_1H_k^2))) = Q_1(Q_3(Q_3^{h-1}({}_1H_k^2))) \cup Q_2(Q_3(Q_3^{h-1}({}_1H_k^2))) \cup Q_4(Q_3(Q_3^{h-1}({}_1H_k^2))),$$

we consider the following three cases.

Case 1: $v \in Q_2(Q_3(Q_3^{h-1}({}_1H_k^2)))$. Consider $v' \in Q_3(Q_3(Q_3^{h-1}({}_1H_k^2)))$ with $\mathrm{x}(v') = \mathrm{x}(v)$, then $d_2(v',u)^2 > d_2(v,u)^2$ and $\delta_C(v',u) < \delta_C(v,u)$, which gives that $s_{C,2}(v',u,v,u) > 0$ in Lemma 1; hence $\mathcal{L}_{C,2}(v,u) < \mathcal{L}_{C,2}(v',u)$.

Case 2: $v \in Q_1(Q_3(Q_3^{h-1}({}_1H_k^2)))$. Consider $v'' \in Q_2(Q_3(Q_3^{h-1}({}_1H_k^2)))$ with $\mathrm{y}(v'') = \mathrm{y}(v)$, then, as in Case 1, we have $s_{C,2}(v'',u,v,u) > 0$ and $\mathcal{L}_{C,2}(v,u) \le \mathcal{L}_{C,2}(v'',u)$. Then from Case 1, there exists $v' \in Q_3(Q_3(Q_3^{h-1}({}_1H_k^2)))$ such that $\mathcal{L}_2(v,u) < \mathcal{L}_{C,2}(v'',u) < \mathcal{L}_{C,2}(v',u)$.

Case 3: $v \in Q_4(Q_3(Q_3^{h-1}({}_1H_k^2)))$. Consider $v' \in Q_3(Q_3(Q_3^{h-1}({}_1H_k^2)))$ with $\mathrm{x}(v') = 1$ and $\mathrm{y}(v') = \mathrm{y}(v)$, we prove that $s_{C,2}(v',u,v,u) > 0$ as follows.

1. We expand $s_{C,2}(v',u,v,u)$ in terms of x- and y-coordinates of relevant grid points:

$$\begin{aligned}
&s_{C,2}(v',u,v,u)\\
&= d_2(v',u)^2\delta_C(v,u) - d_2(v,u)^2\delta_C(v',u)\\
&= ((\mathrm{x}(u)-\mathrm{x}(v'))^2 + (\mathrm{y}(u)-\mathrm{y}(v'))^2)\\
&\quad \cdot (\delta_C(v,\partial_2(Q_3^{h-1}({}_1H_k^2))) + \frac{7\cdot 2^{2k} - 2^{2(k-h-1)}}{3} + \delta_C(u,\partial_1({}_4H_k^2)) + 1)\\
&\quad - ((\mathrm{x}(u)-\mathrm{x}(v))^2 + (\mathrm{y}(u)-\mathrm{y}(v))^2)\\
&\quad \cdot (\delta_C(v',\partial_2(Q_3^{h-1}({}_1H_k^2))) + \frac{7\cdot 2^{2k} - 2^{2(k-h-1)}}{3} + \delta_C(u,\partial_1({}_4H_k^2)) + 1)\\
&= ((\mathrm{x}(u)-1)^2)(\delta_C(v,\partial_2(Q_3^{h-1}({}_1H_k^2))) + \frac{7\cdot 2^{2k} - 2^{2(k-h-1)}}{3}\\
&\quad + \delta_C(u,\partial_1({}_4H_k^2)) + 1)\\
&\quad + (\mathrm{y}(u)-\mathrm{y}(v))^2(\delta_C(v,\partial_2(Q_3^{h-1}({}_1H_k^2))) + \frac{7\cdot 2^{2k} - 2^{2(k-h-1)}}{3} + \delta_C(u,\partial_1({}_4H_k^2)) + 1)\\
&\quad (\text{note that } \mathrm{x}(v')=1, \mathrm{y}(v')=\mathrm{y}(v))\\
&\quad - ((\mathrm{x}(u)-\mathrm{x}(v))^2)(\delta_C(v',\partial_2(Q_3^{h-1}({}_1H_k^2))) + \frac{7\cdot 2^{2k} - 2^{2(k-h-1)}}{3} + \delta_C(u,\partial_1({}_4H_k^2)) + 1)\\
&\quad - (\mathrm{y}(u)-\mathrm{y}(v))^2(\delta_C(v',\partial_2(Q_3^{h-1}({}_1H_k^2))) + \frac{7\cdot 2^{2k} - 2^{2(k-h-1)}}{3} + \delta_C(u,\partial_1({}_4H_k^2)) + 1)\\
&= \mathrm{x}(u)^2(\delta_C(v,\partial_2(Q_3^{h-1}({}_1H_k^2))) - \delta_C(v',\partial_2(Q_3^{h-1}({}_1H_k^2))))\\
&\quad + (-2\mathrm{x}(u) + 1 + 2\mathrm{x}(u)\mathrm{x}(v) - \mathrm{x}(v)^2)(\frac{7\cdot 2^{2k} - 2^{2(k-h-1)}}{3} + 1)\\
&\quad + (-2\mathrm{x}(u) + 1 + 2\mathrm{x}(u)\mathrm{x}(v) - \mathrm{x}(v)^2)\\
&\quad \cdot (\delta_C(v,\partial_2(Q_3^{h-1}({}_1H_k^2))) + \delta_C(u,\partial_1({}_4H_k^2)))\\
&\quad + (2\mathrm{x}(u)\mathrm{x}(v) - \mathrm{x}(v)^2)\\
&\quad \cdot (\delta_C(v',\partial_2(Q_3^{h-1}({}_1H_k^2))) - \delta_C(v,\partial_2(Q_3^{h-1}({}_1H_k^2))))\\
&\quad + (\mathrm{y}(u)-\mathrm{y}(v))^2\delta_C(v,\partial_2(Q_3^{h-1}({}_1H_k^2)))\\
&\quad - (\mathrm{y}(u)-\mathrm{y}(v))^2\delta_C(v',\partial_2(Q_3^{h-1}({}_1H_k^2)))\\
&= \mathrm{x}(u)^2(\delta_C(v,\partial_2(Q_3^{h-1}({}_1H_k^2))) - \delta_C(v',\partial_2(Q_3^{h-1}({}_1H_k^2))))\\
&\quad + (2\mathrm{x}(u) - \mathrm{x}(v) - 1)(\mathrm{x}(v) - 1)(\frac{7\cdot 2^{2k} - 2^{2(k-h-1)}}{3} + 1)\\
&\quad + (2\mathrm{x}(u) - \mathrm{x}(v) - 1)(\mathrm{x}(v) - 1)(\delta_C(v,\partial_2(Q_3^{h-1}({}_1H_k^2))) + \delta_C(u,\partial_1({}_4H_k^2)))\\
&\quad + (2\mathrm{x}(u)\mathrm{x}(v) - \mathrm{x}(v)^2)(\delta_C(v',\partial_2(Q_3^{h-1}({}_1H_k^2))) - \delta_C(v,\partial_2(Q_3^{h-1}({}_1H_k^2))))\\
&\quad + (\mathrm{y}(u)-\mathrm{y}(v))^2(\delta_C(v,\partial_2(Q_3^{h-1}({}_1H_k^2))) - \delta_C(v',\partial_2(Q_3^{h-1}({}_1H_k^2)))).
\end{aligned}$$

2. We bound all the x- and y-coordinate, and index-differences of relevant grid points via:

$$4 \cdot 2^k \geq \mathrm{x}(u) \geq \frac{7}{2} \cdot 2^k + 1, \quad 2^k \geq \mathrm{y}(u) \geq \frac{1}{2} \cdot 2^k + 1;$$
$$2^{k-h} \geq \mathrm{x}(v) \geq 2^{k-h-1} + 1, \quad 2^{k-h-1} \geq \mathrm{y}(v) \geq 1;$$
$$\frac{2^{2k} - 2^{2(k-h)}}{3} + 2^{2(k-h-1)} > \delta_C(v, \partial_2(Q_3^{h-1}({}_1H_k^2))) \geq \frac{2^{2k} - 2^{2(k-h)}}{3},$$
$$\frac{2^{2k} - 2^{2(k-h)}}{3} + 2 \cdot 2^{2(k-h-1)} > \delta_C(v', \partial_2(Q_3^{h-1}({}_1H_k^2))) \geq \frac{2^{2k} - 2^{2(k-h)}}{3} + 2^{2(k-h-1)}.$$

3. The lower and upper bounds in item 2 above yield the following bounds for the five terms appearing in $d_2(v',u)^2\delta_C(v,u) - d_2(v,u)^2\delta_C(v',u)$ in item 1:
   (a) $\mathrm{x}(u)^2(\delta_C(v, \partial_2(Q_3^{h-1}({}_1H_k^2))) - \delta_C(v', \partial_2(Q_3^{h-1}({}_1H_k^2)))) \geq -2^{4k-2h+3}$,
   (b) $(2\mathrm{x}(u) - \mathrm{x}(v) - 1)(\mathrm{x}(v) - 1)(\frac{7 \cdot 2^{2k} - 2^{2k-2h-2}}{3} + 1) \geq (7 \cdot 2^k - 2^{k-h} + 1)(2^{k-h-1})(\frac{7 \cdot 2^{2k} - 2^{2k-2h-2}}{3} + 1) > \frac{46}{3} \cdot 2^{4k-h-1}$,
   (c) $(2\mathrm{x}(u) - \mathrm{x}(v) - 1)(\mathrm{x}(v) - 1)(\delta_C(v, \partial_2(Q_3^{h-1}({}_1H_k^2))) + \delta_C(u, \partial_1({}_4H_k^2))) \geq (7 \cdot 2^k - 2^{k-h} + 1)(2^{k-h-1})(\frac{2^{2k} - 2^{2(k-h)}}{3} + 2^{2(k-h-1)} + \frac{1}{2} \cdot 2^{2k}) > \frac{1}{3} \cdot (35 \cdot 2^{4k-h-2} - 5 \cdot 2^{4k-2h-2} - 7 \cdot 2^{4k-3h-3} + 2^{4k-4h-3})$,
   (d) $(2\mathrm{x}(u)\mathrm{x}(v) - \mathrm{x}(v)^2)(\delta_C(v', \partial_2(Q_3^{h-1}({}_1H_k^2))) - \delta_C(v, \partial_2(Q_3^{h-1}({}_1H_k^2)))) > 0$, and
   (e) $(\mathrm{y}(u) - \mathrm{y}(v))^2(\delta_C(v, \partial_2(Q_3^{h-1}({}_1H_k^2))) - \delta_C(v', \partial_2(Q_3^{h-1}({}_1H_k^2)))) \geq (2^k - 1)^2(-2 \cdot 2^{2(k-h-1)}) > -2^{4k-2h-1}$.

   These five terms together show that the grid point $v' \in Q_3(Q_3(Q_3^{h-1}({}_1H_k^2)))$ with $\mathrm{x}(v') = 1$ and $\mathrm{y}(v') = \mathrm{y}(v)$ satisfies that:

$$\begin{aligned} & s_{C,2}(v', u, v, u) \\ &= d_2(v',u)^2\delta_C(v,u) - d_2(v,u)^2\delta_C(v',u) \\ &> \frac{1}{3} \cdot (2^{4k-h-2}) \cdot (35 - 34 \cdot 2^{-h} - \frac{7}{2} \cdot 2^{-2h} + \frac{1}{2} \cdot 2^{-3h}) \\ &> 0 \text{ (note that } h \geq 1), \end{aligned}$$

   thus, $\mathcal{L}_{C,2}(v,u) < \mathcal{L}_{C,2}(v',u)$.

Combining the three cases, we have proved the lemma. □

Lemma 4 *For all positive integers k and h with $1 \leq h < k$, and all grid-point pairs $v \in Q_3^h({}_1H_k^2) - Q_3^k({}_1H_k^2)$ and $u \in Q_3({}_4H_k^2)$, there exists $v' \in Q_3^k({}_1H_k^2)$ such that $(v,u) \prec (v',u)$ via the comparison: $\mathcal{L}_{C,2}(v,u) < \mathcal{L}_{C,2}(v',u)$.*

Proof We prove the lemma by an induction on $k-h$. For the basis of the induction ($k-h=1$), we apply Lemma 3 with $h = k-1$.

For the induction step, assume that the statement in the lemma is true for all integers h with $1 \leq k-h < n$, where $n > 1$. Consider the case when $k-h = n$. Let $v \in Q_3^h({}_1H_k^2) - Q_3^k({}_1H_k^2)$ and $u \in Q_3({}_4H_k^2)$ be arbitrary. The partition of $Q_3^h({}_1H_k^2) = Q_3(Q_3^h({}_1H_k^2)) \cup (Q_1(Q_3^h({}_1H_k^2)) \cup Q_2(Q_3^h({}_1H_k^2)) \cup Q_4(Q_3^h({}_1H_k^2))) = Q_3^{h+1}({}_1H_k^2) \cup (Q_3^h({}_1H_k^2) - Q_3^{h+1}({}_1H_k^2))$ suggests that we consider the following two cases.

Case 1: $v \in Q_3^{h+1}({}_1H_k^2)$. Notice that $k-(h+1) < n$, and we apply the induction hypothesis for the case of $k-(h+1)$, and obtain a desired grid point v'.

Case 2: $v \in Q_3^h({}_1H_k^2) - Q_3^{h+1}({}_1H_k^2)$. By Lemma 3, there exists $v' \in Q_3^{h+1}({}_1H_k^2))$ such that $\mathcal{L}_{C,2}(v,u) < \mathcal{L}_{C,2}(v',u)$. If $v' \in Q_3^k({}_1H_k^2)$, then v' is a desired grid point. Otherwise ($v \in Q_3^{h+1}({}_1H_k^2) - Q_3^k({}_1H_k^2)$), it is reduced to Case 1.

This completes the induction step, and the lemma is proved. □

Lemma 4 asserts that the lower-left corner grid point v' with coordinates (1, 1) is unique in $Q_3({}_1H_k^2)$ for maximizing the $\mathcal{L}_{C,2}$-value: for arbitrary $u \in Q_3({}_4H_k^2)$, $\mathcal{L}_{C,2}(v',u) = \max\{\mathcal{L}_{C,2}(v,u) \mid v \in Q_3({}_1H_k^2)\}$.

The search for a candidate representative grid-point pair is reduced to a case-analysis for all possible combinations of subcurve pairs: $Q_i({}_1H_k^2) \times Q_j({}_4H_k^2)$ for all $i,j \in [4]$, and their possible systematic reductions. After eliminating symmetrical cases and grouping of underlying subcurves, it suffices to consider the analysis for five major cases: $Q_3({}_1H_k^2) \times Q_2({}_4H_k^2)$, $Q_3({}_1H_k^2) \times Q_3({}_4H_k^2)$, $Q_3({}_1H_k^2) \times Q_4({}_4H_k^2)$, $Q_4({}_1H_k^2) \times {}_4H_k^2$, and $(Q_1({}_1H_k^2) \cup Q_2({}_1H_k^2)) \times (Q_3({}_4H_k^2) \cup Q_4({}_4H_k^2))$. We can further discard the latter four subcurve pairs due to their (strict) reductions to the first subcurve pair $Q_3({}_1H_k^2) \times Q_2({}_4H_k^2)$ in Lemma 5.

Lemma 5 *For all positive integers $k \geq 1$, each of the following four subcurve pairs: $Q_3({}_1H_k^2) \times Q_3({}_4H_k^2)$, $Q_3({}_1H_k^2) \times Q_4({}_4H_k^2)$, $Q_4({}_1H_k^2) \times {}_4H_k^2$, and $(Q_1({}_1H_k^2) \cup Q_2({}_1H_k^2)) \times (Q_3({}_4H_k^2) \cup Q_4({}_4H_k^2))$ is strictly reducible to the subcurve pair $Q_3({}_1H_k^2) \times Q_2({}_4H_k^2)$:*

1. $Q_3({}_1H_k^2) \times Q_3({}_4H_k^2) \prec Q_3({}_1H_k^2) \times Q_2({}_4H_k^2)$,
2. $Q_3({}_1H_k^2) \times Q_4({}_4H_k^2) \prec Q_3({}_1H_k^2) \times Q_2({}_4H_k^2)$,
3. $Q_4({}_1H_k^2) \times {}_4H_k^2 \prec Q_3({}_1H_k^2) \times Q_2({}_4H_k^2)$, and
4. $(Q_1({}_1H_k^2) \cup Q_2({}_1H_k^2)) \times (Q_3({}_4H_k^2) \cup Q_4({}_4H_k^2)) \prec Q_3({}_1H_k^2) \times Q_2({}_4H_k^2)$.

Proof For part 1: $Q_3({}_1H_k^2) \times Q_3({}_4H_k^2) \prec Q_3({}_1H_k^2) \times Q_2({}_4H_k^2)$, we show that: for all positive integers $k \geq 1$, and all $(v, u) \in Q_3({}_1H_k^2) \times Q_3({}_4H_k^2)$, there exists $(v', u') \in Q_3({}_1H_k^2) \times Q_2({}_4H_k^2)$ such that $(v, u) \prec (v', u')$ via the comparison: $\mathcal{L}_{C,2}(v, u) < \mathcal{L}_{C,2}(v', u')$.

Consider $v' \in Q_3^k({}_1H_k^2)$ $(= (1, 1))$ and $u' \in Q_2({}_4H_k^2)$ with $\mathrm{x}(u') = \mathrm{x}(u)$ and $\mathrm{y}(u') = 1$. A case-analysis for $u \in Q_i(Q_3({}_4H_k^2))$ with $i \in [4]$ can show that $\mathcal{L}_{C,2}(v', u) < \mathcal{L}_{C,2}(v', u')$. By Lemma 4, $\mathcal{L}_{C,2}(v, u) \leq \mathcal{L}_{C,2}(v', u)$; therefore, $\mathcal{L}_{C,2}(v, u) < \mathcal{L}_{C,2}(v', u')$.

For part 2: $Q_3({}_1H_k^2) \times Q_4({}_4H_k^2) \prec Q_3({}_1H_k^2) \times Q_2({}_4H_k^2)$, we show that: for all positive integers $k \geq 1$, and all $(v, u) \in Q_3({}_1H_k^2) \times Q_4({}_4H_k^2)$, there exists $(v', u') \in Q_3({}_1H_k^2) \times Q_2({}_4H_k^2)$, such that $(v, u) \prec (v', u')$ via the comparison: $\mathcal{L}_{C,2}(v, u) < \mathcal{L}_{C,2}(v', u')$.

Consider $u'' \in Q_3({}_4H_k^2)$ with $\mathrm{y}(u'') = \mathrm{y}(u)$. Notice that $d_2(v, u'') > d_2(v, u)$ and $\delta_C(v, u'') < \delta_C(v, u)$, we have $\mathcal{L}_{C,2}(v, u) < \mathcal{L}_{C,2}(v, u'')$. By part 1 above, there exists $(v', u') \in Q_3({}_1H_k^2) \times Q_2({}_4H_k^2)$ such that $\mathcal{L}_{C,2}(v, u) < \mathcal{L}_{C,2}(v, u'') < \mathcal{L}_{C,2}(v', u')$.

We develop a strict reduction: $Q_4({}_1H_k^2) \times {}_4H_k^2 \prec Q_3({}_1H_k^2) \times {}_4H_k^2$ in Lemma 6 below that helps derive the strict reductions in the remaining two parts of Lemma 5. □

Lemma 6 *For all positive integers* $k \geq 1$, $Q_4({}_1H_k^2) \times {}_4H_k^2 \prec Q_3({}_1H_k^2) \times {}_4H_k^2$. *We show that: all positive integers* $k \geq 1$, *and all grid-point pairs* $v \in Q_4({}_1H_k^2)$ *and* $u \in {}_4H_k^2 (= Q_1({}_4H_k^2) \cup Q_2({}_4H_k^2) \cup Q_3({}_4H_k^2) \cup Q_4({}_4H_k^2))$, *there exists* $v' \in Q_3({}_1H_k^2)$ *such that* $(v, u) \prec (v', u)$ *via the comparison*: $\mathcal{L}_{C,2}(v, u) < \mathcal{L}_{C,2}(v', u)$.

Proof Consider $v' \in Q_3({}_1H_k^2)$ with $\mathrm{y}(v') = \mathrm{y}(v)$ and $\mathrm{x}(v') = 1$. A case-analysis for $u \in Q_i({}_4H_k^2)$ with $i \in [4]$ can show that $\mathcal{L}_{C,2}(v, u) < \mathcal{L}_{C,2}(v', u)$. □

We continue to part 3 of Lemma 5: $Q_4({}_1H_k^2) \times {}_4H_k^2 \prec Q_3({}_1H_k^2) \times Q_2({}_4H_k^2)$, and show that: for all positive integers $k \geq 1$, and all $(v, u) \in Q_4({}_1H_k^2) \times {}_4H_k^2$, there exists $(v', u') \in Q_3({}_1H_k^2) \times Q_2({}_4H_k^2)$ such that $(v, u) \prec (v', u')$ via the comparison: $\mathcal{L}_{C,2}(v, u) < \mathcal{L}_{C,2}(v', u')$.

Lemma 6 asserts that there exists $v' \in Q_3({}_1H_k^2)$ such that $\mathcal{L}_{C,2}(v, u) < \mathcal{L}_{C,2}(v', u)$. As $u \in {}_4H_k^2 = Q_1({}_4H_k^2) \cup Q_2({}_4H_k^2) \cup Q_3({}_4H_k^2) \cup Q_4({}_4H_k^2)$, we consider the four combinations of subcurve pairs for (v', u): $Q_3({}_1H_k^2) \times Q_i({}_4H_k^2)$ with $i \in [4]$. The analysis for the subcurve pair $Q_3({}_1H_k^2) \times Q_1({}_4H_k^2)$ is equivalent to that for $Q_4({}_1H_k^2) \times Q_2({}_4H_k^2)$, which is strictly reducible to $Q_3({}_1H_k^2) \times Q_2({}_4H_k^2)$ by applying Lemma 6. The subcurve pair $Q_3({}_1H_k^2) \times Q_3({}_4H_k^2)$ is strictly reducible to $Q_3({}_1H_k^2) \times Q_2({}_4H_k^2)$ by part 1 above, and the subcurve pair $Q_3({}_1H_k^2) \times Q_4({}_4H_k^2)$ is strictly reducible to $Q_3({}_1H_k^2) \times Q_2({}_4H_k^2)$ by part 2 above.

For part 4: $(Q_1({}_1H_k^2) \cup Q_2({}_1H_k^2)) \times (Q_3({}_4H_k^2) \cup Q_4({}_4H_k^2)) \prec Q_3({}_1H_k^2) \times Q_2({}_4H_k^2)$, we show that: for all positive integers $k \geq 1$, and all $(v, u) \in (Q_1({}_1H_k^2) \cup Q_2({}_1H_k^2)) \times (Q_3({}_4H_k^2) \cup Q_4({}_4H_k^2))$, there exists $(v', u') \in Q_3({}_1H_k^2) \times Q_2({}_4H_k^2)$ such that $(v, u) \prec (v', u')$ via the comparison: $\mathcal{L}_{C,2}(v, u) < \mathcal{L}_{C,2}(v', u')$.

Consider $v'' \in Q_3({}_1H_k^2) \cup Q_4({}_1H_k^2)$ with $\mathrm{x}(v'') = \mathrm{x}(v)$ and $\mathrm{y}(v'') = \mathrm{y}(v) - 2^{k-1}$, and $u'' \in Q_1({}_4H_k^2) \cup Q_2({}_4H_k^2)$ with $\mathrm{x}(u'') = \mathrm{x}(u)$ and $\mathrm{y}(u'') = \mathrm{y}(u) - 2^{k-1}$. Observe that $d_2(v'', u'') = d_2(v, u)$ and $\delta_C(v'', u'') < \delta_C(v, u)$, and we have $\mathcal{L}_{C,2}(v, u) < \mathcal{L}_{C,2}(v'', u'')$. Hence, it suffices to consider two combinations of subcurve pairs for (v'', u''): $Q_3({}_1H_k^2) \times Q_1({}_4H_k^2)$ and $Q_4({}_1H_k^2) \times (Q_1({}_4H_k^2) \cup Q_2({}_4H_k^2))$. The analysis for the subcurve pair $Q_3({}_1H_k^2) \times Q_1({}_4H_k^2)$ is equivalent to that for $Q_4({}_1H_k^2) \times Q_2({}_4H_k^2)$, which is strictly reducible to $Q_3({}_1H_k^2) \times Q_2({}_4H_k^2)$ by Lemma 6. The subcurve pair $Q_4({}_1H_k^2) \times (Q_1({}_4H_k^2) \cup Q_2({}_4H_k^2))$ is a subcase of part 3 above. Consequently, for these two combinations of subcurve pairs for (v'', u''), there exists $(v', u') \in Q_3({}_1H_k^2) \times Q_2({}_4H_k^2)$ such that $\mathcal{L}_{C,2}(v, u) < \mathcal{L}_{C,2}(v'', u'') < \mathcal{L}_{C,2}(v', u')$, as desired.

This completes the proof of Lemma 5. □

An immediate consequence of Lemma 5 supports and helps prove our geometric intuition that a representative grid-point pair must reside in $Q_3({}_1H_k^2) \times Q_2({}_4H_k^2)$, as stated in Corollary 1.

Corollary 1 *For all positive integers* $k \geq 1$, *and all grid-point pairs* $v \in {}_1H_k^2 - Q_3({}_1H_k^2)$ *and* $u \in {}_4H_k^2 - Q_2({}_4H_k^2)$, *there exist* $v' \in Q_3({}_1H_k^2)$ *and* $u' \in Q_2({}_4H_k^2)$ *such that* $(v, u) \prec (v', u')$ *via the comparison*: $\mathcal{L}_{C,2}(v, u) < \mathcal{L}_{C,2}(v', u')$.

Lemmas 7 and 8 below complement Lemmas 3 and 4, respectively, with analogous proofs. Applying Corollary 1 to reach the subcurve pair $Q_3({}_1H_k^2) \times Q_2({}_4H_k^2)$ for seeking a candidate representative grid-point pair (v', u'), the two lemmas guide the search into successive Q_3-subcurves of ${}_1H_k^2$ for v'. The symmetry in the subcurve pair $Q_3({}_1H_k^2) \times Q_2({}_4H_k^2)$ leads the search into successive Q_2-subcurves of ${}_4H_k^2$ for u'.

Lemma 7 *For all positive integers* k *and* h *with* $1 \leq h < k$, *and all grid-point pairs* $v \in Q_3^h({}_1H_k^2) - Q_3^{h+1}({}_1H_k^2)$ *and* $u \in Q_2({}_4H_k^2)$, *there exists* $v' \in Q_3^{h+1}({}_1H_k^2)$ *such that* $(v, u) \prec (v', u)$ *via the comparison*: $\mathcal{L}_{C,2}(v, u) < \mathcal{L}_{C,2}(v', u)$.

Lemma 8 *For all positive integers* k *and* h *with* $1 \leq h < k$, *and all grid-point pairs* $v \in Q_3^h({}_1H_k^2) - Q_3^k({}_1H_k^2)$ *and* $u \in Q_2({}_4H_k^2)$, *there exists* $v' \in Q_3^k({}_1H_k^2)$ *such that* $(v, u) \prec (v', u)$ *via the comparison*: $\mathcal{L}_{C,2}(v, u) < \mathcal{L}_{C,2}(v', u)$.

We summarize the analyses above in Theorem 1, which asserts that the unique representative grid-point pair reside at the lower-left and lower-right corners of C.

Theorem 1 *For all positive integers $k \geq 1$, and all grid-point pairs $(v, u) \in {}_1H_k^2 \times {}_4H_k^2 - Q_3^k({}_1H_k^2) \times Q_2^k({}_4H_k^2)$, there exist $v' \in Q_3^k({}_1H_k^2)$ and $u' \in Q_2^k({}_4H_k^2)$ such that $(v, u) \prec (v', u')$ via the comparison: $\mathcal{L}_{C,2}(v, u) < \mathcal{L}_{C,2}(v', u')$ and $\mathcal{L}_{C,2}(v', u') = 6 \cdot \frac{2^{2k+3}-2^{k+2}+2^{-1}}{2^{2k+3}+1}$.*

Proof By Corollary 1 and Lemma 8 (and its symmetry), the grid-point pair at the lower-left and lower-right corners of C: $v' \in Q_3^k({}_1H_k^2)$ with coordinates (1, 1) and $u' \in Q_2^k(H_k^2)$ with coordinates $(2^{k+2}, 1)$ maximizes the $\mathcal{L}_{C,2}$-value.

Notice that $\delta_C(v', u') = 2(\sum_{i=0}^{k-1} 2^{2i} + 1 + 2 \cdot 2^{2k}) - 1$, hence, $\mathcal{L}_{C,2}(v', u') = \frac{d_2(v',u')^2}{\delta_C(v',u')} = 6 \cdot \frac{2^{2k+3}-2^{k+2}+2^{-1}}{2^{2k+3}+1}$. □

Exact Formula for $L_2(H_k^2)$

The current best bounds for the two-dimensional Hilbert curve family with respect to L_2 (lower bound in [17] and upper bound in [2]) are:

$$6(1 - O(2^{-k})) \leq L_2(H_k^2) \leq 6\frac{1}{2}.$$

Following the argument in [17] with a more refined combinatorial analysis, together with the above-obtained exact formula for $\mathcal{L}_{C,2}(Q_3({}_1H_k^2), Q_2({}_4H_k^2))(= \mathcal{L}_{C,2}(Q_3^k({}_1H_k^2), Q_2^k({}_4H_k^2)))$ in "L_2-Locality of Four Linearly Contiguous Hilbert Subcurves" section, we close the gaps between the two bounds with an exact formula for $L_2(H_k^2)$.

Theorem 2 *For all positive integers $k \geq 5$,*

$$L_2(H_k^2) = 6 \cdot \frac{2^{2k-3} - 2^{k-1} + 2^{-1}}{2^{2k-3} + 1}.$$

Proof We refine a geometric constraint, from the upper-bound argument in [17], which relates the path-length of a subpath of H_k^2 versus the geometric distance between its initial and terminal grid points. Consider an arbitrary subcurve/subpath P of length l along H_k^2. Note that for arbitrary l, there exists a sufficiently large positive integer r such that $(2^{r-1})^2 < l \leq (2^r)^2$. This gives that P is contained in two adjacent quadrants Q' and Q'', each with size $(2^r)^2$ (grid points). Let D denote the Euclidean diameter (based on the 2-normed metric d_2) of the set of grid points in P. A case-analysis of subpath-containment of P in subquadrants of size $(2^{r-1})^2$ within $Q' \cup Q''$ results in the following six cases:

Case	Lower and upper bounds for l	Upper bounds for D^2 and $\frac{D^2}{l}$
1.	$\frac{4}{16} \cdot 4^r < l \leq \frac{5}{16} \cdot 4^r$:	$D^2 < \frac{5}{4} \cdot 4^r$, hence $\frac{D^2}{l} \leq 5$.
2.	$\frac{5}{16} \cdot 4^r < l \leq \frac{6}{16} \cdot 4^r$:	$D^2 < \frac{29}{16} \cdot 4^r$, hence $\frac{D^2}{l} \leq 5\frac{4}{5}$.
3.	$\frac{6}{16} \cdot 4^r < l \leq \frac{7}{16} \cdot 4^r$:	$D^2 < \frac{10}{4} \cdot 4^r$, hence $\frac{D^2}{l} \leq 6\frac{2}{3}$.
4.	$\frac{7}{16} \cdot 4^r < l \leq \frac{8}{16} \cdot 4^r$:	$D^2 < \frac{10}{4} \cdot 4^r$, hence $\frac{D^2}{l} \leq 5\frac{5}{7}$.
5.	$\frac{8}{16} \cdot 4^r < l \leq \frac{12}{16} \cdot 4^r$:	$D^2 < \frac{13}{4} \cdot 4^r$, hence $\frac{D^2}{l} \leq 6\frac{1}{2}$.
6.	$\frac{12}{16} \cdot 4^r < l \leq 4^r$:	$D^2 < 5 \cdot 4^r$, hence $\frac{D^2}{l} \leq 6\frac{2}{3}$.

To obtain the desired L_2-bound, it suffices to refine the analysis of subpath-containment in Cases 3, 5, and 6 in subquadrants of size $(2^{r-2})^2$.

The refined analysis for Case 3 yields the upper bounds on $\frac{D^2}{l}$: $\frac{29}{6}$, $\frac{137}{25}$, $\frac{141}{26}$, and $\frac{160}{27}$ (the maximum is $\frac{160}{27} < 5.93$). For Case 6, the upper bounds on $\frac{D^2}{l}$ are: $\frac{68}{12}$, $\frac{73}{13}$, $\frac{80}{14}$, and $\frac{80}{15}$ (the maximum is $\frac{80}{14} < 5.72$).

The refined analysis for Case 5 reveals that all but one arrangement (of subquadrants of size $(2^{r-2})^2$) yield upper bounds that are bounded above and away from 6. The exception structure is given by the subcurve C (described in "L_2-Locality of Four Linearly Contiguous Hilbert Subcurves" section) of four linearly-contiguous Hilbert subcurves H_κ^2 of order κ; the maximum possible κ-value is $k - 3$ (embedded in H_k^2). By Theorem 1, the maximum $\frac{D^2}{l}$-value for this case is:

$$6 \cdot \frac{2^{2\kappa+3} - 2^{\kappa+2} + 2^{-1}}{2^{2\kappa+3} + 1}.$$

Observe that the expression $\frac{2^{2\kappa+3}-2^{\kappa+2}+2^{-1}}{2^{2\kappa+3}+1}$ is strictly increasing in $\kappa \geq 0$ (and approaching 1 as $\kappa \to \infty$). Thus, when $\kappa = k - 3$, the maximum $\frac{D^2}{l}$-value, which is $6 \cdot \frac{2^{2\kappa+3}-2^{\kappa+2}+2^{-1}}{2^{2\kappa+3}+1}$, assumes the value:

$$6 \cdot \frac{2^{2k-3} - 2^{k-1} + 2^{-1}}{2^{2k-3} + 1},$$

which is strictly increasing in $k \geq 3$. To show the desired formula for $L_2(H_k^2)$ for all positive integers $k \geq 5$, we further consider the two ranges of k-value: $k \geq 9$ and $0 \leq k \leq 8$, as follows.

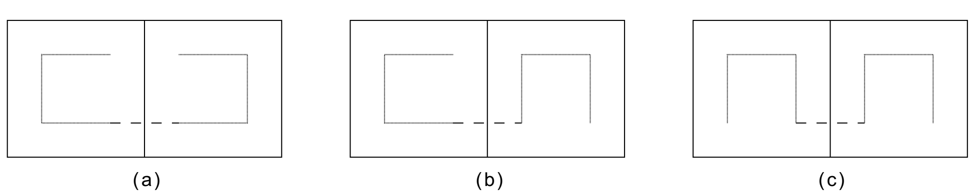


Fig. 4 The three possible adjacent H_κ^2-subcurves: **a** y^--oriented and y^+-oriented subcurves, **b** y^--oriented and x^+-oriented subcurves, **c** x^+-oriented and x^+-oriented subcurves

When $k = 9$, we have $6 \cdot \frac{2^{2k-3}-2^{k-1}+2^{-1}}{2^{2k-3}+1} > 5.953$, which is greater than all the upper bounds on $\frac{D^2}{l}$-value in the above refined analyses for Cases 3 and 6. For $4 \le k \le 8$, exhaustive searches for representative grid-point pairs of H_k^2 show that $L_2(H_k^2) = 6 \cdot \frac{2^{2k-3}-2^{k-1}+2^{-1}}{2^{2k-3}+1}$ for each $k \in \{5, 6, 7, 8\}$ (except for $k = 4$), and the theorem is proved. □

For an x^+-oriented Hilbert curve H_k^2 with $\partial_1(H_k^2) = (1, 1)$, where $k \ge 5$, the representative grid-point pair for H_k^2 with respect to L_2 reside at the lower-left corner (with coordinates $(2^{k-2}+1, 2^{k-1}+1)$) and the lower-right corner (with coordinates $(2^k - 2^{k-2}, 2^{k-1}+1)$) of four linearly-contiguous largest subquadrants (H_{k-3}^2-subcurves).

Exact Formulas for $L_p(H_k^2)$ with $p > 2$

To study L_p for arbitrary real $p > 2$, we first investigate the monotonicity of the underlying p-normed metric.

Lemma 9 *For every positive real constant α, the function $f : (0, \infty) \to (1, \infty)$ defined by $f(p) = (1 + \alpha^p)^{\frac{1}{p}}$ is strictly decreasing over its domain.*

Proof It is equivalent to show that the function $g : (0, \infty) \to (0, \infty)$ defined by $g(p) = \log f(p)$ ("log" denotes the natural logarithm) is strictly decreasing over its domain. We consider the first derivative of g, which is defined on $(0, \infty)$:

$$g'(p) = \frac{\frac{\alpha^p}{1+\alpha^p}\log\alpha^p - \log(1+\alpha^p)}{p^2} = \frac{\log\alpha^p - \log(1+\alpha^p) - \frac{\log\alpha^p}{1+\alpha^p}}{p^2}.$$

Note that: for $0 < \alpha < 1$, $g'(p) = \frac{\frac{\alpha^p}{1+\alpha^p}\log\alpha^p - \log(1+\alpha^p)}{p^2} < 0$, and for $1 \le \alpha$, $g'(p) = \frac{\log\alpha^p - \log(1+\alpha^p) - \frac{\log\alpha^p}{1+\alpha^p}}{p^2} < 0$. This proves the strictly decreasing property of g over its domain, and therefore the lemma. □

An immediate consequence of Lemma 9 is that for all grid points v and u, the p-normed metric $d_p(v, u)$ as a function of $p \in (0, \infty)$ is decreasing over its domain. Hence for a space-filling curve C, $\mathcal{L}_{C,p}(v, u) = \frac{d_p(v,u)^2}{\delta_C(v,u)}$ is decreasing in $p \in (0, \infty)$, as $\delta_C(v, u)$ is independent of p.

Theorem 3 *For all positive integers $k \ge 5$,*

$$L_p(H_k^2) = 6 \cdot \frac{2^{2k-3} - 2^{k-1} + 2^{-1}}{2^{2k-3} + 1} \text{ for all reals } p \ge 2.$$

Proof According to Theorem 2, let (v', u') be the representative grid-point pair for H_k^2 with respect to L_2, with their coordinates $v' = (2^{k-2}+1, 2^{k-1}+1)$ and $u' = (2^k - 2^{k-2}, 2^{k-1}+1)$. Consider an arbitrary real $p \ge 2$, and we show that (v', u') also serves as the unique representative grid-point pair for H_k^2 with respect to L_p, that is, for all $(v, u) \ne (v', u')$, $(v, u) \prec (v', u')$ via $\mathcal{L}_{H_k^2,p}(v, u) < \mathcal{L}_{H_k^2,p}(v', u')$.

Observe that $y(v') = y(u')$, which implies that $d_p(v', u') = d_2(v', u')$. Then for arbitrary grid points $v, u \in H_k^2$ with $(v', u') \ne (v, u)$, we have:

$$\mathcal{L}_{H_k^2,p}(v', u') = \frac{d_p(v', u')^2}{\delta_{H_k^2}(v', u')} = \frac{d_2(v', u')^2}{\delta_{H_k^2}(v', u')} = \mathcal{L}_{H_k^2,2}(v', u')$$

$$> \mathcal{L}_{H_k^2,2}(v, u) \quad ((v', u') : \text{ a representative grid-point pair with respect to } \mathcal{L}_{H_k^2,2})$$

$$\ge \mathcal{L}_{H_k^2,p}(v, u) \quad (\text{by the monotonicity of } \mathcal{L}_{H_k^2,p}).$$

□

Exact Formula for $L_1(H_k^2)$

We develop an argument similar to the one in "Exact Formulas for $L_p(H_k^2)$ with $p \ge 2$" section in establishing $L_2(H_k^2)$ to obtain the exact formula for $L_1(H_k^2)$. Adopting similar denotations in the proof of Theorem 2, consider the

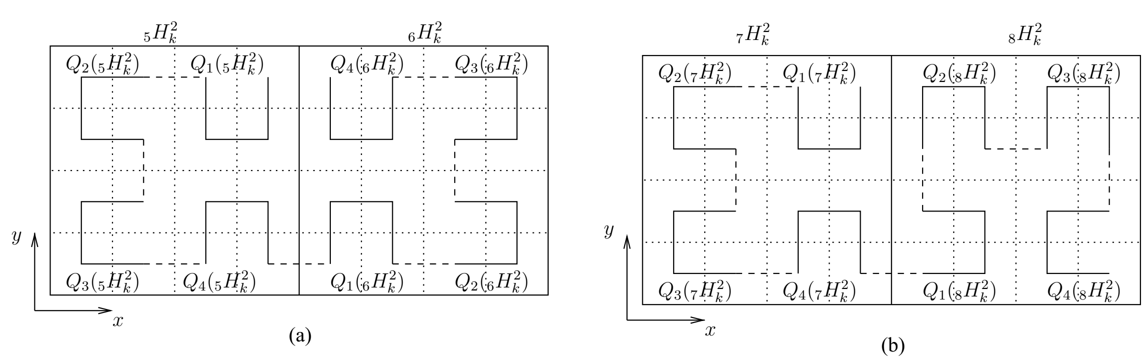


Fig. 5 Two Hilbert subcurves for the refined subpath-containment analysis: **a** two adjacent y^-- and y^+-oriented Hilbert subcurves; **b** two adjacent y^-- and x^+-oriented Hilbert subcurves

subpath-containment analysis with an arbitrary subcurve/subpath P of length l embedded in a two-dimensional Hilbert curve. There exists a sufficiently large positive integer r such that $(2^{r-1})^2 < l \le (2^r)^2$ and P is contained in two adjacent quadrants Q' and Q'' of size $(2^r)^2$ grid points each. Figure 4 provides the three possible arrangements of the two adjacent H^2_κ-subcurves where $\kappa \le r$ (modulo symmetry).

Denote by Δ the rectilinear diameter (based on the 1-normed metric d_1) of the set of grid points in P. A case-analysis of subpath-containment of P in subquadrants of size $(2^{r-1})^2$ within $Q' \cup Q''$ results in the following six cases:

Case	Lower and upper bounds for l	Upper bounds for Δ^2 and $\frac{\Delta^2}{l}$
1.	$\frac{4}{16}\cdot 4^r < l \le \frac{5}{16}\cdot 4^r$:	$\Delta^2 < \frac{36}{16}\cdot 4^r$, hence $\frac{\Delta^2}{l} \le 9$.
2.	$\frac{5}{16}\cdot 4^r < l \le \frac{6}{16}\cdot 4^r$:	$\Delta^2 < \frac{49}{16}\cdot 4^r$, hence $\frac{\Delta^2}{l} \le 9\frac{4}{5}$.
3.	$\frac{6}{16}\cdot 4^r < l \le \frac{7}{16}\cdot 4^r$:	$\Delta^2 < \frac{64}{16}\cdot 4^r$, hence $\frac{\Delta^2}{l} \le 10\frac{2}{3}$.
4.	$\frac{7}{16}\cdot 4^r < l \le \frac{8}{16}\cdot 4^r$:	$\Delta^2 < \frac{64}{16}\cdot 4^r$, hence $\frac{\Delta^2}{l} \le 9\frac{1}{7}$.
5.	$\frac{8}{16}\cdot 4^r < l \le \frac{12}{16}\cdot 4^r$:	$\Delta^2 < \frac{100}{16}\cdot 4^r$, hence $\frac{\Delta^2}{l} \le 12\frac{1}{2}$.
6.	$\frac{12}{16}\cdot 4^r < l \le 4^r$:	$\Delta^2 < \frac{144}{16}\cdot 4^r$, hence $\frac{\Delta^2}{l} \le 12$.

A refined analysis that is based on the entry and exit subquadrants/subcurves of size $(2^{r-2})^2$ or $(2^{r-3})^2$ and their orientations within $Q' \cup Q''$ further partitions the above six cases into subcases as follows:

Case 1. $\frac{4}{16}\cdot 4^r < l \le \frac{5}{16}\cdot 4^r$: $\Delta^2 < \frac{36}{16}\cdot 4^r$, hence $\frac{\Delta^2}{l} \le 9$.

Case 2. $\frac{5}{16}\cdot 4^r < l \le \frac{6}{16}\cdot 4^r$: $\Delta^2 < \frac{36}{16}\cdot 4^r$, hence $\frac{\Delta^2}{l} \le 7\frac{1}{5}$ (entry and exit subcurves on common coordinate axis).

Case 3. $\frac{6}{16}\cdot 4^r < l \le \frac{7}{16}\cdot 4^r$: $\Delta^2 < \frac{49}{16}\cdot 4^r$, hence $\frac{\Delta^2}{l} \le 8\frac{1}{6}$ (entry and exit subcurves on common coordinate axis).

Case 4. $\frac{7}{16}\cdot 4^r < l \le \frac{8}{16}\cdot 4^r$: $(\Delta^2 < \frac{64}{16}\cdot 4^r)$

Case	Lower and upper bounds for l	Upper bounds for Δ^2 and $\frac{\Delta^2}{l}$
4.1.	$\frac{7}{16}\cdot 4^r < l \le \frac{7}{16}\cdot 4^r + \frac{1}{64}\cdot 4^r$:	$\Delta^2 < \frac{225}{64}\cdot 4^r$, hence $\frac{\Delta^2}{l} \le 8\frac{1}{28}$.
4.2.	$\frac{7}{16}\cdot 4^r + \frac{1}{64}\cdot 4^r < l \le \frac{7}{16}\cdot 4^r + \frac{2}{64}\cdot 4^r$:	$\Delta^2 < \frac{64}{16}\cdot 4^r$, hence $\frac{\Delta^2}{l} \le 8\frac{24}{29}$.
4.3.	$\frac{7}{16}\cdot 4^r + \frac{2}{64}\cdot 4^r < l \le \frac{7}{16}\cdot 4^r + \frac{3}{64}\cdot 4^r$:	$\Delta^2 < \frac{64}{16}\cdot 4^r$, hence $\frac{\Delta^2}{l} \le 8\frac{8}{15}$.
4.4.	$\frac{7}{16}\cdot 4^r + \frac{3}{64}\cdot 4^r < l \le \frac{8}{16}\cdot 4^r$:	$\Delta^2 < \frac{64}{16}\cdot 4^r$, hence $\frac{\Delta^2}{l} \le 8\frac{8}{31}$.

Case 5. $\frac{8}{16}\cdot 4^r < l \le \frac{12}{16}\cdot 4^r$: $(\Delta^2 < \frac{100}{16}\cdot 4^r)$

Case	Lower and upper bounds for l	Upper bounds for Δ^2 and $\frac{\Delta^2}{l}$
5.1.	$\frac{8}{16}\cdot 4^r < l \le \frac{9}{16}\cdot 4^r$:	$\Delta^2 < \frac{64}{16}\cdot 4^r$, hence $\frac{\Delta^2}{l} \le 8$.
5.2.	$\frac{9}{16}\cdot 4^r < l \le \frac{10}{16}\cdot 4^r$: (entry and exit subcurves on common coordinate axis).	$\Delta^2 < \frac{64}{16}\cdot 4^r$, hence $\frac{\Delta^2}{l} \le 7\frac{1}{9}$
5.3.	$\frac{10}{16}\cdot 4^r < l \le \frac{11}{16}\cdot 4^r$: (entry and exit subcurves on common coordinate axis).	$\Delta^2 < \frac{81}{16}\cdot 4^r$, hence $\frac{\Delta^2}{l} \le 8\frac{1}{10}$
5.4.	$\frac{11}{16}\cdot 4^r < l \le \frac{12}{16}\cdot 4^r$:	$\Delta^2 < \frac{100}{16}\cdot 4^r$, hence $\frac{\Delta^2}{l} \le 9\frac{1}{11}$.

Case 6. $\frac{12}{16}\cdot 4^r < l \le 4^r$: $(\Delta^2 < \frac{144}{16}\cdot 4^r)$

Case	Lower and upper bounds for l	Upper bounds for Δ^2 and $\frac{\Delta^2}{l}$
6.1.	$\frac{12}{16}\cdot 4^r < l \le \frac{13}{16}\cdot 4^r$:	$\Delta^2 < \frac{100}{16}\cdot 4^r$, hence $\frac{\Delta^2}{l} \le 8\frac{1}{3}$.
6.2.	$\frac{13}{16}\cdot 4^r < l \le \frac{14}{16}\cdot 4^r$: (entry and exit subcurves on common coordinate axis).	$\Delta^2 < \frac{100}{16}\cdot 4^r$, hence $\frac{\Delta^2}{l} \le 7\frac{9}{13}$.
6.3.	$\frac{14}{16}\cdot 4^r < l \le \frac{15}{16}\cdot 4^r$: (entry and exit subcurves on common coordinate axis).	$\Delta^2 < \frac{121}{16}\cdot 4^r$, hence $\frac{\Delta^2}{l} \le 8\frac{9}{14}$.
6.4.	$\frac{15}{16}\cdot 4^r < l \le \frac{16}{16}\cdot 4^r$:	$\Delta^2 < \frac{144}{16}\cdot 4^r$, hence $\frac{\Delta^2}{l} \le 9\frac{3}{5}$.

The exact formula for $L_1(H^2_k)$ proven below is asymptotically (as $k \to \infty$) equal to 9, while the refined analysis shows that all but three (sub)cases (Cases 1, 5.4, and 6.4) yield upper bounds on $\frac{\Delta^2}{l}$ that are bounded above and away from 9.

Each of the Cases 1, 6.4, and 5.4 appears in both arrangements in Fig. 4a, b. Denote the first/left and right/last Hilbert subcurves (in the traversal order) of the two adjacent subcurves in Fig. 4a by ${}_5H^2_k$ (y^--oriented) and ${}_6H^2_k$ (y^+-oriented), respectively, and analogously for Fig. 4b by ${}_7H^2_k$ (y^--oriented) and ${}_8H^2_k$ (x^+-oriented), respectively. Figure 5a, b illustrate the annotations of the H^2_k-subcurves and their quadrants (H^2_{k-1}-subcurves) in Fig. 4a, b, respectively.

Case 1 appears in Fig. 5a, b with $k = r - 1$ (embedding the subpath P from $Q_3({}_5H^2_{r-1})$ to $Q_3({}_6H^2_{r-1})$ and from $Q_3({}_7H^2_{r-1})$ to $Q_3({}_8H^2_{r-1})$, respectively) and Case 6.4 appears in Fig. 5a, b with $k = r$ (embedding the subpath P from $Q_3({}_5H^2_r)$ to $Q_3({}_6H^2_r)$ and from $Q_3({}_7H^2_r)$ to $Q_3({}_8H^2_r)$, respectively); the locality analyses of Cases 1 and 6.4 are studied in "Two Adjacent y^-- and y^+-Oriented Hilbert Subcurves: Direct-Diagonal Corners" and "Two Adjacent y^-- and x^+-Oriented Hilbert Subcurves: Direct- and Slanted-Diagonal Corners" sections. Case 5.4 appears in Fig. 5a, b with $k = r$

(embedding the subpath P from $Q_3(Q_3({}_5H_r^2))$ to $Q_3(Q_2({}_6H_r^2))$ and from $Q_3(Q_3({}_7H_r^2))$ to $Q_3(Q_2({}_8H_r^2))$, respectively); the locality analyses of Case 5.4 are studied in "Two Adjacent y^-- and x^+-Oriented Hilbert Subcurves: Slanted-Diagonal Corners" and "Two Adjacent y^-- and x^+-Oriented Hilbert Subcurves: Direct- and Slanted-Diagonal Corners" sections.

The locality study in each case-analysis for a two-dimensional space-filling curve C involves the seeking of representative grid-point pairs via the comparisons of their $\mathcal{L}_{C,1}$-values. Lemma 10 below provides a sufficient condition for the strict reducibility of $(v, u) \prec (v', u')$ via $\mathcal{L}_{C,1}(v, u) < \mathcal{L}_{C,1}(v', u')$ for two grid-point pairs $(v,\ u)$ and (v', u') indexed by C in restricted forms of coordinate-relationship.

Denote $\hat{s}_{C,1}(v', v, u) = 2d_1(v, v')\delta_C(v, u) - d_1(v, u)\delta_C(v, v')$. The sufficient conditions via $s_{C,2}$ in Lemma 1 and $\hat{s}_{C,1}$ in Lemma 10 play analogous roles in yielding the reducibility conditions for grid-point pairs with respect to the locality measures $\mathcal{L}_{C,2}$ and $\mathcal{L}_{C,1}$, respectively, for the $L_2(H_k^2)$- and $L_1(H_k^2)$-studies, respectively.

Lemma 10 *For two arbitrary grid-point pairs $(v,\ u)$ and (v', u) indexed by a two-dimensional space-filling curve C such that the sequence of the three grid points (v', v, u) satisfies the monotone-coordinate condition: monotone in each coordinate (but may have different monotonicities), if $\hat{s}_{C,1}(v', v, u) > 0$ then $(v, u) \prec (v', u)$ via the comparison: $\mathcal{L}_{C,1}(v, u) < \mathcal{L}_{C,1}(v', u)$.*

By symmetry, for two arbitrary grid-point pairs $(v,\ u)$ and (v, u') indexed by a two-dimensional space-filling curve C such that the sequence of the three grid points (v, u, u') satisfies the monotone-coordinate condition, if $\hat{s}_{C,1}(u', u, v) > 0$ then $(v, u) \prec (v, u')$ via the comparison: $\mathcal{L}_{C,1}(v, u) < \mathcal{L}_{C,1}(v, u')$.

Proof It suffices to prove the case for two arbitrary grid-point pairs $(v,\ u)$ and (v', u) in the stated monotone-coordinate condition. Noting that $\mathcal{L}_{C,1}(v, u) < \mathcal{L}_{C,1}(v', u)$ is equivalent to $d_1(v', u)^2\delta_C(v, u) - d_1(v, u)^2\delta_C(v', u) > 0$, we consider:

$$
\begin{aligned}
&d_1(v', u)^2\delta_C(v, u) - d_1(v, u)^2\delta_C(v', u)\\
&\quad = (d_1(v', v) + d_1(v, u))^2\delta_C(v, u) - d_1(v, u)^2\delta_C(v', u)\\
&\qquad \text{(by the monotone-coordinate condition of } (v', v, u))\\
&\quad \geq (d_1(v', v) + d_1(v, u))^2\delta_C(v, u) - d_1(v, u)^2(\delta_C(v', v) + \delta_C(v, u))\\
&\qquad \text{(by the triangle-inequality of } \delta_C)\\
&\quad = d_1(v', u)^2\delta_C(v, u) + (2d_1(v', v)\delta_C(v, u) - d_1(v, u)\delta_C(v', v))d_1(v, u)\\
&\quad = d_1(v', v)^2\delta_C(v, u) + \hat{s}_{C,1}(v', v, u)d_1(v, u),
\end{aligned}
$$

and then the trivial positivity of $d_1(v', v)$, $\delta_C(v, u)$, and $d_1(v, u)$ (from the non-inequalities of both v versus u and v' versus v) yields the desired sufficient condition. □

Two Adjacent y^-- and y^+-Oriented Hilbert Subcurves: Direct-Diagonal Corners

Figure 5a depicts the labeled arrangement in Cartesian coordinates of a subcurve C that is composed of two adjacent H_k^2-subcurves: the left ${}_5H_k^2$ (y^--oriented) and the right ${}_6H_k^2$ (y^+-oriented). In the following analysis, we identify a pair of grid points at direct-diagonal corners of the subcurve C_1 joining $Q_3({}_5H_k^2)$ and $Q_3({}_6H_k^2)$: $v' \in Q_3({}_5H_k^2)$ and $u' \in Q_3({}_6H_k^2)$ such that $\mathcal{L}_{C,1}(v', u') = \mathcal{L}_{C,1}(Q_3({}_5H_k^2), Q_3({}_6H_k^2))$. Lemmas 11–13 yield the reduction of "$v' \in Q_3({}_5H_k^2)$" in successive Q_3-subcurves of ${}_5H_k^2$, and Lemmas 14–16 do the counterpart for "$u' \in Q_3({}_6H_k^2)$".

Note that the proofs of some lemmas in "Two Adjacent y^-- and y^+-Oriented Hilbert Subcurves: Direct-Diagonal Corners" and "Two Adjacent y^-- and x^+-Oriented Hilbert Subcurves: Slanted-Diagonal Corners" sections are achieved with case-analyses based on the quadrant-decomposition of the underlying subcurves for the membership of a candidate v' or u'. For each membership-case of v' or u', Lemma 10 is employed to justify the candidacy of v' or u'. The case-analysis is summarized in a table completed with non-trivial entries. We demonstrate a typical derivation of a membership-case in the proof/table of Lemma 11.

Lemma 11 *For all positive integers $k \geq 2$, and all grid-point pairs $v \in Q_3({}_5H_k^2) - Q_3(Q_3({}_5H_k^2))$ and $u \in Q_3({}_6H_k^2)$, there exists $v' \in Q_3(Q_3({}_5H_k^2))$ such that $(v, u) \prec (v', u)$ via the comparison: $\mathcal{L}_{C,1}(v, u) < \mathcal{L}_{C,1}(v', u)$.*

Proof With K denoting the subcurve $Q_3({}_5H_k^2)$ in the proof, the case-analysis based on the quadrant-decomposition of K is summarized in the following table.

We show below an example-derivation of the membership-case of $v \in Q_4(K)$ and $v' \in Q_3(K)$ with $(\mathrm{x}(v'), \mathrm{y}(v')) = (1, \mathrm{y}(v))$. Note that $d_1(v, u) < 12 \cdot 2^{k-2}$, $\delta_C(v, u) > 3 \cdot 2^{2k-2}$, $d_1(v, v') \geq 2^{k-2}$, and $\delta_C(v, v') \leq 2 \cdot 2^{2k-4}$, we have:

$$
\begin{aligned}
\hat{s}_{C,1}(v', v, u) &= 2d_1(v, v')\delta_C(v, u) - d_1(v, u)\delta_C(v, v')\\
&> 2 \cdot 2^{k-2} \cdot 3 \cdot 2^{2k-2} - 12 \cdot 2^{k-2} \cdot 2 \cdot 2^{2k-4}\\
&= 0.
\end{aligned}
$$

□

$v \in$	$v' \in$	v'-coordinate(s):	$d_1(v,u) <$	$\delta_C(v,u) >$	$d_1(v,v') \geq$	$\delta_C(v,v') \leq$	$\hat{s}_{C,1}(v',v,u) >$
$Q_2(K)$	$Q_3(K)$	$\mathrm{x}(v') = \mathrm{x}(v)$	$d_1(v',u)$	$\delta_C(v',u)$			
$Q_1(K)$	$Q_2(K)$	$\mathrm{y}(v') = \mathrm{y}(v)$	$d_1(v',u)$	$\delta_C(v',u)$			
$Q_4(K)$	$Q_3(K)$	$(\mathrm{x}(v'),\mathrm{y}(v'))$ $= (1,\mathrm{y}(v))$	$12 \cdot 2^{k-2}$	$3 \cdot 2^{2k-2}$	2^{k-2}	$2 \cdot 2^{2k-4}$	0

Lemma 12 *For all positive integers k and h with* $1 \leq h < k$, *and all grid-point pairs* $v \in Q_3^h({}_5H_k^2) - Q_3^{h+1}({}_5H_k^2)$ *and* $u \in Q_3({}_6H_k^2)$, *there exists* $v' \in Q_3^{h+1}({}_5H_k^2)$ *such that* $(v,u) \prec (v',u)$ *via the comparison*: $\mathcal{L}_{C,1}(v,u) < \mathcal{L}_{C,1}(v',u)$.

Proof With K denoting the subcurve $Q_3^h({}_5H_k^2)$ in the proof, the case-analysis based on the quadrant-decomposition of K is summarized in the following table. □

$v \in$	$v' \in$	v'-coordinate(s):	$d_1(v,u) <$	$\delta_C(v,u) >$	$d_1(v,v') \geq$	$\delta_C(v,v') \leq$	$\hat{s}_{C,1}(v',v,u) >$
$Q_2(K)$	$Q_3(K)$	$\mathrm{x}(v') = \mathrm{x}(v)$	$d_1(v',u)$	$\delta_C(v',u)$			
$Q_1(K)$	$Q_2(K)$	$\mathrm{y}(v') = \mathrm{y}(v)$	$d_1(v',u)$	$\delta_C(v',u)$			
$Q_4(K)$	$Q_3(K)$	$(\mathrm{x}(v'),\mathrm{y}(v'))$ $= (1,\mathrm{y}(v))$	$12 \cdot 2^{k-2}$	$3 \cdot 2^{2k-2}$	2^{k-h-1}	$2 \cdot 2^{2k-2h-2}$	$3 \cdot 2^{3k-h-2}$ $-3 \cdot 2^{3k-2h-1}$ > 0

Lemma 13 *For all positive integers k and h with* $1 \leq h < k$, *and all grid-point pairs* $v \in Q_3^h({}_5H_k^2) - Q_3^k({}_5H_k^2)$ *and* $u \in Q_3({}_6H_k^2)$, *there exists* $v' \in Q_3^k({}_5H_k^2)$ *such that* $(v,u) \prec (v',u)$ *via the comparison*: $\mathcal{L}_{C,1}(v,u) < \mathcal{L}_{C,1}(v',u)$.

Proof Similar to the proof of Lemma 4 for $L_2(H_k^2)$ in "L_2 -Locality of Four Linearly Contiguous Hilbert Subcurves" section. □

Lemma 14 *For all positive integers* $k \geq 2$, *and all grid-point pairs* $v \in Q_3^k({}_5H_k^2)$ *and* $u \in Q_3({}_6H_k^2) - Q_3(Q_3({}_6H_k^2))$, *there exists* $u' \in Q_3(Q_3({}_6H_k^2))$ *such that* $(v,u) \prec (v,u')$ *via the comparison*: $\mathcal{L}_{C,1}(v,u) < \mathcal{L}_{C,1}(v,u')$.

Proof With K denoting the subcurve $Q_3({}_6H_k^2)$ in the proof, the case-analysis based on the quadrant-decomposition of K is summarized in the following table. □

$u \in$	$u' \in$	u'-coordinate(s):	$d_1(v,u) <$	$\delta_C(v,u) >$	$d_1(u,u') \geq$	$\delta_C(u,u') \leq$	$\hat{s}_{C,1}(u',u,v) >$
$Q_2(K)$	$Q_3(K)$	$(\mathrm{x}(u'),\mathrm{y}(u'))$ $= (\mathrm{x}(u),2^k)$	$12 \cdot 2^{k-2}$	$3 \cdot 2^{2k-2}$	2^{k-2}	$2 \cdot 2^{2k-4}$	0
$Q_1(K)$	$Q_2(K)$	$(\mathrm{x}(u'),\mathrm{y}(u'))$ $= (2^{k+1},\mathrm{y}(u))$	$12 \cdot 2^{k-2}$	$3 \cdot 2^{2k-2}$	2^{k-2}	$2 \cdot 2^{2k-4}$	0
$Q_4(K)$	$Q_3(K)$	$\mathrm{y}(u') = \mathrm{y}(u)$	$d_1(v,u')$	$\delta_C(v,u')$			

Lemma 15 *For all positive integers k and h with* $1 \leq h < k$, *and all grid-point pairs* $v \in Q_3^k({}_5H_k^2)$ *and* $u \in Q_3^h({}_6H_k^2) - Q_3^{h+1}({}_6H_k^2)$, *there exists* $u' \in Q_3^{h+1}({}_6H_k^2)$ *such that* $(v,u) \prec (v,u')$ *via the comparison*: $\mathcal{L}_{C,1}(v,u) < \mathcal{L}_{C,1}(v,u')$.

Proof With K denoting the subcurve $Q_3^h({}_6H_k^2)$ in the proof, the case-analysis based on the quadrant-decomposition of K is summarized in the following table. □

$u \in$	$u' \in$	u'-coordinate(s):	$d_1(v,u) <$	$\delta_C(v,u) >$	$d_1(u,u') \geq$	$\delta_C(u,u') \leq$	$\hat{s}_{C,1}(u',u,v) >$
$Q_2(K)$	$Q_3(K)$	$(\mathrm{x}(u'), \mathrm{y}(u')) = (\mathrm{x}(u), 2^k)$	$12 \cdot 2^{k-2}$	$3 \cdot 2^{2k-2}$	2^{k-h-1}	$2 \cdot 2^{2k-2h-2}$	$3 \cdot 2^{3k-h-2} - 3 \cdot 2^{3k-2h-1} > 0$
$Q_1(K)$	$Q_2(K)$	$(\mathrm{x}(u'), \mathrm{y}(u')) = (2^{k+1}, \mathrm{y}(u))$	$12 \cdot 2^{k-2}$	$3 \cdot 2^{2k-2}$	2^{k-h-1}	$2 \cdot 2^{2k-2h-2}$	$3 \cdot 2^{3k-h-2} - 3 \cdot 2^{3k-2h-1} > 0$
$Q_4(K)$	$Q_3(K)$	$\mathrm{y}(u') = \mathrm{y}(u)$	$d_1(v,u')$	$\delta_C(v,u')$			

Lemma 16 *For all positive integers k and h with $1 \leq h < k$, and all grid-point pairs $v \in Q_3^k({}_5H_k^2)$ and $u \in Q_3^h({}_6H_k^2) - Q_3^k({}_6H_k^2)$, there exists $u' \in Q_3^k({}_6H_k^2)$ such that $(v,u) \prec (v,u')$ via the comparison: $\mathcal{L}_{C,1}(v,u) < \mathcal{L}_{C,1}(v,u')$.*

Proof Similar to the proof of Lemma 4 for $L_2(H_k^2)$ in "L_2-Locality of Four Linearly Contiguous Hilbert Subcurves" section. □

The six lemmas (Lemmas 11 – 16) identify the unique representative grid-point pair $(v',u') \in Q_3^k({}_5H_k^2) \times Q_3^k({}_6H_k^2)$ that maximizes the $\mathcal{L}_{C,1}$-value for the subcurve C_1 (joining the direct-diagonal corners $Q_3({}_5H_k^2)$ and $Q_3({}_6H_k^2)$ Hilbert subcurves) — with (v',u') residing at the lower-left and upper-right corners of C_1 with coordinates $v' = (1,1)$ and $u' = (2^{k+1}, 2^k)$, respectively:

$$\mathcal{L}_{C,1}(v',u') = \mathcal{L}_{C,1}(Q_3({}_5H_k^2), Q_3({}_6H_k^2)) = \mathcal{L}_{C,1}(Q_3^k({}_5H_k^2), Q_3^k({}_6H_k^2)) = \frac{(2^{k+1}-1+2^k-1)^2}{2^{2k}} = \frac{(3 \cdot 2^k - 2)^2}{2^{2k}} = 9 - 3 \cdot 2^{-k+2} + 2^{-2k+2}.$$

Two Adjacent y^-- and x^+-Oriented Hilbert Subcurves: Slanted-Diagonal Corners

Analogous to the case of direct-diagonal corners of C_1 in "Two Adjacent y^-- and y^+-Oriented Hilbert Subcurves: Direct-Diagonal Corners" section, we identify a grid-point pair at slanted-diagonal corners of the subcurve C_2 joining $Q_3(Q_3({}_5H_k^2))$ and $Q_3(Q_2({}_6H_k^2))$: $v' \in Q_3(Q_3({}_5H_k^2))$ and $u' \in Q_3(Q_2({}_6H_k^2))$ such that $\mathcal{L}_{C,1}(v',u') = \mathcal{L}_{C,1}(Q_3(Q_3({}_5H_k^2)), Q_3(Q_2({}_6H_k^2)))$.

Lemma 17 *For all positive integers $k \geq 3$, and all grid-point pairs $v \in Q_3^2({}_5H_k^2) - Q_3(Q_3^2({}_5H_k^2))$ and $u \in Q_3(Q_2({}_6H_k^2))$, there exists $v' \in Q_3(Q_3^2({}_5H_k^2))$ such that $(v,u) \prec (v',u)$ via the comparison: $\mathcal{L}_{C,1}(v,u) < \mathcal{L}_{C,1}(v',u)$.*

Proof With K denoting the subcurve $Q_3^2({}_5H_k^2)$ in the proof, the case-analysis based on the quadrant-decomposition of K is summarized in the following table. □

$v \in$	$v' \in$	v'-coordinate(s):	$d_1(v,u) <$	$\delta_C(v,u) >$	$d_1(v,v') \geq$	$\delta_C(v,v') \leq$	$\hat{s}_{C,1}(v',v,u) >$
$Q_2(K)$	$Q_3(K)$	$\mathrm{x}(v') = \mathrm{x}(v)$	$d_1(v',u)$	$\delta_C(v',u)$			
$Q_1(K)$	$Q_2(K)$	$\mathrm{y}(v') = \mathrm{y}(v)$	$d_1(v',u)$	$\delta_C(v',u)$			
$Q_4(K)$	$Q_3(K)$	$(\mathrm{x}(v'), \mathrm{y}(v')) = (1, \mathrm{y}(v))$	$10 \cdot 2^{k-2}$	$\frac{11}{4} \cdot 2^{2k-2}$	2^{k-3}	$2 \cdot 2^{2k-6}$	$6 \cdot 2^{3k-6} > 0$

Lemma 18 *For all positive integers k and h with $2 \leq h < k$, and all grid-point pairs $v \in Q_3^h({}_5H_k^2) - Q_3^{h+1}({}_5H_k^2)$ and $u \in Q_3(Q_2({}_6H_k^2))$, there exists $v' \in Q_3^{h+1}({}_5H_k^2)$ such that $(v,u) \prec (v',u)$ via the comparison: $\mathcal{L}_{C,1}(v,u) < \mathcal{L}_{C,1}(v',u)$.*

Proof With K denoting the subcurve $Q_3^h({}_5H_k^2)$ in the proof, the case-analysis based on the quadrant-decomposition of K is summarized in the following table. □

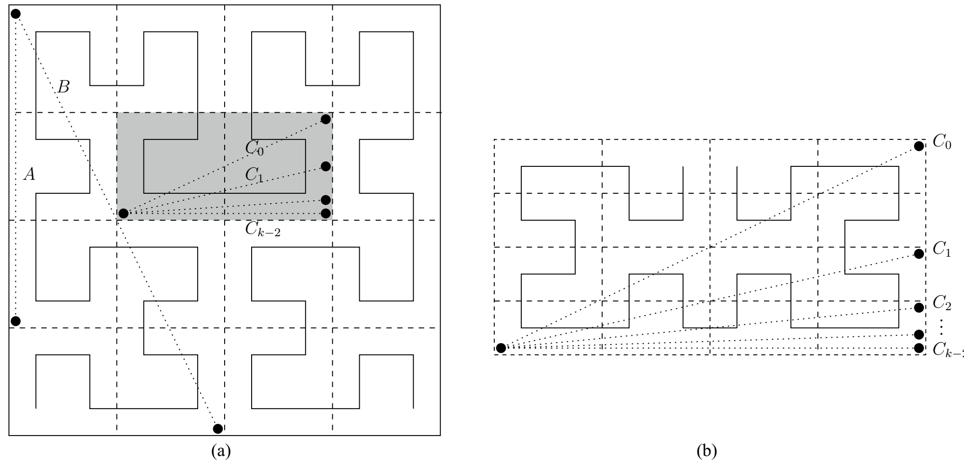


Fig. 6 Candidate representative grid-point pairs for H_k^2 with respect to L_p for $k \geq 2$: **a** three sources $\{A, B, C\}$ of candidate representative grid-point pairs; **b** detailed view of the source C

$v \in$	$v' \in$	v'-coordinate(s):	$d_1(v,u) <$	$\delta_C(v,u) >$	$d_1(v,v') \geq$	$\delta_C(v,v') \leq$	$\hat{s}_{C,1}(v',v,u) >$
$Q_2(K)$	$Q_3(K)$	$\mathrm{x}(v') = \mathrm{x}(v)$	$d_1(v',u)$	$\delta_C(v',u)$			0
$Q_1(K)$	$Q_2(K)$	$\mathrm{y}(v') = \mathrm{y}(v)$	$d_1(v',u)$	$\delta_C(v',u)$			0
$Q_4(K)$	$Q_3(K)$	$(\mathrm{x}(v'), \mathrm{y}(v')) = (1, \mathrm{y}(v))$	$10 \cdot 2^{k-2}$	$\frac{11}{4} \cdot 2^{2k-2}$	2^{k-h-1}	$2 \cdot 2^{2k-2h-2}$	$11 \cdot 2^{3k-h-4}$ $-5 \cdot 2^{3k-2h-2}$ > 0

Lemma 19 *For all positive integers k and h with* $2 \leq h < k$, *and all grid-point pairs* $v \in Q_3^h({}_5H_k^2) - Q_3^k({}_5H_k^2)$ *and* $u \in Q_3(Q_2({}_6H_k^2))$, *there exists* $v' \in Q_3^k({}_5H_k^2)$ *such that* $(v,u) \prec (v',u)$ *via the comparison*: $\mathcal{L}_{C,1}(v,u) < \mathcal{L}_{C,1}(v',u)$.

Proof Similar to the proof of Lemma 4 for $L_2(H_k^2)$ in "L_2-Locality of Four Linearly Contiguous Hilbert Subcurves" section. □

Lemma 20 *For all positive integers* $k \geq 3$, *and all grid-point pairs* $v \in Q_3^k({}_5H_k^2)$ *and* $u \in Q_3(Q_2({}_6H_k^2)) - Q_3(Q_3(Q_2({}_6H_k^2)))$, *there exists* $u' \in Q_3(Q_3(Q_2({}_6H_k^2)))$ *such that* $(v,u) \prec (v,u')$ *via the comparison*: $\mathcal{L}_{C,1}(v,u) < \mathcal{L}_{C,1}(v,u')$.

Proof With K denoting the subcurve $Q_3(Q_2({}_6H_k^2))$ in the proof, the case-analysis based on the quadrant-decomposition of K is summarized in the following table. □

$u \in$	$u' \in$	u'-coordinate(s):	$d_1(v,u) <$	$\delta_C(v,u) >$	$d_1(u,u') \geq$	$\delta_C(u,u') \leq$	$\hat{s}_{C,1}(u',u,v) >$
$Q_2(K)$	$Q_3(K)$	$(\mathrm{x}(u'), \mathrm{y}(u')) = (\mathrm{x}(u), 2^{k-1})$	$10 \cdot 2^{k-2}$	$\frac{11}{4} \cdot 2^{2k-2}$	2^{k-3}	$\leq 2 \cdot 2^{2k-6}$	$6 \cdot 2^{3k-6}$ > 0
$Q_1(K)$	$Q_2(K)$	$(\mathrm{x}(u'), \mathrm{y}(u')) = (2^{k+1}, \mathrm{y}(u))$	$10 \cdot 2^{k-2}$	$\frac{11}{4} \cdot 2^{2k-2}$	2^{k-3}	$\leq 2 \cdot 2^{2k-6}$	$6 \cdot 2^{3k-6}$ > 0
$Q_4(K)$	$Q_3(K)$	$\mathrm{y}(u') = \mathrm{y}(u)$	$d_1(v,u')$	$\delta_C(v,u')$			

Lemma 21 *For all positive integers k and h with* $2 \leq h < k$, *and all grid-point pairs* $v \in Q_3^k({}_5H_k^2)$ *and* $u \in Q_3^{h-1}(Q_2({}_6H_k^2)) - Q_3^h(Q_2({}_6H_k^2))$, *there exists* $u' \in Q_3^h(Q_2({}_6H_k^2))$ *such that* $(v,u) \prec (v,u')$ *via the comparison*: $\mathcal{L}_{C,1}(v,u) < \mathcal{L}_{C,1}(v,u')$.

Proof With K denoting the subcurve $Q_3^{h-1}(Q_2({}_6H_k^2))$ in the proof, the case-analysis based on the quadrant-decomposition of K is summarized in the following table. □

Table 1 Representative grid-point pairs for H_k^2 with respect to L_p for $k \in \{2, 3, \dots, 16\}$ and $p \in [1.00, 2.00]$ with granularity of 0.01

k	p	(x, y)-Coordinates	Representative grid-point pair coordinates in terms of k	Source
2	[1.00, 2.00]	((2, 1), (1, 4))	$((2^{k-1}, 1), (1, 2^k))$	B
3	[1.00, 2.00]	((4, 1), (1, 8))	$((2^{k-1}, 1), (1, 2^k))$	B
4	[1.00, 1.82]	((8, 1), (1, 16))	$((2^{k-1}, 1), (1, 2^k))$	B
	[1.83, 2.00]	((1, 5), (1, 16))	$((1, \frac{1}{4} \cdot 2^k + 1), (1, 2^k))$	A
5	[1.00, 1.61]	((16, 1), (1, 32))	$((2^{k-1}, 1), (1, 2^k))$	B
	[1.62, 2.00]	((9, 17), (24, 17))	$((\frac{1}{4} \cdot 2^k + 1, 2^{k-1} + 1), (\frac{3}{4} \cdot 2^k, 2^{k-1} + 1))$	C_3
6	[1.00, 1.51]	((32, 1), (1, 64))	$((2^{k-1}, 1), (1, 2^k))$	B
	[1.52, 1.55]	((17, 33), (48, 40))	$((\frac{1}{4} \cdot 2^k + 1, 2^{k-1} + 1), (\frac{3}{4} \cdot 2^k, 2^{k-1} + 2^{k-3}))$	C_1
	[1.56, 1.60]	((17, 33), (48, 36))	$((\frac{1}{4} \cdot 2^k + 1, 2^{k-1} + 1), (\frac{3}{4} \cdot 2^k, 2^{k-1} + 2^{k-4}))$	C_2
	[1.61, 2.00]	((17, 33), (48, 33))	$((\frac{1}{4} \cdot 2^k + 1, 2^{k-1} + 1), (\frac{3}{4} \cdot 2^k, 2^{k-1} + 1))$	C_4
7	[1.00, 1.41]	((64, 1), (1, 128))	$((2^{k-1}, 1), (1, 2^k))$	B
	[1.42, 1.57]	((33, 65), (96, 80))	$((\frac{1}{4} \cdot 2^k + 1, 2^{k-1} + 1), (\frac{3}{4} \cdot 2^k, 2^{k-1} + 2^{k-3}))$	C_1
	[1.58, 1.66]	((33, 65), (96, 72))	$((\frac{1}{4} \cdot 2^k + 1, 2^{k-1} + 1), (\frac{3}{4} \cdot 2^k, 2^{k-1} + 2^{k-4}))$	C_2
	[1.67, 1.67]	((33, 65), (96, 68))	$((\frac{1}{4} \cdot 2^k + 1, 2^{k-1} + 1), (\frac{3}{4} \cdot 2^k, 2^{k-1} + 2^{k-5}))$	C_3
	[1.68, 2.00]	((33, 65), (96, 65))	$((\frac{1}{4} \cdot 2^k + 1, 2^{k-1} + 1), (\frac{3}{4} \cdot 2^k, 2^{k-1} + 1))$	C_5
8	[1.00, 1.36]	((128, 1), (1, 256))	$((2^{k-1}, 1), (1, 2^k))$	B
	[1.37, 1.57]	((65, 129), (192, 160))	$((\frac{1}{4} \cdot 2^k + 1, 2^{k-1} + 1), (\frac{3}{4} \cdot 2^k, 2^{k-1} + 2^{k-3}))$	C_1
	[1.58, 1.68]	((65, 129), (192, 144))	$((\frac{1}{4} \cdot 2^k + 1, 2^{k-1} + 1), (\frac{3}{4} \cdot 2^k, 2^{k-1} + 2^{k-4}))$	C_2
	[1.69, 1.72]	((65, 129), (192, 136))	$((\frac{1}{4} \cdot 2^k + 1, 2^{k-1} + 1), (\frac{3}{4} \cdot 2^k, 2^{k-1} + 2^{k-5}))$	C_3
	[1.73, 2.00]	((65, 129), (192, 129))	$((\frac{1}{4} \cdot 2^k + 1, 2^{k-1} + 1), (\frac{3}{4} \cdot 2^k, 2^{k-1} + 1))$	C_6
9	[1.00, 1.33]	((256, 1), (1, 512))	$((2^{k-1}, 1), (1, 2^k))$	B
	[1.34, 1.58]	((129, 257), (384, 320))	$((\frac{1}{4} \cdot 2^k + 1, 2^{k-1} + 1), (\frac{3}{4} \cdot 2^k, 2^{k-1} + 2^{k-3}))$	C_1
	[1.59, 1.69]	((129, 257), (384, 288))	$((\frac{1}{4} \cdot 2^k + 1, 2^{k-1} + 1), (\frac{3}{4} \cdot 2^k, 2^{k-1} + 2^{k-4}))$	C_2
	[1.70, 1.75]	((129, 257), (384, 272))	$((\frac{1}{4} \cdot 2^k + 1, 2^{k-1} + 1), (\frac{3}{4} \cdot 2^k, 2^{k-1} + 2^{k-5}))$	C_3
	[1.76, 1.77]	((129, 257), (384, 264))	$((\frac{1}{4} \cdot 2^k + 1, 2^{k-1} + 1), (\frac{3}{4} \cdot 2^k, 2^{k-1} + 2^{k-6}))$	C_4
	[1.78, 2.00]	((129, 257), (384, 257)	$((\frac{1}{4} \cdot 2^k + 1, 2^{k-1} + 1), (\frac{3}{4} \cdot 2^k, 2^{k-1} + 1))$	C_7
10	[1.00, 1.32]	((512, 1), (1, 1024))	$((2^{k-1}, 1), (1, 2^k))$	B
	[1.33, 1.58]	((257, 513), (768, 640))	$((\frac{1}{4} \cdot 2^k + 1, 2^{k-1} + 1), (\frac{3}{4} \cdot 2^k, 2^{k-1} + 2^{k-3}))$	C_1
	[1.59, 1.70]	((257, 513), (768, 576))	$((\frac{1}{4} \cdot 2^k + 1, 2^{k-1} + 1), (\frac{3}{4} \cdot 2^k, 2^{k-1} + 2^{k-4}))$	C_2
	[1.71, 1.76]	((257, 513), (768, 544))	$((\frac{1}{4} \cdot 2^k + 1, 2^{k-1} + 1), (\frac{3}{4} \cdot 2^k, 2^{k-1} + 2^{k-5}))$	C_3
	[1.77, 1.79]	((257, 513), (768, 528))	$((\frac{1}{4} \cdot 2^k + 1, 2^{k-1} + 1), (\frac{3}{4} \cdot 2^k, 2^{k-1} + 2^{k-6}))$	C_4
	[1.80, 1.80]	((257, 513), (768, 520))	$((\frac{1}{4} \cdot 2^k + 1, 2^{k-1} + 1), (\frac{3}{4} \cdot 2^k, 2^{k-1} + 2^{k-7}))$	C_5
	[1.81, 2.00]	((257, 513), (768, 513))	$((\frac{1}{4} \cdot 2^k + 1, 2^{k-1} + 1), (\frac{3}{4} \cdot 2^k, 2^{k-1} + 1))$	C_8
11	[1.00, 1.31]	((1024, 1), (1, 2048))	$((2^{k-1}, 1), (1, 2^k))$	B
	[1.32, 1.58]	((513, 1025), (1536, 1280))	$((\frac{1}{4} \cdot 2^k + 1, 2^{k-1} + 1), (\frac{3}{4} \cdot 2^k, 2^{k-1} + 2^{k-3}))$	C_1
	[1.59, 1.70]	((513, 1025), (1536, 1152))	$((\frac{1}{4} \cdot 2^k + 1, 2^{k-1} + 1), (\frac{3}{4} \cdot 2^k, 2^{k-1} + 2^{k-4}))$	C_2
	[1.71, 1.76]	((513, 1025), (1536, 1088))	$((\frac{1}{4} \cdot 2^k + 1, 2^{k-1} + 1), (\frac{3}{4} \cdot 2^k, 2^{k-1} + 2^{k-5}))$	C_3
	[1.77, 1.80]	((513, 1025), (1536, 1056))	$((\frac{1}{4} \cdot 2^k + 1, 2^{k-1} + 1), (\frac{3}{4} \cdot 2^k, 2^{k-1} + 2^{k-6}))$	C_4
	[1.81, 1.82]	((513, 1025), (1536, 1040))	$((\frac{1}{4} \cdot 2^k + 1, 2^{k-1} + 1), (\frac{3}{4} \cdot 2^k, 2^{k-1} + 2^{k-7}))$	C_5
	[1.83, 2.00]	((513, 1025), (1536, 1025))	$((\frac{1}{4} \cdot 2^k + 1, 2^{k-1} + 1), (\frac{3}{4} \cdot 2^k, 2^{k-1} + 1))$	C_9
12	[1.00, 1.31]	((2048, 1), (1, 4096))	$((2^{k-1}, 1), (1, 2^k))$	B
	[1.32, 1.58]	((1025, 2049), (3072, 2560))	$((\frac{1}{4} \cdot 2^k + 1, 2^{k-1} + 1), (\frac{3}{4} \cdot 2^k, 2^{k-1} + 2^{k-3}))$	C_1
	[1.59, 1.70]	((1025, 2049), (3072, 2304))	$((\frac{1}{4} \cdot 2^k + 1, 2^{k-1} + 1), (\frac{3}{4} \cdot 2^k, 2^{k-1} + 2^{k-4}))$	C_2
	[1.71, 1.77]	((1025, 2049), (3072, 2176))	$((\frac{1}{4} \cdot 2^k + 1, 2^{k-1} + 1), (\frac{3}{4} \cdot 2^k, 2^{k-1} + 2^{k-5}))$	C_3

Table 1 (continued)

k	p	(x, y)-Coordinates	Representative grid-point pair coordinates in terms of k	Source
	[1.78, 1.81]	((1025, 2049), (3072, 2112))	$((\frac{1}{4}\cdot 2^k+1, 2^{k-1}+1), (\frac{3}{4}\cdot 2^k, 2^{k-1}+2^{k-6}))$	C_4
	[1.82, 1.83]	((1025, 2049), (3072, 2080))	$((\frac{1}{4}\cdot 2^k+1, 2^{k-1}+1), (\frac{3}{4}\cdot 2^k, 2^{k-1}+2^{k-7}))$	C_5
	[1.84, 1.84]	((1025, 2049), (3072, 2064))	$((\frac{1}{4}\cdot 2^k+1, 2^{k-1}+1), (\frac{3}{4}\cdot 2^k, 2^{k-1}+2^{k-8}))$	C_6
	[1.85, 2.00]	((1025, 2049), (3072, 2049))	$((\frac{1}{4}\cdot 2^k+1, 2^{k-1}+1), (\frac{3}{4}\cdot 2^k, 2^{k-1}+1))$	C_{10}
13	[1.00, 1.30]	((4096, 1), (1, 8192))	$((2^{k-1}, 1), (1, 2^k))$	B
	[1.31, 1.58]	((2049, 4097), (6144, 5120))	$((\frac{1}{4}\cdot 2^k+1, 2^{k-1}+1), (\frac{3}{4}\cdot 2^k, 2^{k-1}+2^{k-3}))$	C_1
	[1.59, 1.70]	((2049, 4097), (6144, 4608))	$((\frac{1}{4}\cdot 2^k+1, 2^{k-1}+1), (\frac{3}{4}\cdot 2^k, 2^{k-1}+2^{k-4}))$	C_2
	[1.71, 1.77]	((2049, 4097), (6144, 4352))	$((\frac{1}{4}\cdot 2^k+1, 2^{k-1}+1), (\frac{3}{4}\cdot 2^k, 2^{k-1}+2^{k-5}))$	C_3
	[1.78, 1.81]	((2049, 4097), (6144, 4224))	$((\frac{1}{4}\cdot 2^k+1, 2^{k-1}+1), (\frac{3}{4}\cdot 2^k, 2^{k-1}+2^{k-6}))$	C_4
	[1.82, 1.83]	((2049, 4097), (6144, 4160))	$((\frac{1}{4}\cdot 2^k+1, 2^{k-1}+1), (\frac{3}{4}\cdot 2^k, 2^{k-1}+2^{k-7}))$	C_5
	[1.84, 1.85]	((2049, 4097), (6144, 4128))	$((\frac{1}{4}\cdot 2^k+1, 2^{k-1}+1), (\frac{3}{4}\cdot 2^k, 2^{k-1}+2^{k-8}))$	C_6
	[1.86, 1.86]	((2049, 4097), (6144, 4112))	$((\frac{1}{4}\cdot 2^k+1, 2^{k-1}+1), (\frac{3}{4}\cdot 2^k, 2^{k-1}+2^{k-9}))$	C_7
	[1.87, 2.00]	((2049, 4097), (6144, 4097))	$((\frac{1}{4}\cdot 2^k+1, 2^{k-1}+1), (\frac{3}{4}\cdot 2^k, 2^{k-1}+1))$	C_{11}
14	[1.00, 1.30]	((8192, 1), (1, 16384))	$((2^{k-1}, 1), (1, 2^k))$	B
	[1.31, 1.58]	((4097, 8193), (12288, 10240))	$((\frac{1}{4}\cdot 2^k+1, 2^{k-1}+1), (\frac{3}{4}\cdot 2^k, 2^{k-1}+2^{k-3}))$	C_1
	[1.59, 1.70]	((4097, 8193), (12288, 9216))	$((\frac{1}{4}\cdot 2^k+1, 2^{k-1}+1), (\frac{3}{4}\cdot 2^k, 2^{k-1}+2^{k-4}))$	C_2
	[1.71, 1.77]	((4097, 8193), (12288, 8704))	$((\frac{1}{4}\cdot 2^k+1, 2^{k-1}+1), (\frac{3}{4}\cdot 2^k, 2^{k-1}+2^{k-5}))$	C_3
	[1.78, 1.81]	((4097, 8193), (12288, 8448))	$((\frac{1}{4}\cdot 2^k+1, 2^{k-1}+1), (\frac{3}{4}\cdot 2^k, 2^{k-1}+2^{k-6}))$	C_4
	[1.82, 1.84]	((4097, 8193), (12288, 8320))	$((\frac{1}{4}\cdot 2^k+1, 2^{k-1}+1), (\frac{3}{4}\cdot 2^k, 2^{k-1}+2^{k-7}))$	C_5
	[1.85, 1.86]	((4097, 8193), (12288, 8256))	$((\frac{1}{4}\cdot 2^k+1, 2^{k-1}+1), (\frac{3}{4}\cdot 2^k, 2^{k-1}+2^{k-8}))$	C_6
	[1.87, 1.87]	((4097, 8193), (12288, 8224))	$((\frac{1}{4}\cdot 2^k+1, 2^{k-1}+1), (\frac{3}{4}\cdot 2^k, 2^{k-1}+2^{k-9}))$	C_7
	[1.88, 1.88]	((4097, 8193), (12288, 8208))	$((\frac{1}{4}\cdot 2^k+1, 2^{k-1}+1), (\frac{3}{4}\cdot 2^k, 2^{k-1}+2^{k-10}))$	C_8
	[1.89, 2.00]	((4097, 8193), (12288, 8193))	$((\frac{1}{4}\cdot 2^k+1, 2^{k-1}+1), (\frac{3}{4}\cdot 2^k, 2^{k-1}+1))$	C_{12}
15	[1.00, 1.30]	((16384, 1), (1, 32768))	$((2^{k-1}, 1), (1, 2^k))$	B
	[1.31, 1.58]	((8193, 16385), (24576, 20480))	$((\frac{1}{4}\cdot 2^k+1, 2^{k-1}+1), (\frac{3}{4}\cdot 2^k, 2^{k-1}+2^{k-3}))$	C_1
	[1.59, 1.70]	((8193, 16385), (24576, 18432))	$((\frac{1}{4}\cdot 2^k+1, 2^{k-1}+1), (\frac{3}{4}\cdot 2^k, 2^{k-1}+2^{k-4}))$	C_2
	[1.71, 1.77]	((8193, 16385), (24576, 17408))	$((\frac{1}{4}\cdot 2^k+1, 2^{k-1}+1), (\frac{3}{4}\cdot 2^k, 2^{k-1}+2^{k-5}))$	C_3
	[1.78, 1.81]	((8193, 16385), (24576, 16896))	$((\frac{1}{4}\cdot 2^k+1, 2^{k-1}+1), (\frac{3}{4}\cdot 2^k, 2^{k-1}+2^{k-6}))$	C_4
	[1.82, 1.84]	((8193, 16385), (24576, 16640))	$((\frac{1}{4}\cdot 2^k+1, 2^{k-1}+1), (\frac{3}{4}\cdot 2^k, 2^{k-1}+2^{k-7}))$	C_5
	[1.85, 1.86]	((8193, 16385), (24576, 16512))	$((\frac{1}{4}\cdot 2^k+1, 2^{k-1}+1), (\frac{3}{4}\cdot 2^k, 2^{k-1}+2^{k-8}))$	C_6
	[1.87, 1.87]	((8193, 16385), (24576, 16448))	$((\frac{1}{4}\cdot 2^k+1, 2^{k-1}+1), (\frac{3}{4}\cdot 2^k, 2^{k-1}+2^{k-9}))$	C_7
	[1.88, 1.88]	((8193, 16385), (24576, 16416))	$((\frac{1}{4}\cdot 2^k+1, 2^{k-1}+1), (\frac{3}{4}\cdot 2^k, 2^{k-1}+2^{k-10}))$	C_8
	[1.89, 1.89]	((8193, 16385), (24576, 16400))	$((\frac{1}{4}\cdot 2^k+1, 2^{k-1}+1), (\frac{3}{4}\cdot 2^k, 2^{k-1}+2^{k-11}))$	C_9
	[1.90, 2.00]	((8193, 16385), (24576, 16385))	$((\frac{1}{4}\cdot 2^k+1, 2^{k-1}+1), (\frac{3}{4}\cdot 2^k, 2^{k-1}+1))$	C_{13}
16	[1.00, 1.30]	((32768, 1), (1, 65536))	$((2^{k-1}, 1), (1, 2^k))$	B
	[1.31, 1.58]	((16385, 32769), (49152, 40960))	$((\frac{1}{4}\cdot 2^k+1, 2^{k-1}+1), (\frac{3}{4}\cdot 2^k, 2^{k-1}+2^{k-3}))$	C_1
	[1.59, 1.70]	((16385, 32769), (49152, 36864))	$((\frac{1}{4}\cdot 2^k+1, 2^{k-1}+1), (\frac{3}{4}\cdot 2^k, 2^{k-1}+2^{k-4}))$	C_2
	[1.71, 1.77]	((16385, 32769), (49152, 34816))	$((\frac{1}{4}\cdot 2^k+1, 2^{k-1}+1), (\frac{3}{4}\cdot 2^k, 2^{k-1}+2^{k-5}))$	C_3
	[1.78, 1.81]	((16385, 32769), (49152, 33792))	$((\frac{1}{4}\cdot 2^k+1, 2^{k-1}+1), (\frac{3}{4}\cdot 2^k, 2^{k-1}+2^{k-6}))$	C_4
	[1.82, 1.84]	((16385, 32769), (49152, 33280))	$((\frac{1}{4}\cdot 2^k+1, 2^{k-1}+1), (\frac{3}{4}\cdot 2^k, 2^{k-1}+2^{k-7}))$	C_5
	[1.85, 1.86]	((16385, 32769), (49152, 33024))	$((\frac{1}{4}\cdot 2^k+1, 2^{k-1}+1), (\frac{3}{4}\cdot 2^k, 2^{k-1}+2^{k-8}))$	C_6
	[1.87, 1.87]	((16385, 32769), (49152, 32896))	$((\frac{1}{4}\cdot 2^k+1, 2^{k-1}+1), (\frac{3}{4}\cdot 2^k, 2^{k-1}+2^{k-9}))$	C_7
	[1.88, 1.89]	((16385, 32769), (49152, 32832))	$((\frac{1}{4}\cdot 2^k+1, 2^{k-1}+1), (\frac{3}{4}\cdot 2^k, 2^{k-1}+2^{k-10}))$	C_8
	[1.90, 1.90]	((16385, 32769), (49152, 32784))	$((\frac{1}{4}\cdot 2^k+1, 2^{k-1}+1), (\frac{3}{4}\cdot 2^k, 2^{k-1}+2^{k-12}))$	C_{10}

Table 1 (continued)

k	p	(x, y)-Coordinates	Representative grid-point pair coordinates in terms of k	Source
	[1.91, 2.00]	((16385, 32769), (49152, 32769))	$((\frac{1}{4} \cdot 2^k + 1, 2^{k-1} + 1), (\frac{3}{4} \cdot 2^k, 2^{k-1} + 1))$	C_{14}

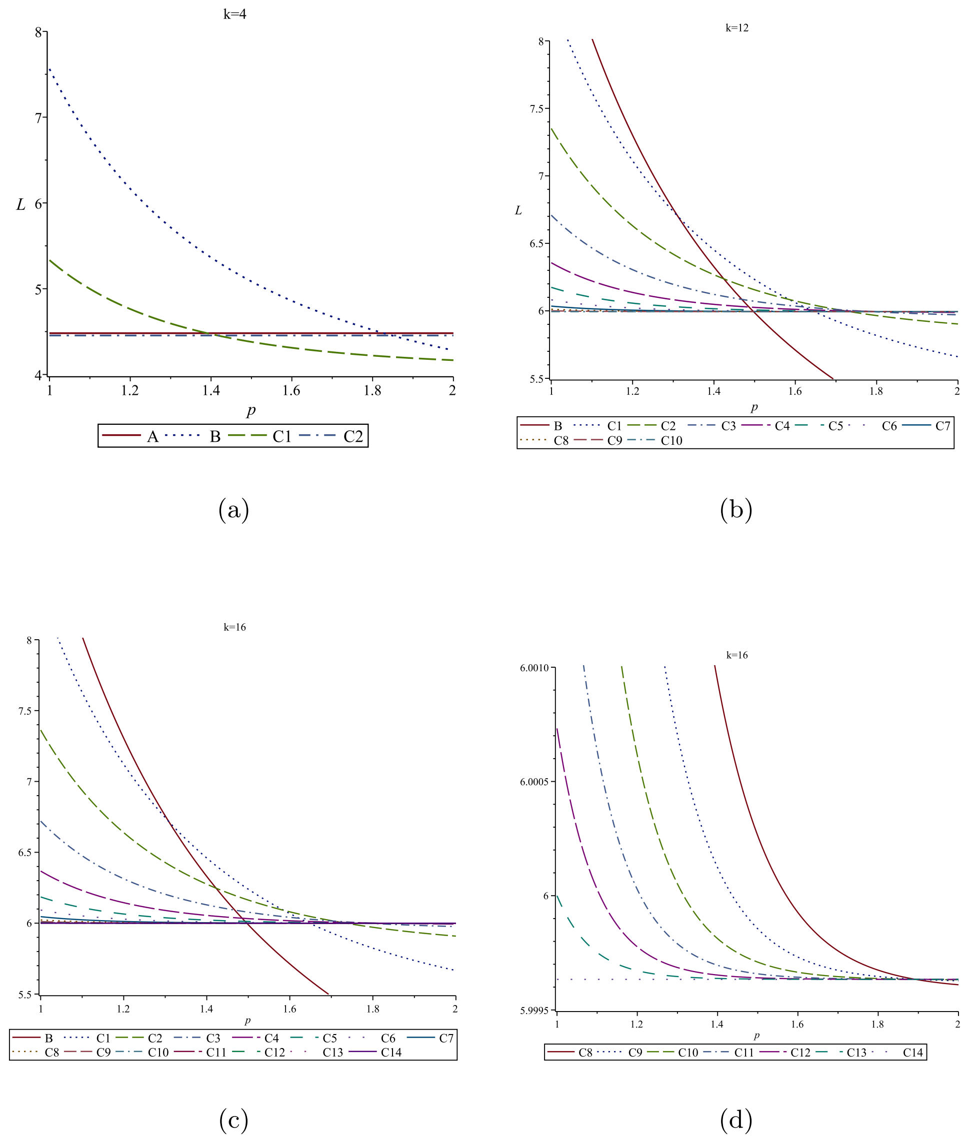


Fig. 7 Locality measures corresponding to the grid-point pairs in: **a** A, B, and $C = \{C_2\}$ for $k = 4$ and p-granularity of 0.01; **b** B and $C = \{C_t \mid 1 \leq t \leq k-2\}$ for $k = 12$ and p-granularity of 0.01; **c**, **d** B and $C = \{C_t \mid 1 \leq t \leq k-2\}$ for $k = 16$ and p-granularity of 0.01

$u \in$	$u' \in$	u'-coordinate(s):	$d_1(v, u) <$	$\delta_C(v, u) >$	$d_1(u, u') \geq$	$\delta_C(u, u') \leq$	$\hat{s}_{C,1}(u', u, v) >$
$Q_2(K)$	$Q_3(K)$	$(\mathrm{x}(u'), \mathrm{y}(u')) = (\mathrm{x}(u), 2^{k-1})$	$10 \cdot 2^{k-2}$	$\frac{11}{4} \cdot 2^{2k-2}$	2^{k-h-1}	$2 \cdot 2^{2k-2h-2}$	$11 \cdot 2^{3k-h-4} - 5 \cdot 2^{3k-2h-2} > 0$
$Q_1(K)$	$Q_2(K)$	$(\mathrm{x}(u'), \mathrm{y}(u')) = (2^{k+1}, \mathrm{y}(u))$	$10 \cdot 2^{k-2}$	$\frac{11}{4} \cdot 2^{2k-2}$	2^{k-h-1}	$2 \cdot 2^{2k-2h-2}$	$11 \cdot 2^{3k-h-4} - 5 \cdot 2^{3k-2h-2} > 0$
$Q_4(K)$	$Q_3(K)$	$\mathrm{y}(u') = \mathrm{y}(u)$	$d_1(v, u')$	$\delta_C(v, u')$			

Table 2 For selected k-values $k \in \{12, 16\}$: enumeration of intersections in $p \in (1, 2)$ of two functions $\mathcal{L}_{H_k^2,p}(v, u)$ for (v, u) in B versus C_1 and C_i versus C_j for some $i < j$ in $\{1, 2, \ldots, k-2\}$, which yield consecutive p-subintervals $([1, p_1], [p_1, p_2], \ldots)$ partitioning [1, 2] with their dominant grid-point pairs

$k = 12$		$k = 16$	
Two sources	Intersection (in p)	Two sources	Intersection (in p)
B, C_1	$p_1 = 1.308506668$	B, C_1	$p_1 = 1.308144712$
C_1, C_2	$p_2 = 1.584954815$	C_1, C_2	$p_2 = 1.585292219$
C_2, C_3	$p_3 = 1.704624651$	C_2, C_3	$p_3 = 1.705738029$
C_3, C_4	$p_4 = 1.770094088$	C_3, C_4	$p_4 = 1.772316180$
C_4, C_5	$p_5 = 1.810228346$	C_4, C_5	$p_5 = 1.814308770$
C_5, C_6	$p_6 = 1.835689535$	C_5, C_6	$p_6 = 1.843073443$
C_6, C_7	1.850364304	C_6, C_7	$p_7 = 1.863837864$
C_7, C_8	1.854042783	C_7, C_8	$p_8 = 1.879247199$
C_8, C_9	1.840799205	C_8, C_9	1.890629924
C_9, C_{10}	1.780373868	C_9, C_{10}	1.898437578
		C_{10}, C_{11}	1.902231935
C_6, C_{10}	$p_6 = 1.849641746$	C_{11}, C_{12}	1.900104562
C_7, C_{10}	1.847782860	C_{12}, C_{13}	1.886347004
C_8, C_{10}	1.829317612	C_{13}, C_{14}	1.835908289
C_9, C_{10}	1.780373868		
		C_8, C_{10}	$p_9 = 1.892362171$
		C_9, C_{10}	1.898437578
		C_{10}, C_{14}	$p_{10} = 1.900238177$
		C_{11}, C_{14}	1.894655955
		C_{12}, C_{14}	1.877334050
		C_{13}, C_{14}	1.835908289

Lemma 22 *For all positive integers k and h with $2 \le h < k$, and all grid-point pairs $v \in Q_3^k({}_5H_k^2)$ and $u \in Q_3^{h-1}(Q_2({}_6H_k^2)) - Q_3^{k-1}(Q_2({}_6H_k^2))$, there exists $u' \in Q_3^{k-1}(Q_2({}_6H_k^2))$ such that $(v, u) \prec (v, u')$ via the comparison: $\mathcal{L}_{C,1}(v, u) < \mathcal{L}_{C,1}(v, u')$.*

Proof Similar to the proof of Lemma 4 for $L_2(H_k^2)$ in "L_2-Locality of Four Linearly Contiguous Hilbert Subcurves" section. □

The six lemmas (Lemmas 17–22) identify the unique representative grid-point pair $(v', u') \in Q_3^k({}_5H_k^2) \times Q_3^{k-1}(Q_2({}_6H_k^2))$ that maximizes the $\mathcal{L}_{C,1}$-value for the subcurve C_2 (joining the slanted-diagonal corners $Q_3(Q_3({}_5H_k^2))$ and $Q_3(Q_2({}_6H_k^2))$ Hilbert subcurves) — with (v', u') residing at the lower-left and middle-right corners of C_2 with coordinates $v' = (1, 1)$ and $u' = (2^{k+1}, 2^{k-1})$, respectively:

$$\begin{aligned}\mathcal{L}_{C,1}(v', u') &= \mathcal{L}_{C,1}(Q_3(Q_3({}_5H_k^2)), Q_3(Q_2({}_6H_k^2))) \\ &= \mathcal{L}_{C,1}(Q_3^k({}_5H_k^2), Q_3^{k-1}(Q_2({}_6H_k^2))) \\ &= \frac{(2^{k+1} - 1 + 2^{k-1} - 1)^2}{3 \cdot 2^{2k-2}} = \frac{(\frac{5}{2} \cdot 2^k - 2)^2}{3 \cdot 2^{2k-2}} \\ &= \frac{25}{3} - \frac{5}{3} \cdot 2^{-k+3} + \frac{1}{3} \cdot 2^{-2k+4}.\end{aligned}$$

Two Adjacent y^-- and x^+-Oriented Hilbert Subcurves: Direct- and Slanted-Diagonal Corners

Figure 5b illustrates the labeled arrangement in Cartesian coordinates of a subcurve C' that is composed of two adjacent H_k^2-subcurves: the left ${}_7H_k^2$ (y^--oriented) and the right ${}_8H_k^2$ (x^+-oriented). Through translation and symmetry (with respect to the 1-normed metric d_1 and the index-difference functions $\delta_C/\delta_{C'}$), the treatments in locating candidate representative grid-point pairs for C' are equivalent to those for C in the two cases C_1 (in "Two Adjacent y^-- and y^+-Oriented Hilbert Subcurves: Direct-Diagonal Corners" section) and C_2 (in "Two Adjacent y^-- and x^+-Oriented Hilbert Subcurves: Slanted-Diagonal Corners" section), which result in the following Lemmas 23 and 24, respectively.

Lemma 23 *For all positive integers $k \ge 2$, and all grid-point pairs $(v, u) \in Q_3({}_7H_k^2) \times Q_3({}_8H_k^2) - Q_3^k({}_7H_k^2) \times Q_3^k({}_8H_k^2)$, there exist $v' \in Q_3^k({}_7H_k^2)$ and $u' \in Q_3^k({}_8H_k^2)$ such that $(v, u) \prec (v', u')$ via the comparison: $\mathcal{L}_{C',1}(v, u) < \mathcal{L}_{C',1}(v', u')$.*

Lemma 23 now yields the unique representative grid-point pair $(v', u') \in Q_3^k({}_7H_k^2) \times Q_3^k({}_8H_k^2)$ that maximizes the $\mathcal{L}_{C',1}$-value for the subcurve C'_1 joining the direct-diagonal corners $Q_3({}_7H_k^2)$ and $Q_3({}_8H_k^2)$ Hilbert subcurves — with (v', u') residing at the lower-left and upper-right corners of C'_1 with coordinates $v' = (1, 1)$ and $u' = (2^{k+1}, 2^k)$, respectively:

$$\begin{aligned}\mathcal{L}_{C',1}(v', u') &= \mathcal{L}_{C',1}(Q_3({}_7H_k^2), Q_3({}_8H_k^2)) \\ &= \mathcal{L}_{C',1}(Q_3^k({}_7H_k^2), Q_3^k({}_8H_k^2)) \\ &= \frac{(2^{k+1} - 1 + 2^k - 1)^2}{2^{2k}} \\ &= \frac{(3 \cdot 2^k - 2)^2}{2^{2k}} = 9 - 3 \cdot 2^{-k+2} + 2^{-2k+2}.\end{aligned}$$

Lemma 24 *For all positive integers $k \ge 3$, and all grid-point pairs $(v, u) \in Q_3(Q_3({}_7H_k^2)) \times Q_3(Q_2({}_8H_k^2)) - Q_3^k({}_7H_k^2) \times Q_3^{k-1}(Q_2({}_8H_k^2))$, there exist $v' \in Q_3^k({}_7H_k^2)$ and $u' \in Q_3^{k-1}(Q_2({}_8H_k^2))$ such that $(v, u) \prec (v', u')$ via the comparison: $\mathcal{L}_{C,1}(v, u) < \mathcal{L}_{C,1}(v', u')$.*

Lemma 24 now yields the unique representative grid-point pair $(v', u') \in Q_3^k({}_7H_k^2) \times Q_3^{k-1}(Q_2({}_8H_k^2))$ that maximizes the $\mathcal{L}_{C',1}$-value for the subcurve C'_2 joining the direct-slanted corners $Q_3(Q_3({}_7H_k^2))$ and $Q_3(Q_2({}_8H_k^2))$ Hilbert subcurves — with (v', u') residing at the lower-left and upper-middle corners of C_2' with coordinates $v' = (1, 1)$ and $u' = (\frac{3}{2} \cdot 2^k, 2^k)$, respectively:

$$\begin{aligned}\mathcal{L}_{C',1}(v', u') &= \mathcal{L}_{C',1}(Q_3(Q_3({}_7H_k^2)), Q_3(Q_2({}_8H_k^2)))\\ &= \mathcal{L}_{C',1}(Q_3^k({}_7H_k^2), Q_3^{k-1}(Q_2({}_8H_k^2)))\\ &= \frac{(2^{k+1} - 1 + 2^{k-1} - 1)^2}{3 \cdot 2^{2k-2}} = \frac{(\frac{5}{2} \cdot 2^k - 2)^2}{3 \cdot 2^{2k-2}}\\ &= \frac{25}{3} - \frac{5}{3} \cdot 2^{-k+3} + \frac{1}{3} \cdot 2^{-2k+4}.\end{aligned}$$

Representative Grid-Point Pairs for $L_1(H_k^2)$

We follow a uniform approach to identifying all representative grid-point pairs that realize the $L_1(H_k^2)$-values for $p \in \{1, 2\}$, and obtain the same matching lower and upper bounds for $L_1(H_k^2)$ in [10, 28], respectively: for all $k \geq 2$,

1. $\mathcal{L}_{C,1}(Q_3({}_5H_\kappa^2), Q_3({}_6H_\kappa^2)) = \mathcal{L}_{C,1}(Q_3^\kappa({}_5H_\kappa^2), Q_3^\kappa({}_6H_\kappa^2))$
   $= \mathcal{L}_{C,1}((1, 1), (2^{\kappa+1}, 2^\kappa))$
   $= 9 - 3 \cdot 2^{-\kappa+2} + 2^{-2\kappa+2}$
   — maximum possible κ-value is $k - 2$ (embedded in H_k^2);
2. $\mathcal{L}_{C,1}(Q_3(Q_3({}_5H_\kappa^2)), Q_3(Q_2({}_6H_\kappa^2))) = \mathcal{L}_{C,1}(Q_3^\kappa({}_5H_\kappa^2), Q_3^{\kappa-1}(Q_2({}_6H_\kappa^2)))$
   $= \mathcal{L}_{C,1}((1, 1), (2^{\kappa+1}, 2^{\kappa-1}))$
   $= \frac{25}{3} - \frac{5}{3} \cdot 2^{-\kappa+3} + \frac{1}{3} \cdot 2^{-2\kappa+4}$
   — maximum possible κ-value is $k - 2$ (embedded in H_k^2);
3. $\mathcal{L}_{C',1}(Q_3({}_7H_\kappa^2), Q_3({}_8H_\kappa^2)) = \mathcal{L}_{C',1}(Q_3^\kappa({}_7H_\kappa^2), Q_3^\kappa({}_8H_\kappa^2))$
   $= \mathcal{L}_{C',1}((1, 1), (2^{\kappa+1}, 2^\kappa))$
   $= 9 - 3 \cdot 2^{-\kappa+2} + 2^{-2\kappa+2}$
   — maximum possible κ-value is $k - 1$ (embedded in H_k^2); and
4. $\mathcal{L}_{C',1}(Q_3(Q_3({}_7H_\kappa^2)), Q_3(Q_2({}_8H_\kappa^2))) = \mathcal{L}_{C',1}(Q_3^\kappa({}_7H_\kappa^2), Q_3^{\kappa-1}(Q_2({}_8H_\kappa^2)))$
   $= \mathcal{L}_{C',1}((1, 1), (\frac{3}{2} \cdot 2^\kappa, 2^\kappa))$
   $= \frac{25}{3} - \frac{5}{3} \cdot 2^{-\kappa+3} + \frac{1}{3} \cdot 2^{-2\kappa+4}$
   — maximum possible κ-value is $k - 1$ (embedded in H_k^2).

$$L_1(H_k^2) = \frac{(3 \cdot 2^{k-1} - 2)^2}{4^{k-1}}.$$

The refined subpath-containment analysis in establishing $L_1(H_k^2)$ developed above suffices us to consider three cases (Cases 1, 6.4, and 5.4) whose locality analyses are studied in "Two Adjacent y^-- and y^+-Oriented Hilbert Subcurves: Direct-Diagonal Corners"–"Two Adjacent y^-- and x^+-Oriented Hilbert Subcurves: Direct- and Slanted-Diagonal Corners" sections, and we summarize their results with an exact formula for $L_1(H_k^2)$ below.

Theorem 4 *For all positive integers $k \geq 2$,*

$$L_1(H_k^2) = 9 - 3 \cdot 2^{-k+3} + 2^{-2k+4}.$$

Proof The locality analyses of the three cases: Cases 1, 6.4, and 5.4 (introduced in "Exact Formula for $L_1(H_k^2)$" section) in the refined subpath-containment analysis produce two candidate maximum $\frac{\Delta^2}{l}$-value (from four sources):

Note that both $f_1(\kappa) = 9 - 3 \cdot 2^{-\kappa+2} + 2^{-2\kappa+2}$ and $f_2(\kappa) = \frac{25}{3} - \frac{5}{3} \cdot 2^{-\kappa+3} + \frac{1}{3} \cdot 2^{-2\kappa+4}$ are strictly increasing in $\kappa \geq 0$; therefore, f_1 and f_2 attain their maximum value at $\kappa = k - 1$ with

$$f_1(k-1) = 9 - 3 \cdot 2^{-k+3} + 2^{-2k+4}, \text{ and}$$
$$f_2(k-1) = \frac{25}{3} - \frac{5}{3} \cdot 2^{-k+4} + \frac{1}{3} \cdot 2^{-2k+6}.$$

Observe that, for all positive integers k, $f_1(k-1) > f_2(k-1)$, hence the maximum $\frac{\Delta^2}{l}$-value assumes the value of $f_1(k-1)$. When $k = 8$, we have $9 - 3 \cdot 2^{-k+3} + 2^{-2k+4} > 8.906$, which is greater than all the upper bounds on $\frac{\Delta^2}{l}$-value in the above refined analyses for Case 5.4. For $2 \leq k \leq 7$, exhaustive searches for representative grid-point pairs of H_k^2 show that $L_1(H_k^2) = 9 - 3 \cdot 2^{-k+3} + 2^{-2k+4}$ for each $k \in \{2, 3, \ldots, 7\}$; and this completes the theorem. □

For an x^+-oriented Hilbert curve H_k^2 with $\partial_1(H_k^2) = (1, 1)$, where $k \geq 2$, the two representative grid-point pairs for H_k^2 with respect to L_1 reside at: (1) $Q_2^{k-1}(Q_1(H_k^2)) \times Q_2^k(H_k^2)$ with coordinates $((2^{k-1}, 1), (1, 2^k))$, and (2) their symmetry $Q_3^k(H_k^2) \times Q_3^{k-1}(Q_4(H_k^2))$ with coordinates $((2^k, 2^k), (2^{k-1}+1, 1))$.

Empirical Study on $L_p(H_k^2)$ with $p \in [1, 2]$

To complement the analytical results for $L_p(H_k^2)$ for all reals $p = 1$ and $p \geq 2$, we conduct an empirical study on $L_p(H_k^2)$ for all $k \in \{2, 3, \ldots, 16\}$ and a discrete spectrum of real values of $p \in [1, 2]$. With respect to the canonical orientation of H_k^2 shown in Fig. 2a, we cover the two-dimensional order-k grid space $[2^k]^2$ of H_k^2 in Cartesian coordinates: 2^k columns (respectively, rows) indexed by x-coordinates (respectively, y-coordinates) $1, 2, \ldots, 2^k$. The exhaustive verification requires a two-dimensional $2^{16} \times 2^{16}$ array in main memory. The implementation is in C-language, and is available upon request from the authors.

For every grid-order $k \in \{2, 3, \ldots, 16\}$ and real $p \in [1, 2]$ with granularity of 0.01 (for $2 \leq k \leq 16$), we locate with computer programs all representative pairs of grid points for H_k^2 with respect to L_p. Fig. 6a illustrates the three sources $\{A, B, C\}$ of candidate representative grid-point pairs for $k \geq 2$, which are elaborated below:

1. Source A identifies the grid-point pair $(v_A, u_A) = ((1, \frac{1}{4} \cdot 2^k + 1), (1, 2^k))$ and its symmetry-pair. The pair (v_A, u_A) serves as the representative grid-point pair "briefly" — for $k = 4$ and $1.83 \leq p \leq 2.00$.
2. Source B identifies the grid-point pair $(v_B, u_B) = ((2^{k-1}, 1), (1, 2^k))$ and its symmetry-pair. The pair (v_B, u_B) serves as the representative grid-point pair for every $k \in \{2, 3, \ldots, 16\}$ and all reals p of a (shrinking) prefix-interval $[1, \rho_k) \subseteq [1, 2]$— where, empirically, ρ_k decreases and stabilizes as k increases in $\{2, 3, \ldots, 12\}$ and in $\{13, 14, 15, 16\}$, respectively.
3. Source C identifies a sequence $(C_1, C_2, \ldots, C_{k-2})$ of grid-point pairs:

$$C_t = (v_{C_t}, u_{C_t}) = ((\frac{1}{4} \cdot 2^k + 1, 2^{k-1} + 1), (\frac{3}{4} \cdot 2^k, 2^{k-1} + 2^{k-2-t})),$$

for $t = 1, 2, \ldots, k-2$, and their symmetry-pairs, with

$$\mathrm{x}(u_{C_{t+1}}) = \mathrm{x}(u_{C_t}), \text{ and}$$
$$\mathrm{y}(u_{C_{t+1}}) - 2^{k-1} = \frac{\mathrm{y}(u_{C_t}) - 2^{k-1}}{2}$$

and eventually u_{C_t} converges to $u_{C_{k-2}}$.

Note that, for $t = 0$, the grid-point pair $C_0 = (v_{C_0}, u_{C_0}) = ((\frac{1}{4} \cdot 2^k + 1, 2^{k-1} + 1), (\frac{3}{4} \cdot 2^k, 2^{k-1} + 2^{k-2}))$ is not included in C since C_0 can not be a candidate representative grid-point pair (for any k and real $p \in [1, 2]$):

$$\mathcal{L}_{H_k^2,p}(v_B, u_B) = \frac{((2^{k-1}-1)^p + (2^k-1)^p)^{\frac{2}{p}}}{2^{2k-2}}$$
$$> \mathcal{L}_{H_k^2,p}(v_{C_0}, u_{C_0}) = \frac{((2^{k-1}-1)^p + (2^{k-2}-1)^p)^{\frac{2}{p}}}{\frac{1}{3} \cdot 2^{2k-3} + \frac{1}{3} \cdot 2^{2k-4}}.$$

Empirically, for all $k \in \{5, 6, \ldots, 16\}$ and all reals p in the (growing and stabilized) suffix-interval $(\rho_k, 2] \subseteq [1, 2]$, all the representative grid-point pairs form a subsequence C' of C composed of: (1) a prefix of C and (2) isolated grid-point pair(s) of C including $(v_{C_{k-2}}, u_{C_{k-2}})$. The suffix-interval $(\rho_k, 2]$ is partitioned into disjoint successive p-subintervals, each of which supports a grid-point pair in the subsequence C' as the representative grid-point pair for $L_p(H_k^2)$ (for all reals p of the subinterval). The length of C' (number of all representative grid-point pairs from the source C) should depend on k in general, and on the p-granularity in our empirical setting. Figure 6b depicts the sequence of candidate representative grid-point pairs from the source C.

Table 1 tabulates the following statistics: (1) for each $k \in \{2, 3, \ldots, 16\}$, the partitioning p-subintervals of [1, 2], and the corresponding representative grid-point pair and its source; and (2) $\mathcal{L}_{H_k^2,p}(v, u)$ $(= L_p(H_k^2))$ for a representative grid-point pair (v, u) in the three sources A, B, and C:

$$\mathcal{L}_{H_k^2,p}(v,u) = \begin{cases} \frac{(3\cdot 2^{k-2}-1)^2}{\frac{5}{3}\cdot 2^{2k-4}+\frac{1}{3}} & \text{if } (v,u) \text{is in } A \\ \frac{((2^{k-1}-1)^p+(2^k-1)^p)^{\frac{2}{p}}}{2^{2k-2}} & \text{if } (v,u) \text{ is in } B \\ \frac{((2^{k-1}-1)^p+(2^{k-2-t}-1)^p)^{\frac{2}{p}}}{\frac{1}{3}\cdot 2^{2k-3}+\frac{1}{3}\cdot 2^{2k-4-2t}} & \text{if } (v,u) = (v_{C_t}, u_{C_t}) \text{ in } C, \\ & \text{where } t = 1, 2, \dots, k-2. \end{cases}$$

Figure 7a–d shows the graphs, using the mathematical software Maple, of the locality measure $\mathcal{L}_{H_k^2,p}(v,u)$ for selected grid-order k-values: $k \in \{4, 12, 16\}$, respectively, for all reals $p \in [1, 2]$ and all (v, u) in the three sources A, B, and C. Our future work will involve determining, for each k, the dominant functions/measures over successive subintervals of [1, 2], whose piece-wise combination yields the (overall) locality measure $L_p(H_k^2)$ for all reals $p \in [1, 2]$.

For selected grid-order k-values: $k \in \{4, 12, 16\}$, we elaborate below the empirical statistics that relate the p-subintervals partitioning [1, 2] to their dominant grid-point pairs — subject to the underlying p-granularity and numerical approximation:

1. For the extreme case of $k = 4$ with p-granularity of 0.01, two representative grid-point pairs emerge from the sources B and A over the partitioning subintervals [1.00, 1.82] and [1.83, 2.00], respectively.
2. For the case of $k = 12$ with p-granularity of 0.01, the representative grid-point pairs are from the sources B and C over the partitioning subintervals [1.00, 1.31] and [1.32, 2.00], respectively. Observe that the subsequence C' of all representative grid-point pairs (from the source $C = \{C_t \mid 1 \le t \le 10\}$) is the prefix $\{C_1, C_2, C_3, C_4, C_5, C_6\}$ of C with the isolated grid-point pair C_{10}.

   To highlight the consecutive p-subintervals $([1, p_1], [p_1, p_2], \dots)$ partitioning [1, 2] with their dominant grid-point pairs, we tabulate in Table 2 the intersections (in $p \in (1, 2)$) of two functions $\mathcal{L}_{H_k^2,p}(v,u)$ for: (1) (v, u) in $B \times C_1$, and $C_t \times C_{t+1}$ for $t \in \{1, 2, \dots, 9\}$, and (2) (v, u) in $C_6 \times C_{10}$, $C_7 \times C_{10}$, $C_8 \times C_{10}$, and $C_9 \times C_{10}$.

   The seven intersections $p_1, p_2, \dots, p_7$ correspond to seven p-subintervals:

   $$[1.00, 1.31], [1.32, 1.58], \dots, [1.84, 1.84]$$

   dominated by $B, C_1, \dots, C_6$, respectively — as shown in Table 1. The consideration of the remaining intersections in the tabulation and the monotonicity of the underlying $\mathcal{L}_{H_k^2,p}$-functions indicates the dominance of C_{10} over the last p-subinterval [1.85, 2.00].
3. For the case of $k = 16$ with p-granularity of 0.01, the representative grid-point pairs are from the sources B and C over the partitioning subintervals [1.00, 1.30] and [1.31, 2.00], respectively. Analogous to the case of $k = 12$ subject to the underlying p-granularity and numerical approximation, the subsequence C' of all representative grid-point pairs (from the source $C = \{C_t \mid 1 \le t \le 14\}$) is the prefix $\{C_1, C_2, C_3, C_4, C_5, C_6, C_7, C_8\}$ of C with the isolated grid-point pairs C_{10} and C_{14}. We also tabulate similar statistics in Table 2 for the consecutive intersections that yield the p-subintervals $([1, p_1], [p_1, p_2], \dots)$ partitioning [1, 2] with their dominant grid-point pairs.

Conclusion

Our analytical study of the locality properties of the Hilbert curve family, $\{H_k^2 \mid k = 1, 2, \dots\}$, is based on the locality measure L_p, which is the maximum ratio of $d_p(v,u)^m$ to $d_p(\tilde{v}, \tilde{u})$ over all corresponding grid-point pairs (v, u) and $(\tilde{v}, \tilde{u})$ in the m-dimensional grid space and index space, respectively. Our analytical results close the gaps between the current best lower and upper bounds with exact formulas for norm-parameter $p \in \{1, 2\}$, and extend to all reals $p \ge 2$. In addition, we identify all the representative grid-point pairs (which realize $L_p(H_k^2)$) for $p = 1$ and all reals $p \ge 2$. We also verify the results with computer programs over various p-values ($p \in \{1, 2, 3\}$) and grid-orders ($k \in \{4, 5, \dots, 10\}$). For all real norm-parameters $p \in [1, 2]$ with sufficiently small granularity and grid-orders $k \in \{2, 3, \dots, 16\}$, our empirical study reveals the three major sources (A, B, and C) of representative grid-point pairs (v, u) that give $\mathcal{L}_{H_k^2,p}(v,u) = L_p(H_k^2)$. The empirical results also suggest that, subject to the underlying p-granularity and numerical approximation, all the representative grid-point pairs of B and C are from B and C', which is a prefix-subsequence of C together with some isolated grid-point pair(s) of C including C_{k-2} for some sufficiently large grid-orders $k \in \{5, 6, \dots, 16\}$. The study will shed some light on an analytical study for determining the exact formulas for $L_p(H_k^2)$ for all reals $p \in (1, 2)$ and/or in arbitrary dimensions.

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Declarations

Conflict of interest: The authors declare that they have no conflict of interest.

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