Research Article

Synthesis of multi-positions 3-prismatic–revolute–spherical manipulator

Srinivasa Rao Pundru[1] · Mohan Rao Nalluri[2]

Abstract

In this work, a procedure for synthesis of a 3-prismatic–revolute–spherical parallel manipulator for any number of prescribed positions is presented. The procedure is based on a set of geometrical constraints of the manipulator and involves determination of dimensions of the architectural parameters such that the mobile platform passes through a set of prescribed positions. The synthesis procedure of a 3-PRS manipulator is based on least square technique to minimize the functional errors in a set of obtained positions and orientations of a point on a mobile platform of 3-PRS manipulator. The existing procedures have compatibility approach and exact method approach of synthesis of 3-revolute–prismatic–spherical and 3-PRS manipulators respectively (Kim and Tsai in J ASME 125:92–97, 2003. https://doi.org/10.1115/1.1539505; Pundru and Nalluri in J Inst Eng India Ser C, 2018. https://doi.org/10.1007/s40032-018-0499-6). But the above procedures show only the limited number of prescribed positions and orientations of the mobile platform. The proposed work is the synthesis of 3-PRS manipulator based on constraints of revolute joints and is solved by least square technique and this synthesis procedure can be applied to any number of prescribed positions and orientations of the mobile platform of 3-PRS manipulator. The synthesis procedure for nine positions is demonstrated through a numerical example. The synthesis procedure lead to system of nonlinear equations which are very difficult to solve, therefore by using MATLAB applications the nonlinear coupled algebraic equations are solved. The 3-PRS manipulator is suitable for alignment applications like air craft simulation and robot manipulators where tip, tilt and position motions are significant.

Keywords Synthesis of manipulator · Multi-positions · 3-Degrees of freedom · Least square technique

List of symbols

F	Mobility of manipulator
n	Number of links in a mechanism
B_j	Base vertex of fixed platform
$P_{R_{ji}}$	Position of integrated prismatic and revolute joint
ξ_j	Angle of inclination of fixed limb w.r.t. $\overrightarrow{OB}_j$
$\overrightarrow{P}_i$	Position of reference frame of mobile platform w.r.t. fixed reference frame
ϕ_i, θ_i, ψ_i	Euler angles about Z–X–Y fixed axes
$o_{R_{P_i}}$	Rotation matrix of mobile platform
$o_{R_{P_{R_{ji}}}}$	Orientation matrix of length vector of moving limb
S_{ji}	Coordinates of spherical joint
$\overrightarrow{q_{ji}}$	Position vector of spherical joint
$\overrightarrow{h}_j$	Design variables of spherical joint of mobile platform w.r.t. reference coordinate system
$\tilde{j}$	Axis vector of revolute joint
E	Functional error
$\overrightarrow{g_j}$	Coordinates of fixed base platform vertex
$\overrightarrow{l}_j$	Position vector of moving limb
β_{ji}	Moving limb angular position w.r.t. X-axis

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$\overrightarrow{d_{ji}}$ Position displacement vector of prismatic actuator w.r.t. fixed base vertex

α_{ji} Angle between the ith position of the moving limb and ith actuation of actuator

1 Introduction

Parallel manipulators gained prominence because of their large load carrying capacity, precision in position, high speed and high stiffness. Several parallel manipulators with 6-degrees of freedom were developed for variety of applications like air craft simulation, robot manipulator etc. The 6-degrees of freedom parallel manipulator proposed by Gough and Whitehall [1] for universal tyre test machine was a six-linear jack system and built a platform with closed loop kinematic structure in order to respond to the problems of aero-landing loads. Stewart [2] described a platform with 6-degrees of freedom mechanism used in flight simulators. Majid et al. [3] presented the workspace of 6-degrees of freedom 3-PPSR parallel manipulator and the effects of joint limits were considered and the workspace is then investigated. But the 6-degrees of freedom parallel manipulators suffer due to laborious in kinematic analysis and complex mechanical design. Moreover several applications require less than 6-degrees of freedom. Therefore parallel manipulators with less than 6-degrees of freedom have been developed. Fang and Tsai [4] and Fang et al. [5] have demonstrated the synthesis of low degree of freedom symmetrical parallel manipulators based on theory of reciprocal screws. Rao and Rao [6, 7] discussed the kinematic synthesis of RPS manipulators based on Newton–Raphson method and considered the limitations on cone angles of ball joints. Kim and Tsai [8] have carried out the synthesis of RPS manipulator to show that at the most six positions and orientations of the mobile platform are considered. Zhao et al. [9] the forward–inverse displacement-singularities of spatial parallel manipulator are discussed based on numerical method and Newton–Raphson method. Li and Xu [10, 11] proposed translational parallel manipulators which are capable of moving in Cartesian coordinates. Tsai et al. [12] performed solution to the direct kinematic problem of manipulator based on Bezout's elimination method. Joshi and Tsai [13] used triceps manipulator in which one of the legs is passive and remaining legs is driven by a prismatic actuator. Carretero et al. [14], presented kinematic analysis and optimisation of an inclined PRS parallel manipulator in which prismatic joints are the actuators. Liu and Cheng [15] presented a procedure to obtain direct singularity position of RPS parallel manipulator. Pond and Carretero [16] performed singularity analysis and work space optimization of inclined parallel manipulator. Gallardo et al. [17] presented the forward position kinematic analysis of RPS parallel manipulator by means of screw theory and are carried out by Sylvester dialytic elimination method. Li and Xu [18] introduced a 3-PRS parallel manipulator in which the prismatic joints were actuated. Ji et al. [19] presented kinematic analysis and applications for 6-degrees of freedom parallel manipulator of 3-RRPS type with three legs. Tahmasebi [20] the kinematics of high precision manipulator with inextensible limbs and base mounted actuators were discussed. Pundru and Nalluri [21] presented the exact method of synthesis of manipulator in which the actuated prismatic joints are attached to inclined base platform. The existing procedures have compatibility approach and exact method approach of synthesis of 3-RPS and 3-PRS manipulators respectively [8, 21]. But the above procedures show only the limited number of prescribed positions and orientations of the mobile platform. Therefore, this study attempts the synthesis procedure of 3-PRS manipulator for any number of prescribed positions and is solved by using least square technique.

The work presents synthesis of 3-degrees of freedom manipulator represented as 3-PRS, which consists of three kinematic branches of identical configuration. Each branch consist of an actuated prismatic–revolute–spherical linkages, the actuated prismatic joints are attached to inclined base platform via revolute joints. Each limb is connected from revolute joint to moving platform by spherical joint. The design variables are dimensions of fixed and moving platforms and the length of limb between corresponding revolute and spherical joints. The manipulator is suitable for alignment applications where tip, tilt and position motions are significant. The synthesis of manipulator involves determination of the dimensions of the manipulator such that a point on mobile platform passes through a number of specified positions in space. The work presents a procedure for the synthesis of manipulator based on physical constraints of the manipulator. The constraint equations for different prescribed positions in space are used for the determination of the dimensions of the manipulator through least square technique and the procedure is suitable for any number of prescribed positions.

2 Geometry description of 3-PRS manipulator

The 3-PRS parallel manipulator consists of a mobile platform which is connected to a fixed base by three identical supporting limbs $P_{R_j} S_j$ with symmetrical kinematic structure as shown in Fig. 1. In the selected closed loop structure the number of limbs, number of joints, type of joints and number of actuated joints are identical and equal to the mobility of the mobile platform. The prismatic joints P_{R_j} are actuated on all the three limbs and the actuated prismatic joint of each limb is inclined from the fixed limb of base platform by an angle α_{ji} for ith position of limb. The prismatic joint is actuated on a

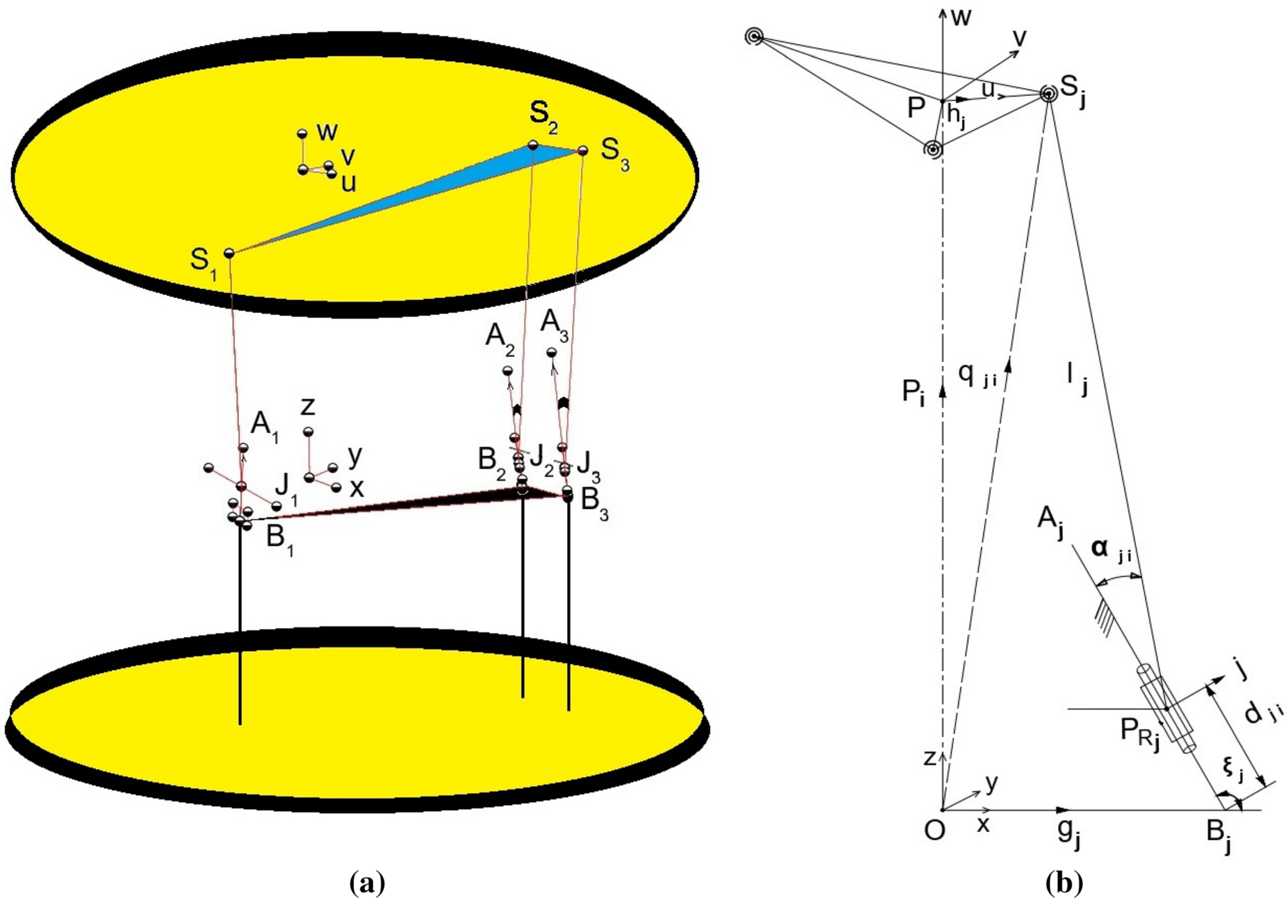

Fig. 1 3-PRS manipulator, **a** Geometry, **b** position geometry

limb of fixed length via a revolute joint. Finally each limb is connected to the mobile platform by a spherical joint S_j. Thus the mobile platform connects to the base platform by means of three identical revolute, prismatic and spherical linkages. The distance between corresponding vertices (B_j) of the fixed platform and the location of the prismatic actuated joint which is on base platform is denoted by $\overrightarrow{d_{ji}}$ as in Fig. 1. Consider $\overrightarrow{d_{ji}}$ are the linear displacement vectors of the three actuated input joint variables along the inclined fixed limbs at an angle of ξ_j with respect to $\overrightarrow{OB_j}$ and α_{ji} be the angle made by the ith position of the limb $\overrightarrow{P_{R_j}S_j}$ with respect to ith actuation along the inclined fixed limb $\overrightarrow{B_jA_j}$.

3 Degree of freedom of 3-PRS manipulator

The manipulator consists of 8-links, 3-revolute joints, 3-prismatic joints and 3-spherical joints. The degree of freedom of manipulator can be calculated by using spatial Kutzbach criterion.

$$F = \lambda(n - j - 1) + \sum_i f_i = 3 \tag{1}$$

F degree of freedom of a manipulator, λ degree of freedom of space in which a mechanism is intended to function, n number of links in a mechanism, including fixed link, j number of joints in a mechanism, f_i degree of relative motion permitted by joint i.

It is demonstrated that the selected parallel manipulator has 3-independent degree of freedom, may be defined as a translation freedom along z-axis and two degrees of rotational freedom about x and y-axis respectively.

4 Synthesis of 3-PRS manipulator for prescribed multi-positions

The constraint condition for the integrated prismatic and revolute joint axis for ith position is

$$\overrightarrow{B_jP_{R_j}}\tilde{j} = 0 \tag{2}$$

$$\left[\overrightarrow{q_{ji}} - \overrightarrow{L_j} - \overrightarrow{g_j}\right]^T \tilde{j} = 0 \tag{3}$$

$$\left[\overrightarrow{P_i} + o_{R_{P_i}}\overrightarrow{h_j} - o_{R_{P_{R_{ji}}}}\overrightarrow{l_j} - \overrightarrow{g_j}\right]^T \tilde{j} = 0 \quad for\, j = 1,2,3\, and\, i = 1,2,3,4\, etc. \tag{4}$$

$$\left[\overrightarrow{P_i}' + o_{R'_{P_i}}\overrightarrow{h_j} - o_{R'_{P_{R_{ji}}}}\overrightarrow{l_j}\right]^T \tilde{j} = 0 \quad for\, j = 1,2,3\, and\, i = 2,3,4\, etc. \tag{5}$$

where $o_{R_{P_i}}, o_{R'_{P_i}}, \overrightarrow{P_i}, \overrightarrow{P_i}', \overrightarrow{L_j}, \overrightarrow{L_j}', o_{R_{P_{R_{ji}}}}, o_{R'_{P_{R_{ji}}}}, \tilde{j}$ are given in "Appendix".

In Eq. (5), the total no. of design variables were eight, which are $h_{xj}, h_{yj}, h_{zj}, \tilde{j}_{xj}, \tilde{j}_{yj}, l_{xj}, l_{yj}, l_{zj}$. If the values of $h_j, \tilde{j}, l_j$ do not satisfy the Eq. (5), then the left hand side of Eq. (5) will yield a non-zero value. Consider the sum of squares of all these non-zero values be the error E is given by

$$E = \sum_{i=2}^{n}\left[\left[\overrightarrow{p_i}' + o_{R'_{P_i}}\overrightarrow{h_j} - o_{R'_{P_{R_{ji}}}}\overrightarrow{l_j}\right]^T \tilde{j}\right]^2 \tag{6}$$

The error E should be minimized, if the partial derivative of error (E) with respect to $h_{xj}, h_{yj}, h_{zj}, \tilde{j}_{xj}, \tilde{j}_{yj}, l_{xj}, l_{yj}, l_{zj}$ should be equated to zero. i.e.,

$$\frac{\partial E}{\partial h_{xj}} = \frac{\partial E}{\partial h_{yj}} = \frac{\partial E}{\partial h_{zj}} = \frac{\partial E}{\partial \tilde{j}_{xj}} = \frac{\partial E}{\partial \tilde{j}_{yj}} = \frac{\partial E}{\partial l_{xj}} = \frac{\partial E}{\partial l_{yj}} = \frac{\partial E}{\partial l_{zj}} = 0 \tag{7}$$

That means

$$\begin{aligned}\sum_{i=2}^{n} & \left[(P'_{xi} + u'_{xi}h_{xj} + v'_{xi}h_{yj} + w'_{xi}h_{zj} - u'_{ui}l_{xj} - v'_{ui}l_{yj} - w'_{ui}l_{zj})\tilde{j}_{xj}\right. \\ & + \left(P'_{yi} + u'_{yi}h_{xj} + v'_{yi}h_{yj} + w'_{yi}h_{zj} - u'_{vi}l_{xj} - v'_{vi}l_{yj} - w'_{vi}l_{zj}\right)\tilde{j}_{yj} \\ & \left. + (P'_{zi} + u'_{zi}h_{xj} + v'_{zi}h_{yj} + w'_{zi}h_{zj} - u'_{wi}l_{xj} - v'_{wi}l_{yj} - w'_{wi}l_{zj})\right] \\ & \left(u'_{xi}\tilde{j}_{xj} + u'_{yi}\tilde{j}_{yj} + u'_{zi}\right)|_{j=1,2,3} = 0\end{aligned} \tag{8}$$

$$\begin{aligned}\sum_{i=2}^{n} & \left[(P'_{xi} + u'_{xi}h_{xj} + v'_{xi}h_{yj} + w'_{xi}h_{zj} - u'_{ui}l_{xj} - v'_{ui}l_{yj} - w'_{ui}l_{zj})\tilde{j}_{xj}\right. \\ & + \left(P'_{yi} + u'_{yi}h_{xj} + v'_{yi}h_{yj} + w'_{yi}h_{zj} - u'_{vi}l_{xj} - v'_{vi}l_{yj} - w'_{vi}l_{zj}\right)\tilde{j}_{yj} \\ & \left. + (P'_{zi} + u'_{zi}h_{xj} + v'_{zi}h_{yj} + w'_{zi}h_{zj} - u'_{wi}l_{xj} - v'_{wi}l_{yj} - w'_{wi}l_{zj})\right] \\ & \left(v'_{xi}\tilde{j}_{xj} + v'_{yi}\tilde{j}_{yj} + v'_{zi}\right)|_{j=1,2,3} = 0\end{aligned} \tag{9}$$

$$\begin{aligned}\sum_{i=2}^{n} & \left[(P'_{xi} + u'_{xi}h_{xj} + v'_{xi}h_{yj} + w'_{xi}h_{zj} - u'_{ui}l_{xj} - v'_{ui}l_{yj} - w'_{ui}l_{zj})\tilde{j}_{xj}\right. \\ & + \left(P'_{yi} + u'_{yi}h_{xj} + v'_{yi}h_{yj} + w'_{yi}h_{zj} - u'_{vi}l_{xj} - v'_{vi}l_{yj} - w'_{vi}l_{zj}\right)\tilde{j}_{yj} \\ & \left. + (P'_{zi} + u'_{zi}h_{xj} + v'_{zi}h_{yj} + w'_{zi}h_{zj} - u'_{wi}l_{xj} - v'_{wi}l_{yj} - w'_{wi}l_{zj})\right] \\ & \left(w'_{xi}\tilde{j}_{xj} + w'_{yi}\tilde{j}_{yj} + w'_{zi}\right)|_{j=1,2,3} = 0\end{aligned} \tag{10}$$

$$\begin{aligned}\sum_{i=2}^{n} & \left[(P'_{xi} + u'_{xi}h_{xj} + v'_{xi}h_{yj} + w'_{xi}h_{zj} - u'_{ui}l_{xj} - v'_{ui}l_{yj} - w'_{ui}l_{zj})\tilde{j}_{xj}\right. \\ & + \left(P'_{yi} + u'_{yi}h_{xj} + v'_{yi}h_{yj} + w'_{yi}h_{zj} - u'_{vi}l_{xj} - v'_{vi}l_{yj} - w'_{vi}l_{zj}\right)\tilde{j}_{yj} \\ & \left. + (P'_{zi} + u'_{zi}h_{xj} + v'_{zi}h_{yj} + w'_{zi}h_{zj} - u'_{wi}l_{xj} - v'_{wi}l_{yj} - w'_{wi}l_{zj})\right] \\ & (P'_{xi} + u'_{xi}h_{xj} + v'_{xi}h_{yj} + w'_{xi}h_{zj} - u'_{ui}l_{xj} - v'_{ui}l_{yj} - w'_{ui}l_{zj})|_{j=1,2,3} = 0\end{aligned} \tag{11}$$

$$\begin{aligned}\sum_{i=2}^{n} & \left[(P'_{xi} + u'_{xi}h_{xj} + v'_{xi}h_{yj} + w'_{xi}h_{zj} - u'_{ui}l_{xj} - v'_{ui}l_{yj} - w'_{ui}l_{zj})\tilde{j}_{xj}\right. \\ & + \left(P'_{yi} + u'_{yi}h_{xj} + v'_{yi}h_{yj} + w'_{yi}h_{zj} - u'_{vi}l_{xj} - v'_{vi}l_{yj} - w'_{vi}l_{zj}\right)\tilde{j}_{yj} \\ & \left. + (P'_{zi} + u'_{zi}h_{xj} + v'_{zi}h_{yj} + w'_{zi}h_{zj} - u'_{wi}l_{xj} - v'_{wi}l_{yj} - w'_{wi}l_{zj})\right] \\ & \left(P'_{yi} + u'_{yi}h_{xj} + v'_{yi}h_{yj} + w'_{yi}h_{zj} - u'_{vi}l_{xj} - v'_{vi}l_{yj} - w'_{vi}l_{zj}\right)|_{j=1,2,3} = 0\end{aligned} \tag{12}$$

$$\begin{aligned}\sum_{i=2}^{n} & \left[(P'_{xi} + u'_{xi}h_{xj} + v'_{xi}h_{yj} + w'_{xi}h_{zj} - u'_{ui}l_{xj} - v'_{ui}l_{yj} - w'_{ui}l_{zj})\tilde{j}_{xj}\right. \\ & + \left(P'_{yi} + u'_{yi}h_{xj} + v'_{yi}h_{yj} + w'_{yi}h_{zj} - u'_{vi}l_{xj} - v'_{vi}l_{yj} - w'_{vi}l_{zj}\right)\tilde{j}_{yj} \\ & \left. + (P'_{zi} + u'_{zi}h_{xj} + v'_{zi}h_{yj} + w'_{zi}h_{zj} - u'_{wi}l_{xj} - v'_{wi}l_{yj} - w'_{wi}l_{zj})\right] \\ & (u'_{ui}\tilde{j}_{xj} + u'_{vi}\tilde{j}_{yj} + u'_{wi})|_{j=1,2,3} = 0\end{aligned} \tag{13}$$

$$\begin{aligned}\sum_{i=2}^{n} & \left[(P'_{xi} + u'_{xi}h_{xj} + v'_{xi}h_{yj} + w'_{xi}h_{zj} - u'_{ui}l_{xj} - v'_{ui}l_{yj} - w'_{ui}l_{zj})\tilde{j}_{xj}\right. \\ & + \left(P'_{yi} + u'_{yi}h_{xj} + v'_{yi}h_{yj} + w'_{yi}h_{zj} - u'_{vi}l_{xj} - v'_{vi}l_{yj} - w'_{vi}l_{zj}\right)\tilde{j}_{yj} \\ & \left. + (P'_{zi} + u'_{zi}h_{xj} + v'_{zi}h_{yj} + w'_{zi}h_{zj} - u'_{wi}l_{xj} - v'_{wi}l_{yj} - w'_{wi}l_{zj})\right. \\ & (v'_{ui}\tilde{j}_{xj} + v'_{vi}\tilde{j}_{yj} + v'_{wi})|_{j=1,2,3} = 0\end{aligned} \tag{14}$$

$$\begin{aligned}\sum_{i=2}^{n} & \left[(P'_{xi} + u'_{xi}h_{xj} + v'_{xi}h_{yj} + w'_{xi}h_{zj} - u'_{ui}l_{xj} - v'_{ui}l_{yj} - w'_{ui}l_{zj})\tilde{j}_{xj}\right. \\ & + \left(P'_{yi} + u'_{yi}h_{xj} + v'_{yi}h_{yj} + w'_{yi}h_{zj} - u'_{vi}l_{xj} - v'_{vi}l_{yj} - w'_{vi}l_{zj}\right)\tilde{j}_{yj} \\ & \left. + (P'_{zi} + u'_{zi}h_{xj} + v'_{zi}h_{yj} + w'_{zi}h_{zj} - u'_{wi}l_{xj} - v'_{wi}l_{yj} - w'_{wi}l_{zj})\right] \\ & (w'_{ui}\tilde{j}_{xj} + w'_{vi}\tilde{j}_{yj} + w'_{wi})|_{j=1,2,3} = 0\end{aligned} \tag{15}$$

But the constraint conditions of all revolute joints are $\vec{L}.\tilde{j} = 0$, i.e. by least square technique the Eq. (7) can be written as

$$\sum_{i=2}^{n}\left[\left(P'_{xi}+u'_{xi}h_{xj}+v'_{xi}h_{yj}+w'_{xi}h_{zj}\right)\tilde{J}_{xj}+\left(P'_{yi}+u'_{yi}h_{xj}+v'_{yi}h_{yj}+w'_{yi}h_{zj}\right)\tilde{J}_{yj}+\left(P'_{zi}+u'_{zi}h_{xj}+v'_{zi}h_{yj}+w'_{zi}h_{zj}\right)\right]\left(u'_{xi}\tilde{J}_{xj}+u'_{yi}\tilde{J}_{yj}+u'_{zi}\right)|_{j=1,2,3}=0 \tag{16}$$

$$\sum_{i=2}^{n}\left[\left(P'_{xi}+u'_{xi}h_{xj}+v'_{xi}h_{yj}+w'_{xi}h_{zj}\right)\tilde{J}_{xj}+\left(P'_{yi}+u'_{yi}h_{xj}+v'_{yi}h_{yj}+w'_{yi}h_{zj}\right)\tilde{J}_{yj}+\left(P'_{zi}+u'_{zi}h_{xj}+v'_{zi}h_{yj}+w'_{zi}h_{zj}\right)\right]\left(v'_{xi}\tilde{J}_{xj}+v'_{yi}\tilde{J}_{yj}+v'_{zi}\right)|_{j=1,2,3}=0 \tag{17}$$

$$\sum_{i=2}^{n}\left[\left(P'_{xi}+u'_{xi}h_{xj}+v'_{xi}h_{yj}+w'_{xi}h_{zj}\right)\tilde{J}_{xj}+\left(P'_{yi}+u'_{yi}h_{xj}+v'_{yi}h_{yj}+w'_{yi}h_{zj}\right)\tilde{J}_{yj}+\left(P'_{zi}+u'_{zi}h_{xj}+v'_{zi}h_{yj}+w'_{zi}h_{zj}\right)\right]\left(w'_{xi}\tilde{J}_{xj}+w'_{yi}\tilde{J}_{yj}+w'_{zi}\right)|_{j=1,2,3}=0 \tag{18}$$

$$\sum_{i=2}^{n}\left[\left(P'_{xi}+u'_{xi}h_{xj}+v'_{xi}h_{yj}+w'_{xi}h_{zj}\right)\tilde{J}_{xj}+\left(P'_{yi}+u'_{yi}h_{xj}+v'_{yi}h_{yj}+w'_{yi}h_{zj}\right)\tilde{J}_{yj}+\left(P'_{zi}+u'_{zi}h_{xj}+v'_{zi}h_{yj}+w'_{zi}h_{zj}\right)\right]\left(P'_{xi}+u'_{xi}h_{xj}+v'_{xi}h_{yj}+w'_{xi}h_{zj}\right)|_{j=1,2,3}=0 \tag{19}$$

$$\sum_{i=2}^{n}\left[\left(P'_{xi}+u'_{xi}h_{xj}+v'_{xi}h_{yj}+w'_{xi}h_{zj}\right)\tilde{J}_{xj}+\left(P'_{yi}+u'_{yi}h_{xj}+v'_{yi}h_{yj}+w'_{yi}h_{zj}\right)\tilde{J}_{yj}+\left(P'_{zi}+u'_{zi}h_{xj}+v'_{zi}h_{yj}+w'_{zi}h_{zj}\right)\right]\left(P'_{yi}+u'_{yi}h_{xj}+v'_{yi}h_{yj}+w'_{yi}h_{zj}\right)|_{j=1,2,3}=0 \tag{20}$$

By solving Eqs. (16)–(20), the unknowns $h_{xj}, h_{yj}, h_{zj}, \tilde{J}_{xj}, \tilde{J}_{yj}$ can be determined. By substituting the values of $h_{xj}, h_{yj}, h_{zj}, \tilde{J}_{xj}, \tilde{J}_{yj}$ in Eq. (4) then the unknowns $l_{xj}, l_{yj}, l_{zj}, g_{xj}, g_{yj}, g_{zj}$ can be determined.

5 Numerical example

The above procedures for synthesis are demonstrated through the following numerical example. The positions considered for the synthesis of the manipulator are given in Table 1.

The sets of position (p_{xi}, p_{yi}, p_{zi}) and orientation $(\psi_i, \theta_i, \phi_i)$ of mobile platform of manipulator in Table 1 are chosen arbitrarily. Using the least square technique procedure the manipulator can be designed for any number of arbitrary positions and orientations. From Eqs. (16) to (20) which contains the unknown parameters $(h_{xj}, h_{yj}, h_{zj}, \tilde{J}_{xj}, \tilde{J}_{yj})$ and known parameters $\left(P'_{xi}, P'_{yi}, P'_{zi}, u'_{xi}, u'_{yi}, u'_{zi}, v'_{xi}, v'_{yi}, v'_{zi}, w'_{xi}, w'_{yi}, w'_{zi}\right)$ (given in "Appendix") are determined by substituting the values of prescribed sets of position (p_{xi}, p_{yi}, p_{zi}) and orientation $(\psi_i, \theta_i, \phi_i)$ of the mobile platform of Table 1, then substituting the values of known parameters $\left(P'_{xi}, P'_{yi}, P'_{zi}, u'_{xi}, u'_{yi}, u'_{zi}, v'_{xi}, v'_{yi}, v'_{zi}, w'_{xi}, w'_{yi}, w'_{zi}\right)$ in Eqs. (16)–(20) and solved using MATLAB and the unknown parameters $(h_{xj}, h_{yj}, h_{zj}, \tilde{J}_{xj}, \tilde{J}_{yj})$ are determined and results in mobile platform design variable (h_{xj}, h_{yj}, h_{zj}) and the direction of the revolute joint $(\tilde{J}_{xj}, \tilde{J}_{yj})$ of the 3-PRS manipulator. By substituting the values of $(h_{xj}, h_{yj}, h_{zj}, \tilde{J}_{xj}, \tilde{J}_{yj})$ and known parameters $\left(P'_{xi}, P'_{yi}, P'_{zi}, u'_{xi}, u'_{yi}, u'_{zi}, v'_{xi}, v'_{yi}, v'_{zi}, w'_{xi}, w'_{yi}, w'_{zi}\right)$ in Eq. (4) then the unknowns $(g_{xj}, g_{yj}, g_{zj}, l_{xj}, l_{yj}, l_{zj})$ are determined and results in location of revolute joint (g_{xj}, g_{yj}, g_{zj}) and dimensions of limb (l_{xj}, l_{yj}, l_{zj}) of the 3-PRS manipulator. The three sets of $(h_{xj}, h_{yj}, h_{zj}, \tilde{J}_{xj}, \tilde{J}_{yj}, g_{xj}, g_{yj}, g_{zj}, l_{xj}, l_{yj}, l_{zj})$ solutions are required to construct a 3-PRS manipulator. The results are given in Table 2.

The position of spherical joints $(q_{xji}, q_{yji}, q_{zji})$ for ith number of position of $\overrightarrow{q_{ji}}$ are to be resolved by using $\overrightarrow{q_{ji}} = \overrightarrow{P_i} + o_{R_{P_i}} h_j = \left[\, q_{xji} \;\; q_{yji} \;\; q_{zji} \,\right]^T$ which are given in "Appendix". By substituting the values of prescribed sets of position (P_{xi}, P_{yi}, P_{zi}) and orientation $(\psi_i, \theta_i, \phi_i)$ of the mobile platform of Table 1 and the mobile platform design variable (h_{xj}, h_{yj}, h_{zj}) of Table 2 in the spherical joint position $\overrightarrow{q_{ji}}$ which are given in "Appendix" then the positions of each spherical joint $\overrightarrow{q_{ji}}$ for ith number of position are given below in the Table 3.

Table 1 Prescribed sets of position and orientation of the mobile platform

Set no	p_{xi}	p_{yi}	p_{zi}	ϕ_i	θ_i	ψ_i
1	0	0	1	0	0	0
2	-7.76×10^{-5}	-6.97×10^{-6}	1.00325	4×10^{-4}	1.784	1.471
3	-2.23×10^{-4}	-1.24×10^{-5}	1.01366	7.12×10^{-4}	2.523	1.853
4	-3.84×10^{-4}	-1.74×0^{-5}	1.03209	9.99×10^{-4}	3.09	2.122
5	-5.54×10^{-4}	-2.19×10^{-5}	1.05928	1.26×10^{-3}	3.568	2.335
6	-7.3×10^{-4}	-2.67×10^{-5}	1.09582	1.53×10^{-3}	3.989	2.515
7	-9.1×10^{-4}	-3.1×10^{-5}	1.14208	1.78×10^{-3}	4.37	2.673
8	-1.09×10^{-3}	-3.53×10^{-5}	1.19827	2.02×10^{-3}	4.72	2.814
9	-1.28×10^{-3}	-3.94×10^{-5}	1.26435	2.26×10^{-3}	5.046	2.942

p_{xi}, p_{yi}, p_{zi} are in metres and ϕ_i, θ_i, ψ_i are in degrees

6 Inverse kinematic analysis

The position displacement vector of the three prismatic actuated joint variables for ith number of positions of $\overrightarrow{d_{ji}}$ are to be resolved by inverse kinematic analysis. The prismatic actuators are actuated along the inclined fixed limbs. The fixed limbs chosen at an angle of 120° is arbitrary because the displacement positions occupied by the prismatic actuators $\overrightarrow{d_{ji}}$ with respect to fixed triangular base platform vertices of the manipulator along the position of the inclined fixed limb were determined, so the fixed limbs are at an angle of $\xi_j(120°)$ is considered with respect to $\overrightarrow{OB_j}$ then the value of d_{ji} can be calculated as

$$\begin{aligned}
&\big(p_{xi} + \left(\sin\psi_i \sin\theta_i \sin\phi_i + \cos\theta_i \cos\phi_i\right)h_{xj} \\
&\quad + \left(\sin\psi_i \sin\theta_i \cos\phi_i - \cos\theta_i \sin\phi_i\right)h_{yj} + \left(\sin\theta_i \cos\psi_i\right)h_{zj} - g_{xj}\big)^2 \\
&\quad + \left(p_{yi} + \left(\cos\psi_i \sin\phi_i\right)h_{xj} + \left(\cos\psi_i \cos\phi_i\right)h_{yj} - \sin\psi_i h_{zj} - g_{yj}\right)^2 \\
&\quad + \big(p_{zi} + \left(\sin\psi_i \cos\theta_i \sin\phi_i - \sin\theta_i \cos\phi_i\right)h_{xj} \\
&\quad + \left(\sin\psi_i \cos\theta_i \cos\phi_i + \sin\theta_i \sin\phi_i\right)h_{yj} + \left(\cos\theta_i \cos\psi_i\right)h_{zj} - g_{zj}\big)^2 \\
&\quad - 2d_{ji}\left(\cos\beta_j \cos\xi\right)\big(p_{xi} + \left(\sin\psi_i \sin\theta_i \sin\phi_i + \cos\theta_i \cos\phi_i\right)h_{xj} \\
&\quad + \left(\sin\psi_i \sin\theta_i \cos\phi_i - \cos\theta_i \sin\phi_i\right)h_{yj} + \left(\sin\theta_i \cos\psi_i\right)h_{zj} - g_{xj}\big) \\
&\quad - L_{xji}^2 - L_{yji}^2 - L_{zji}^2 - 2d_{ji}\left(\sin\beta_j \cos\xi\right) \\
&\quad \left(p_{yi} + \left(\cos\psi_i \sin\phi_i\right)h_{xj} + \left(\cos\psi_i \cos\phi_i\right)h_{yj} - \sin\psi_i h_{zj} - g_{yj}\right) \\
&\quad - 2d_{ji}(\sin\xi)\big(p_{zi} + \left(\sin\psi_i \cos\theta_i \sin\phi_i - \sin\theta_i \cos\phi_i\right)h_{xj} \\
&\quad + \left(\sin\psi_i \cos\theta_i \cos\phi_i + \sin\theta_i \sin\phi_i\right)h_{yj} + \left(\cos\theta_i \cos\psi_i\right)h_{zj} - g_{zj}\big) \\
&\quad + d_{ji}^2 = 0
\end{aligned} \tag{21}$$

The displacement positions occupied by the prismatic actuators $\overrightarrow{d_{ji}}$ with respect to fixed triangular base platform vertices of the manipulator along the inclined fixed limb are determined by substituting the values of prescribed sets of position $\left(P_{xi}, P_{yi}, P_{zi}\right)$ and orientation $\left(\psi_i, \theta_i, \phi_i\right)$ of the mobile platform of Table 1 and the synthesized mobile platform design variable $\left(h_{xj}, h_{yj}, h_{zj}\right)$, location of revolute joints $\left(g_{xj}, g_{yj}, g_{zj}\right)$ and dimensions of limbs $\left(l_{xj}, l_{yj}, l_{zj}\right)$ of Table 2 in Eq. (21) and solved by using inverse kinematics and the results of positions occupied by the prismatic actuators $\overrightarrow{d_{ji}}$ are shown in Table 4.

7 The positions of 3-PRS manipulator

The synthesis of 3-PRS manipulator for 1st to 9th position of prescribed sets of Table 1 of position $\left(p_{xi}, p_{yi}, p_{zi}\right)$ and orientation $\left(\psi_i, \theta_i, \phi_i\right)$ of mobile platform obtained with the three sets of solutions of Tables 2, 3, and 4 (the three sets of solutions are required to construct a 3-PRS manipulator). i.e. solutions of the synthesized mobile platform design variable $\left(h_{xj}, h_{yj}, h_{zj}\right)$, direction of revolute joints

Table 2 Location of mobile platform design variable, direction and location of the revolute joint and limb length of 3-PRS manipulator

Design variables	Solution 1	Solution 2	Solution 3
$\begin{bmatrix} h_{xj} \\ h_{yj} \\ h_{zj} \end{bmatrix}$	$\begin{bmatrix} 2.9658 \\ 2.3471 \\ 0.2434 \end{bmatrix}$	$\begin{bmatrix} 2.9288 \\ 1.5677 \\ 0.1646 \end{bmatrix}$	$\begin{bmatrix} 3.8398 \\ 2.7571 \\ 0.2912 \end{bmatrix}$
$\begin{bmatrix} \tilde{j}_{xj} \\ \tilde{j}_{yj} \\ \tilde{j}_{zj} \end{bmatrix}$	$\begin{bmatrix} -0.6206 \\ 0.7842 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 0.4719 \\ -0.8816 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 0.5832 \\ -0.8123 \\ 1 \end{bmatrix}$
$\begin{bmatrix} g_{xj} \\ g_{yj} \\ g_{zj} \end{bmatrix}$	$\begin{bmatrix} 2.9658 \\ 2.3471 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 2.9288 \\ 1.5677 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 3.8398 \\ 2.7571 \\ 0 \end{bmatrix}$
$\begin{bmatrix} l_{xj} \\ l_{yj} \\ l_{zj} \end{bmatrix}$	$\begin{bmatrix} 1.19 \times 10^{-6} \\ 9.45 \times 10^{-7} \\ 1.2434 \end{bmatrix}$	$\begin{bmatrix} 1.26 \times 10^{-6} \\ 6.73 \times 10^{-7} \\ 1.1646 \end{bmatrix}$	$\begin{bmatrix} 1.28 \times 10^{-6} \\ 9.22 \times 10^{-7} \\ 1.2912 \end{bmatrix}$

$h_{xj}, h_{yj}, h_{zj}, g_{xj}, g_{yj}, g_{zj}, l_{xj}, l_{yj}, l_{zj}$ are in metres

Table 3 Position of spherical joints $\overrightarrow{q_{ji}}$ of 3-PRS manipulator

3-PRS	$\overrightarrow{q_{1i}}$	$\overrightarrow{q_{2i}}$	$\overrightarrow{q_{3i}}$
Position 1	$\begin{bmatrix} 2.9658 \\ 2.3471 \\ 1.2434 \end{bmatrix}$	$\begin{bmatrix} 2.9288 \\ 1.5677 \\ 1.1646 \end{bmatrix}$	$\begin{bmatrix} 3.8398 \\ 2.7571 \\ 1.2912 \end{bmatrix}$
Position 2	$\begin{bmatrix} 2.97372 \\ 2.34009 \\ 1.21434 \end{bmatrix}$	$\begin{bmatrix} 2.93367 \\ 1.562978 \\ 1.11676 \end{bmatrix}$	$\begin{bmatrix} 3.84911 \\ 2.74874 \\ 1.24541 \end{bmatrix}$
Position 3	$\begin{bmatrix} 2.97672 \\ 2.33802 \\ 1.20197 \end{bmatrix}$	$\begin{bmatrix} 2.93519 \\ 1.56158 \\ 1.09973 \end{bmatrix}$	$\begin{bmatrix} 3.85256 \\ 2.74628 \\ 1.22447 \end{bmatrix}$
Position 4	$\begin{bmatrix} 2.97886 \\ 2.33651 \\ 1.20185 \end{bmatrix}$	$\begin{bmatrix} 2.93613 \\ 1.56057 \\ 1.0964 \end{bmatrix}$	$\begin{bmatrix} 3.85497 \\ 2.74448 \\ 1.21759 \end{bmatrix}$
Position 5	$\begin{bmatrix} 2.98053 \\ 2.33528 \\ 1.21287 \end{bmatrix}$	$\begin{bmatrix} 2.93675 \\ 1.55973 \\ 1.1049 \end{bmatrix}$	$\begin{bmatrix} 3.85684 \\ 2.74301 \\ 1.22282 \end{bmatrix}$
Position 6	$\begin{bmatrix} 2.98191 \\ 2.33421 \\ 1.23482 \end{bmatrix}$	$\begin{bmatrix} 2.93716 \\ 1.55902 \\ 1.12474 \end{bmatrix}$	$\begin{bmatrix} 3.85835 \\ 2.74174 \\ 1.23961 \end{bmatrix}$
Position 7	$\begin{bmatrix} 2.98306 \\ 2.33326 \\ 1.26767 \end{bmatrix}$	$\begin{bmatrix} 2.93743 \\ 1.55838 \\ 1.15576 \end{bmatrix}$	$\begin{bmatrix} 3.8596 \\ 2.74061 \\ 1.26775 \end{bmatrix}$
Position 8	$\begin{bmatrix} 2.98405 \\ 2.33239 \\ 1.31134 \end{bmatrix}$	$\begin{bmatrix} 2.93758 \\ 1.5578 \\ 1.19781 \end{bmatrix}$	$\begin{bmatrix} 3.86066 \\ 2.73958 \\ 1.30706 \end{bmatrix}$
Position 9	$\begin{bmatrix} 2.98491 \\ 2.33159 \\ 1.36562 \end{bmatrix}$	$\begin{bmatrix} 2.93764 \\ 1.55726 \\ 1.25064 \end{bmatrix}$	$\begin{bmatrix} 3.86156 \\ 2.73863 \\ 1.35726 \end{bmatrix}$

$\overrightarrow{q_{ji}}$ are in metres

Table 4 Position displacement vectors of three prismatic actuators of 3-PRS manipulator

3-PRS	$\vec{d}_{ji\|j=solution\ 1,2\ \&\ 3\ and\ i=1\ to\ 9\|} = \left[\vec{d_1}\ \vec{d_2}\ \vec{d_3}\right]^T$
Position 1	$\left[0\ 0\ 0\right]^T$
Position 2	$\left[-0.0334\ \ -0.0549\ \ -0.05255\right]^T$
Position 3	$\left[-0.04755\ \ -0.07428\ \ -0.07641\right]^T$
Position 4	$\left[-0.04766\ \ -0.07806\ \ -0.08421\right]^T$
Position 5	$\left[-0.03501\ \ -0.0684\ \ -0.07825\right]^T$
Position 6	$\left[-0.00972\ \ -0.04577\ \ -0.05909\right]^T$
Position 7	$\left[0.02841\ \ -0.01013\ \ -0.02678\right]^T$
Position 8	$\left[0.07963\ \ 0.03868\ \ 0.01874\right]^T$
Position 9	$\left[0.14419\ \ 0.10084\ \ 0.07756\right]^T$

$\overrightarrow{d_{ji}}$ are in metres

$\left(\tilde{j}_{xj}, \tilde{j}_{yj}\right)$, location of revolute joints $\left(g_{xj}, g_{yj}, g_{zj}\right)$ and dimensions of limbs $\left(l_{xj}, l_{yj}, l_{zj}\right)$ are given in the Table 2 and the position of spherical joints $\left(q_{xji}, q_{yji}, q_{zji}\right)$ of mobile platform are given in the Table 3 and the position displacement vectors of three prismatic actuators $\left(\overrightarrow{d_1}, \overrightarrow{d_2}, \overrightarrow{d_3}\right)$ w.r.t. fixed base vertex are given in the Table 4. The positions of the manipulator constructed using three sets of solutions of the Tables 2, 3 and 4 are shown in Figs. 2, 3, 4, 5 and 6. Figure 2 represents position 1 and position 2, Fig. 3 represents position 4, Fig. 4 represents position 7 and position 8, Fig. 5 represents position 9 and combination of positions 1 and 4, Fig. 6 represents combination of positions 4 and 9 and positions 1, 4 and 9 of the 3-PRS parallel manipulator.

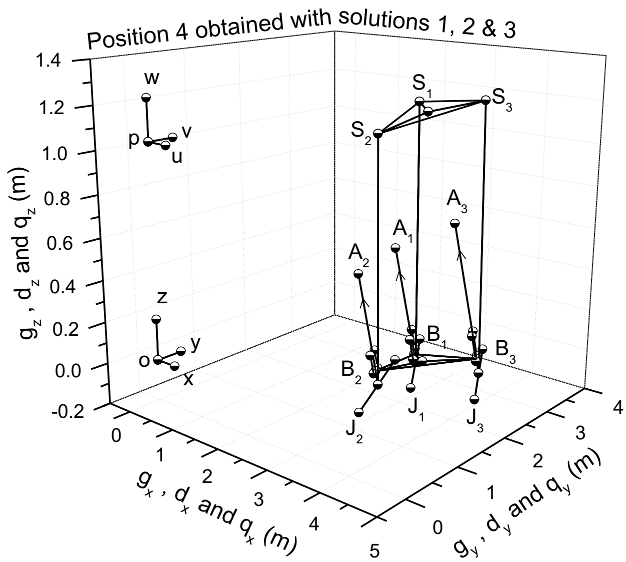

Fig. 3 Design of 3-PRS manipulator for 4th position

8 Summary

In this work, a procedure for synthesis of a 3-PRS manipulator for any number of prescribed positions is presented. The procedure overcomes the problem of limited number of positions of the synthesis procedures presented in the literature.

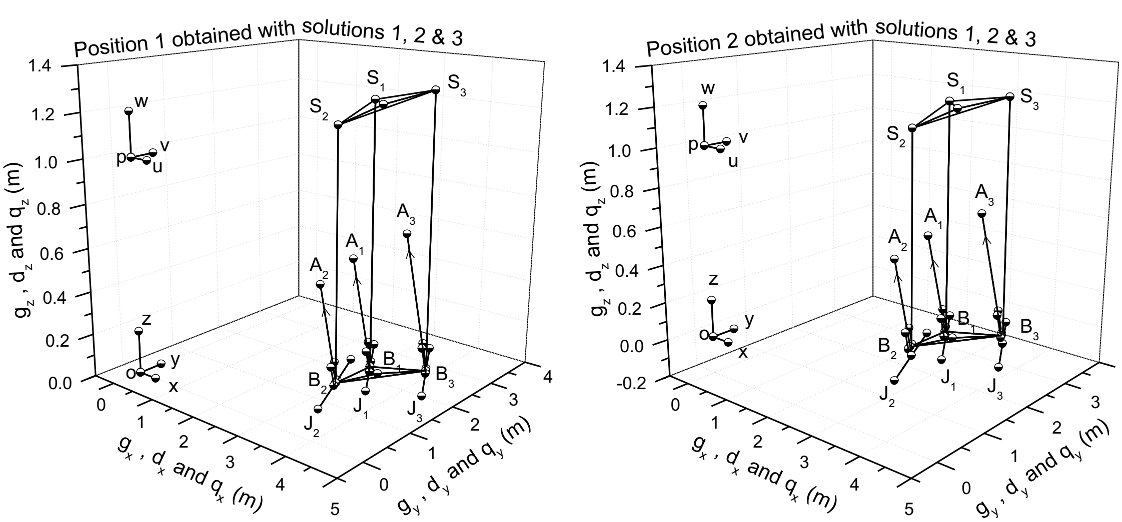

Fig. 2 Design of 3-PRS manipulator for 1st and 2nd position

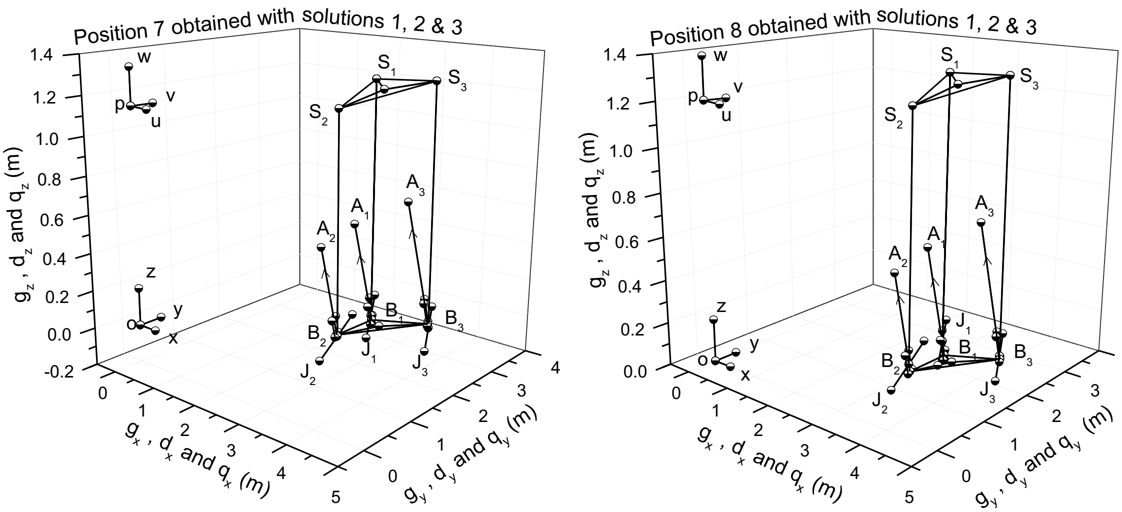

Fig. 4 Design of 3-PRS manipulator for 7th and 8th position

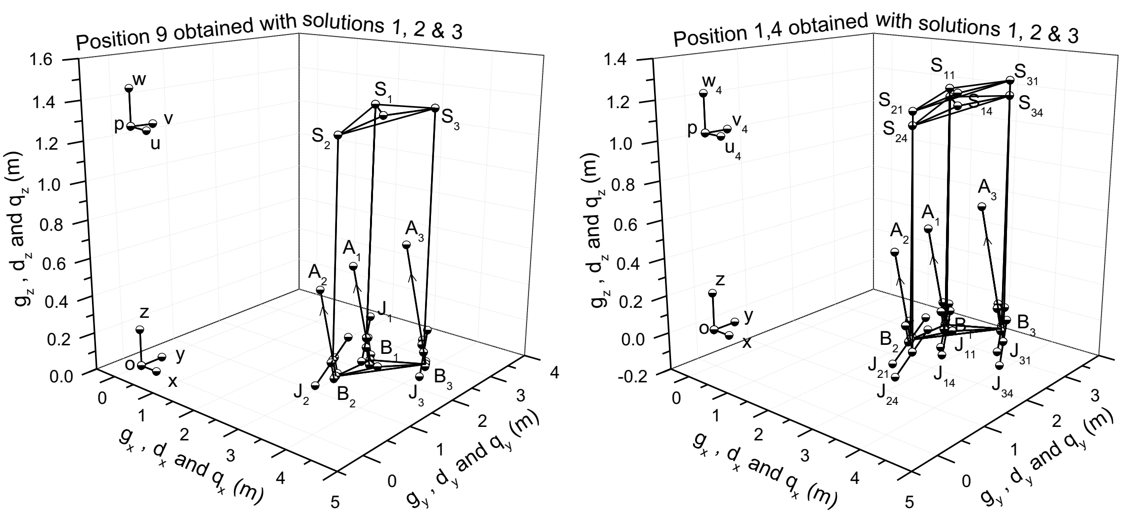

Fig. 5 Design of 3-PRS manipulator for 9th and 1st and 4th position

The procedure is based on a set of geometrical constraints of the manipulator and involves determination of dimensions of the architectural parameters such that the mobile platform passes through a set of prescribed positions. The synthesis procedure used is based on least square technique for nine positions is demonstrated through a numerical example and it can be applied for the synthesis of similar spatial 3-degrees of freedom parallel manipulators.

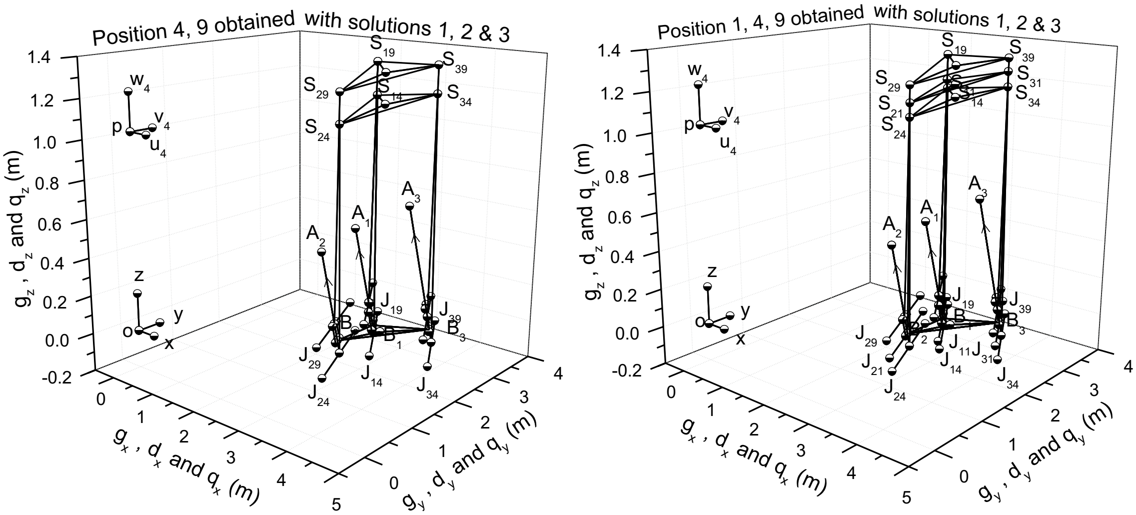

Fig. 6 Design of 3-PRS manipulator for 4th and 9th and 1st, 4th and 9th position

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

Appendix: an appendix Sect. 4

Equations (3)–(5) contain $o_{R_{P_i}}, o_{R'_{P_i}}, \vec{P_i}, \vec{P_i}', \vec{L_j}, \vec{L_j}', o_{R_{P_{R_{ji}}}}, o_{R'_{P_{R_{ji}}}}, \tilde{j}$ which are represented by

$$o_{R_{P_i}} = \begin{bmatrix} u_{xi} & v_{xi} & w_{xi} \\ u_{yi} & v_{yi} & w_{yi} \\ u_{zi} & v_{zi} & w_{zi} \end{bmatrix} = \begin{bmatrix} c\theta_i c\phi_i + s\psi_i s\theta_i s\phi_i & -c\theta_i s\phi_i + s\psi_i s\theta_i c\phi_i & s\theta_i c\psi_i \\ c\psi_i s\phi_i & c\psi_i c\phi_i & -s\psi_i \\ -s\theta_i c\phi_i + s\psi_i c\theta_i s\phi_i & s\theta_i s\phi_i + s\psi_i c\theta_i c\phi_i & c\theta_i c\psi_i \end{bmatrix}_{i=1,2,3,4\,etc}$$

where c, s represents cosine, sine and ϕ_i, θ_i, ψ_i are Euler angles which indicate about Z–X–Y fixed axes.

$$o_{R'_{P_i}} = o_{R_{P_i}} - o_{R_{P_1}}, \text{ i.e.,}$$

$$\begin{bmatrix} u'_{xi} & v'_{xi} & w'_{xi} \\ u'_{yi} & v'_{yi} & w'_{yi} \\ u'_{zi} & v'_{zi} & w'_{zi} \end{bmatrix}_{i=2,3,4etc} = \begin{bmatrix} c\theta_i c\phi_i + s\psi_i s\theta_i s\phi_i & -c\theta_i s\phi_i + s\psi_i s\theta_i c\phi_i & s\theta_i c\psi_i \\ c\psi_i s\phi_i & c\psi_i c\phi_i & -s\psi_i \\ -s\theta_i c\phi_i + s\psi_i c\theta_i s\phi_i & s\theta_i s\phi_i + s\psi_i c\theta_i c\phi_i & c\theta_i c\psi_i \end{bmatrix} - \begin{bmatrix} c\theta_1 c\phi_1 + s\psi_1 s\theta_1 s\phi_1 & -c\theta_1 s\phi_1 + s\psi_1 s\theta_1 c\phi_1 & s\theta_1 c\psi_1 \\ c\psi_1 s\phi_1 & c\psi_1 c\phi_1 & -s\psi_1 \\ -s\theta_1 c\phi_1 + s\psi_1 c\theta_1 s\phi_1 & s\theta_1 s\phi_1 + s\psi_1 c\theta_1 c\phi_1 & c\theta_1 c\psi_1 \end{bmatrix}$$

$$\vec{P_i} = \begin{bmatrix} P_{xi} & P_{yi} & P_{zi} \end{bmatrix}^T_{i=1,2,3,4\,etc}$$

$$\vec{P_i}' = \vec{P_i} - \vec{P_1}, \text{ i.e.,}$$

$$\begin{bmatrix} P'_{xi} & P'_{yi} & P'_{zi} \end{bmatrix}^T_{i=2,3,4etc} = \begin{bmatrix} P_{xi} & P_{yi} & P_{zi} \end{bmatrix}^T - \begin{bmatrix} P_{x1} & P_{y1} & P_{z1} \end{bmatrix}^T$$

$$\vec{L_j} = o_{R_{P_{R_{ji}}}} \vec{l_j} \Big|_{j=1,2,3 \text{ and } i=1,2,3,4\,etc}$$

$$\vec{l_j} = \begin{bmatrix} l_{xj} & l_{yj} & l_{zj} \end{bmatrix}^T$$

$$\vec{L_j} = \left[l_j \cos\left(\xi_j - \alpha_{ji}\right) \cos\beta_j\right] i + \left[l_j \cos\left(\xi_j - \alpha_{ji}\right) \sin\beta_j\right] j + l_j \sin\left(\xi_j - \alpha_{ji}\right) k$$

$$\begin{bmatrix} L_{xj} \\ L_{yj} \\ L_{zj} \end{bmatrix} = \begin{bmatrix} u_{ui} & v_{ui} & w_{ui} \\ u_{vi} & v_{vi} & w_{vi} \\ u_{wi} & v_{wi} & w_{wi} \end{bmatrix} \begin{bmatrix} l_{xj} \\ l_{yj} \\ l_{zj} \end{bmatrix}$$

$$\vec{L_j}' = o_{R'_{P_{R_{ji}}}} \vec{l_j} \Big|_{j=1,2,3 \text{ and } i=2,3,4\,etc}$$

$$\vec{L_j}' = \begin{bmatrix} u'_{ui} & v'_{ui} & w'_{ui} \\ u'_{vi} & v'_{vi} & w'_{vi} \\ u'_{wi} & v'_{wi} & w'_{wi} \end{bmatrix} \begin{bmatrix} l_{xj} \\ l_{yj} \\ l_{zj} \end{bmatrix}$$

$$o_{R'_{P_{R_{ji}}}} = o_{R_{P_{R_{ji}}}} - o_{R_{P_{R_{j1}}}}$$

$$\tilde{j} = \begin{bmatrix} \tilde{J}_{xj} & \tilde{J}_{yj} & 1 \end{bmatrix}^T = \begin{bmatrix} \frac{J_{xj}}{J_{zj}} & \frac{J_{yj}}{J_{zj}} & 1 \end{bmatrix}^T$$

$\overrightarrow{q_{ji}} = \overrightarrow{P_i} + o_{R_{P_i}} \overrightarrow{h_j}$, i.e.,

$$\begin{bmatrix} q_{xji} \\ q_{yji} \\ q_{zji} \end{bmatrix} = \begin{bmatrix} p_{xi} \\ p_{yi} \\ p_{zi} \end{bmatrix} + \begin{bmatrix} c\theta_i c\phi_i + s\psi_i s\theta_i s\phi_i & -c\theta_i s\phi_i + s\psi_i s\theta_i c\phi_i & s\theta_i c\psi_i \\ c\psi_i s\phi_i & c\psi_i c\phi_i & -s\psi_i \\ -s\theta_i c\phi_i + s\psi_i c\theta_i s\phi_i & s\theta_i s\phi_i + s\psi_i c\theta_i c\phi_i & c\theta_i c\psi_i \end{bmatrix} \begin{bmatrix} h_{xj} \\ h_{yj} \\ h_{zj} \end{bmatrix}$$

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