Iranian Journal of Science
https://doi.org/10.1007/s40995-024-01693-x

RESEARCH PAPER

Constrained Bayesian Methods for Union-Intersection and Intersection-Union Hypotheses Testing Problems

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Received: 28 January 2024 / Accepted: 28 July 2024

Abstract

The Union-Intersection and Intersection-Union hypotheses testing problems are considered for all possible combinations of united and intersected sub-sets of hypotheses. Constrained Bayesian Method is developed for solving these problems. Optimal decision rules are derived for all stated combinations of hypotheses. Theorems on the optimality of the derived decision rules in the sense of the restrictions on Type-I and Type-II error rates to the desired levels are proved. The proposed theoretical methods are enhanced for practical examples. Extensive simulation results are presented to confirm the theoretical results and to illustrate the properties of the proposed procedures for a finite sample.

Keywords Constrained Bayesian method · Intersection-Union hypotheses · Statistical hypothesis · Type I and Type II error rates · Union-Intersection hypotheses

1 Introduction

The consideration of the Union-Intersection (UI) problem where the basic hypothesis H_0 states the simultaneous occurrence of several disjoint sub-hypotheses, i.e. when $H_0 = \bigcap_{i=1}^{S} H_{0i}$, started in the middle of the last century (Roy 1953). The reverse scenario where the basic hypothesis H_0 states the occurrence of at least one of the sub-hypotheses, i.e. when $H_0 = \bigcup_{i=1}^{S} H_{0i}$, was considered some time later (SenGupta 1991; SenGupta and Pal 2000, 2001) and was termed the Intersection-Union (IU) testing of hypotheses problem. Both statements of the hypotheses testing problem deserve attention as they appear in many practical applications. For example, UI situations arise when one considers multi-parameter testing problems in multivariate distributions, while IU testing problems arise in, e.g., one-parameter situations such as "equivalence" testing problems, acceptance sampling in statistical process control, reliability and multivariate analysis (Pal and SenGupta 2000), directional statistics (Jammalamadaka and SenGupta 2001, Section 6.3.3; SenGupta and Pal 2001), multi-parameter problems like contaminated or mixture models (Berger 1982; Choudhary and Nagaraja 2004; Madallaz and Mau 1981; SenGupta 2007), multiple comparisons in verbal fluency-disorder studies (Soulakova 2017), group sequential clinical trials (Peng et al. 2018), etc.

The intersection of the separate critical regions obtained by the standard separate tests for each H_{0i} for testing H_0 is considered in Choudhary and Nagaraja (2004). The general uniformly most powerful (UMP) test is presented in (Lehmann 1986) for solving this problem. An approach based on a Pivotal Parametric Product (P^3) as given by

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Published online: 24 August 2024

SenGupta (2007) is exemplified by computations for several practical examples and by comparisons of the obtained results with the results given in (Berger 1982).

To pursue the above problem, more general statements of the UI and IU problems and application of Constrained Bayesian Method (CBM) for solving it are presented below. General statement of the problem is given in Sect. 2, whereas its general solution is given in Sect. 3. Some particular examples are considered in Sect. 4. Numerical results for these examples are presented in Sect. 5. These are computed for concrete data and are given in Sect. 6. Final results are discussed in Sect. 7.

2 Statement of the Problem

Let's consider the problem of testing a basic hypothesis against alternative one when one of them is union or intersection of a sub-set of hypotheses and another is negation of the first one (SenGupta 2007; Roy 1953). Since these two cases easily can be transformed to each other by changing the basic (null) and alternative hypothesis and vice versa, we will consider the case when the null (basic) hypothesis is union or intersection of a sub-set of hypotheses and the alternative is negation of the null, i.e.

$$\text{a) } H_0 \equiv \bigcup_{i=1}^{S_0} H_{0i} \text{ and } H_1 \equiv no\, H_0 \quad \text{or} \quad \text{b) } H_0 \equiv \bigcap_{i=1}^{S_0} H_{0i} \text{ and } H_1 \equiv no\, H_0 \tag{1a}$$

In general, this problem can be stated as follows: To test the basic hypothesis H_0, against the alternative one H_1, where H_0 and H_1 are the union or intersection of some subsets of hypotheses $H_{01}, H_{02}, \ldots, H_{0S_0}$ and $H_{11}, H_{12}, \ldots, H_{1S_1}$, respectively. Here $H_0 \bigcap H_1 = \emptyset$ (or more generally $H_{0i} \bigcap H_{1j} = \emptyset$, $i = 1, .., S_0$, $j = 1, .., S_1$) and the fulfillment of the condition $H_0 \bigcup H_1 = R^m$, where R^m is m dimensional parametrical space, is not obligatory in contrast to the classical case. Hypotheses from one sub-set can intersect with each other but hypotheses from different sub-sets do not intersect. These suppositions make the statistical hypotheses similar to the hypotheses usually encountered in real-life and, therefore, make them more natural.

In general, here we consider the following combinations of testing of hypotheses:

$$\begin{aligned} &\text{a) } H_0 \equiv \bigcap_{i=1}^{S_0} H_{0i} \text{ vs. } H_1 \equiv \bigcap_{i=1}^{S_1} H_{1i};\\ &\text{b) } H_0 \equiv \bigcup_{i=1}^{S_0} H_{0i} \text{ vs. } H_1 \equiv \bigcup_{i=1}^{S_1} H_{1i};\\ &\text{c) } H_0 \equiv \bigcap_{i=1}^{S_0} H_{0i} \text{ vs. } H_1 \equiv \bigcup_{i=1}^{S_1} H_{1i};\\ &\text{d) } H_0 \equiv \bigcup_{i=1}^{S_0} H_{0i} \text{ vs. } H_1 \equiv \bigcap_{i=1}^{S_1} H_{1i}. \end{aligned} \tag{1b}$$

It is obvious that hypotheses Eq. (1a) are particular cases of hypotheses Eq. (1b).

The standard separate tests for each couple H_{0i} and H_{1j}, offered in Choudhary and Nagaraja (2004), which yield a test for H_0 and H_1 with the acceptance regions given by the intersection or union of the separate acceptance regions, has the following drawback. The information, that may be contained in the hypotheses H_{0i} and/or H_{1j} concerning other sub-hypotheses, are lost in such separate considerations. Application of CBM for testing these hypotheses is free from such drawback. It does not need the derivations of a new test statistic for every concrete case and its distribution law (as P^3 test needs) (see SenGupta 2007, 1991), which may be non-trivial in many cases. Besides, it is free from the necessity to have "exact separate tests" (SenGupta 2007).

Let us adopt the following notations for the application of CBM to testing of hypotheses Eq. (1a) or Eq. (1b). Denote $H'_i \equiv H_{0i}$, $i = 1, \ldots, S_0$, $H'_i \equiv H_{1i}$, $i = S_0 + 1, \ldots, S_0 + S_1$. Then we have to test $S = S_0 + S_1$ hypotheses $H'_1, H'_2, \ldots, H'_S$ (instead of $S_0 \cdot S_1$ separate tests in pairs). Let's henceforth omit the upper index for simplicity. Let a sample $x^T = (x_1, \ldots, x_n)$ be generated from probability distribution density $p(x; \theta)$ and the problem of interest is to test hypotheses $H_i : \theta_i \in \Theta_i$, $i = 1, \ldots, S$, where $\Theta_i \in \Omega^m$, $i = 1, \ldots, S$, Ω^m is m dimensional parameter space and the requirement of being disjoint subsets of Θ_i is not obligatory. Let the prior on θ be denoted by $\sum_{i=1}^{S} \pi(\theta|H_i)p(H_i)$, where for each $i = 1, \ldots, S$, $p(H_i)$ is the a priori probability of hypothesis H_i and $\pi(\theta|H_i)$ is a prior density with support Θ_i; $p(x|H_i)$ denotes the marginal density of x given H_i, i.e. $p(x|H_i) = \int_{\Theta_i} p(x|\theta)\pi(\theta|H_i)d\theta$; $D = \{d\}$ is the set of solutions, where $d = \{d_1, \ldots, d_S\}$,

$$d_i = \begin{cases} 1, & \text{if hypothesis } H_i \text{ is accepted} \\ 0, & \text{otherwise} \end{cases}$$

Let $\delta(x) = \{\delta_1(x), \delta_2(x), \ldots, \delta_S(x)\}$ is the decision function that associates each observation vector x with a certain decision

$$x \xrightarrow{\delta(x)} d \in D;$$

(notation: depending upon the choice of x, there is a possibility that $\delta_j(x) = 1$ for more than one j or $\delta_j(x) = 0$ for all $j = 1, \ldots, S$).

Let $\Gamma_i = \{x : \delta_i(x) = 1\}$, $i = 1, \ldots, S$, denote the acceptance region of hypothesis H_i. Let $L_1(H_i, \delta_j(x) = 1)$ and $L_2(H_i, \delta_j(x) = 0)$ be the losses of incorrectly accepted and incorrectly rejected hypotheses, respectively.

One of the aims of CBM (Task 2) is the following (Kachiashvili 2018): To determine Γ_i, $i = 1, \ldots, S$, which minimize the average loss of incorrectly accepted hypotheses

$$r_\delta = \min_{\{\Gamma_j\}}\left\{\sum_{i=1}^{S} p(H_i)\sum_{j=1}^{S}\int_{\Gamma_j} L_1(H_i,\delta_j(x)=1)p(x|H_i)dx\right\} \quad (2)$$

subject to the conditional probabilities of incorrectly rejected hypotheses

$$p(H_i)\cdot\sum_{j=1}^{S}\int_{R^n-\Gamma_j} L_2(H_i,\delta_j(x)=0)p(x|H_i)dx \le r_2^i, i = 1,\dots,S \quad (3)$$

where r_2^i, $i=1,\dots,S$, are some real numbers determining the levels of the losses of incorrectly rejected hypotheses.

Remarks 1a Statement of the problem as in Eqs. (2 and 3) is one of the possible forms that can be modified to other forms depending on the specific hypotheses testing technique applied. So depending on the imposed restrictions and minimizing kinds of errors, we can formulate nine different statements of hypotheses testing, similar to Eqs. (2 and 3) (see, for example, Kachiashvili 2011, 2018; Kachiashvili et al. 2012).

Remarks 1b This problem may be viewed as a generalization of the optimization problem of Dantzig and Wald which was considered in Lehmann (1986) for obtaining the most powerful test.

3 General Solution of the Stated Problem

The solution of the problem Eqs. (2 and 3) by Lagrange method gives

$$\Gamma_j = \left\{x : \sum_{i=1}^{S} L_1(H_i,\delta_j(x)=1)p(H_i)p(x|H_i) < \sum_{i=1}^{S}\lambda_i L_2(H_i,\delta_j(x)=0)p(H_i)p(x|H_i)\right\}, j = 1,\dots,S \quad (4)$$

where Lagrange multipliers λ_i, $i=1,\dots,S$, are determined so that in conditions Eq. (3) the equalities take place.

Using the concept of a posteriori probability, decision making regions Eq. (4) can be rewritten in a more compact form

$$\Gamma_j = \left\{x : \sum_{i=1}^{S} L_1(H_i,\delta_j(x)=1)p(H_i|x) < \sum_{i=1}^{S}\lambda_i L_2(H_i,\delta_j(x)=0)p(H_i|x)\right\}, j = 1,\dots,S \quad (5)$$

Let's consider the following losses

$$L_1(H_i,\delta_j(x)=1) = \begin{cases} 0 & at\, i=j, \\ K_1 & at\, i\neq j \end{cases} \text{and}$$
$$L_2(H_i,\delta_j(x)=0) = \begin{cases} K_0 & at\, i=j, \\ 0 & at\, i\neq j \end{cases} \quad (6)$$

where K_1 and K_0 are the values of the losses of incorrectly accepted and incorrectly rejected hypotheses.

Then restriction conditions Eq. (3) take the form

$$p(H_i)\cdot K_0\left(1-\int_{\Gamma_i} p(x|H_i)dx\right) \le r_2^i, i=1,\dots,S;$$

$$\text{i.e.} \int_{\Gamma_i} p(x|H_i)dx \ge 1 - \frac{r_2^i}{p(H_i)\cdot K_0}, i=1,\dots,S \quad (7)$$

and expression Eq. (5) takes the form

$$\Gamma_j = \left\{x : K_1\cdot\sum_{i=1,i\neq j}^{S} p(H_i|x) < K_0\cdot\lambda_j\cdot p(H_j|x)\right\}, j = 1,\dots,S \quad (8)$$

where λ_j, $j=1,\dots,S$, are determined so that in conditions Eq. (7) the equalities take place.

For testing hypotheses (1), decision making rules are defined on the basis of the regions Eq. (8) as follows:

- for hypotheses of (1a)

(a) accept H_0 if x belongs only to the union of the regions Γ_i, $i=1,..,S_0$ $(x\in\bigcup_{i=1}^{S_0}\Gamma_i)$;
accept H_1 if x belongs only to the region Γ_{S_0+1};
do not make a decision in any other case.
(b) accept H_0 if x belongs only to the intersection of the regions Γ_i, $i=1,..,S_0$ $(x\in\bigcap_{i=1}^{S_0}\Gamma_i)$;
accept H_1 if x belongs only to the region Γ_{S_0+1};
do not make a decision in any other case.

- for hypotheses of Eq. (1b)

(a) accept H_0 if x belongs only to the intersection of the regions Γ_i, $i=1,..,S_0$ $(x\in\bigcap_{i=1}^{S_0}\Gamma_i)$;
accept H_1 if x belongs only to the intersection of the regions Γ_i, $i=S_0+1,\dots,S_0+S_1$ $(x\in\bigcap_{i=S_0+1}^{S_0+S_1}\Gamma_i)$;
do not make a decision in any other case.
(b) accept H_0 if x belongs only to the union of the regions Γ_i, $i=1,..,S_0$ $(x\in\bigcup_{i=1}^{S_0}\Gamma_i)$;
accept H_1 if x belongs only to the union of the regions Γ_i, $i=S_0+1,\dots,S_0+S_1$ $(x\in\bigcup_{i=S_0+1}^{S_0+S_1}\Gamma_i)$;
do not make a decision in any other case.
(c) accept H_0 if x belongs only to the intersection of the regions Γ_i, $i=1,..,S_0$ $(x\in\bigcap_{i=1}^{S_0}\Gamma_i)$;
accept H_1 if x belongs only to the union of the regions Γ_i, $i=S_0+1,\dots,S_0+S_1$ $(x\in\bigcup_{i=S_0+1}^{S_0+S_1}\Gamma_i)$;
do not make a decision in any other case.
(d) accept H_0 if x belongs only to the union of the regions Γ_i, $i=1,..,S_0$ $(x\in\bigcup_{i=1}\Gamma_i)$;
accept H_1 if x belongs only to the intersection of the regions Γ_i, $i=S_0+1,\dots,S_0+S_1$ $(x\in\bigcap_{i=S_0+1}^{S_0+S_1}\Gamma_i)$;
do not make a decision in any other case.

Remarks 2a In all the above situations, the statement "do not make a decision in any other case" was made since it is impossible (see Eq. (3)) to make decision at the desired levels on the basis of existing information.

Remarks 2b If making a decision on the basis of existing information, i.e. on the basis of existing observations, is impossible, then there are two ways of actions: to change restriction levels r_2^i, $i = 1, ..., S$, in Eq. (3), until a decision will not be made, or to continue the sampling, i.e. to pass to the sequential experiment, until a decision will not be made (Kachiashvili 2014, 2018).

Theorem 1 *CBM 2 defined in Eqs. (2 and 3), for hypotheses (b) of (1a) and losses Eq. (6), ensures a decision rule with the error rates Type-I (alpha) and Type-II (beta) restricted by the following inequalities.*

$$\begin{aligned} \alpha &\le \sum_{i=1}^{S_0} \frac{r_2^i}{K_0 \cdot p(H_i)}, \\ \beta &\le \sum_{i=S_0+1}^{S} \frac{r_2^i}{K_0 \cdot p(H_i)} \end{aligned} \tag{9}$$

Proof The Type-I and Type-II error rates for hypotheses Eq. (1b) are the following.

$$\alpha = \int_{\Gamma_1} p(x|H_0)dx \text{ and } \beta = \int_{\Gamma_0} p(x|H_1)dx \tag{10}$$

For hypotheses of Eq. (1b) (b), expressions Eq. (10) can be rewritten as follows

$$\begin{aligned} \alpha &= \int_{\bigcup_{j=S_0+1}^{S} \Gamma_j} \bigcup_{i=1}^{S_0} p(x|H_i)dx \\ &= \sum_{i=1}^{S_0} \int_{\bigcup_{j=S_0+1}^{S} \Gamma_j} p(x|H_i)dx \\ &\le \sum_{i=1}^{S_0} \sum_{j=S_0+1}^{S} \int_{\Gamma_j} p(x|H_i)dx \\ &= \sum_{i=1}^{S_0} \sum_{j=S_0+1}^{S} p(x \in \Gamma_j|H_i) \end{aligned} \tag{11}$$

and

$$\begin{aligned} \beta &= \int_{\bigcup_{j=1}^{S_0} \Gamma_j} \bigcup_{i=S_0+1}^{S} p(x|H_i)dx = \sum_{i=S_0+1}^{S} \int_{\bigcup_{j=1}^{S_0} \Gamma_j} p(x|H_i)dx \\ &\le \sum_{i=S_0+1}^{S} \sum_{j=1}^{S_0} \int_{\Gamma_j} p(x|H_i)dx \\ &= \sum_{i=S_0+1}^{S} \sum_{j=1}^{S_0} p(x \in \Gamma_j|H_i) \end{aligned} \tag{12}$$

Since the following condition holds in CBM (Kachiashvili 2018) when decision is made

$$\sum_{j=1}^{S_0} p(x \in \Gamma_j|H_i) + \sum_{j=S_0+1}^{S} p(x \in \Gamma_j|H_i) = 1, i = 1, ..., S \tag{13}$$

conditions Eqs. (11 and 12) can be rewritten as follows

$$\begin{aligned} \alpha &\le \sum_{i=1}^{S_0} \left[1 - \sum_{j=1}^{S_0} p(x \in \Gamma_j|H_i)\right] \\ &\le \sum_{i=1}^{S_0} \left[\frac{r_2^i}{p(H_i) \cdot K_0} - \sum_{j=1, j \ne i}^{S_0} p(x \in \Gamma_j|H_i)\right] \\ &\le \sum_{i=1}^{S_0} \frac{r_2^i}{p(H_i) \cdot K_0} \end{aligned}$$

and

$$\begin{aligned} \beta &\le \sum_{i=S_0+1}^{S} \left[1 - \sum_{j=S_0+1}^{S} p(x \in \Gamma_j|H_i)\right] \\ &\le \sum_{i=S_0+1}^{S} \left[\frac{r_2^i}{p(H_i) \cdot K_0} - \sum_{j=S_0+1, j \ne i}^{S} p(x \in \Gamma_j|H_i)\right] \\ &\le \sum_{i=S_0+1}^{S} \frac{r_2^i}{p(H_i) \cdot K_0} \end{aligned}$$

Making the same transformations for other combinations of hypotheses Eq. (1b), it is easily seen that the theorem holds for all these hypotheses.

3.1 Another Loss Function

Let us, instead of losses Eq. (6), consider the following loss functions for hypotheses of Eq. (1b):

$$L_1(H_i, \delta_j(x) = 1) = \begin{cases} 0, \text{ at } i,j \in (1, ..., S_0) \text{ or } i,j \in (S_0 + 1, ..., S), \\ K_1, \text{ at } i \in (1, ..., S_0) \text{ and } j \in (S_0 + 1, ..., S) \text{ or} \\ \quad \text{at } i \in (S_0 + 1, ..., S) \text{ and } j \in (1, ..., S_0); \end{cases} \tag{14}$$

and

$$L_2(H_i, \delta_j(x) = 0) = \begin{cases} K_0, \text{ at } i,j \in (1, ..., S_0) \text{ or } i,j \in (S_0 + 1, ..., S), \\ 0, \text{ at } i \in (1, ..., S_0) \text{ and } j \in (S_0 + 1, ..., S) \text{ or}, \\ \quad \text{at } i \in (S_0 + 1, ..., S) \text{ and } j \in (1, ..., S_0). \end{cases} \tag{15}$$

Then CBM 2, i.e. statement of the problem (1)-(2), for hypotheses (b) of Eq. (1b), takes the following form

$$r_\delta = \min_{\{\Gamma_j\}} \left\{ K_1 \cdot \left[\sum_{i=1}^{S_0} p(H_i) \sum_{j=S_0+1}^{S} \int_{\Gamma_j} p(x|H_i)dx + \sum_{i=S_0+1}^{S} p(H_i) \sum_{j=1}^{S_0} \int_{\Gamma_j} p(x|H_i)dx \right] \right\} \tag{16}$$

subject to

$$K_0 \cdot p(H_i) \cdot \sum_{j=1}^{S_0} \int_{R^n-\Gamma_j} p(x|H_i)dx \le r_2^i, i = 1, ..., S_0,$$
$$K_0 \cdot p(H_i) \cdot \sum_{j=S_0+1}^{S} \int_{R^n-\Gamma_j} p(x|H_i)dx \le r_2^i, i = S_0 + 1, ..., S \tag{17}$$

Application of the Lagrange method for solving of the constrained optimization problem Eqs. (16 and 17), gives

$$\Gamma_j = \left\{ x : K_1 \cdot \sum\nolimits_{i=S_0+1}^{S} p(H_i)p(x|H_i) < K_0 \cdot \sum_{i=1}^{S_0} \lambda_i \cdot p(H_i)p(x|H_i) \right\},$$
$$j = 1, ..., S_0,$$
$$\Gamma_j = \left\{ x : K_1 \cdot \sum\nolimits_{i=1}^{S_0} p(H_i)p(x|H_i) < K_0 \cdot \sum_{i=S_0+1}^{S} \lambda_i \cdot p(H_i)p(x|H_i) \right\},$$
$$j = S_0 + 1, ..., S \tag{18}$$

where Lagrange multipliers $\lambda_i, i = 1, ..., S$, are determined so that equality holds in Eq. (17). Thus, we have $\Gamma_1 \equiv ... \equiv \Gamma_{S_0} = \Gamma_0$ and $\Gamma_{S_0+1} \equiv ... \equiv \Gamma_S = \Gamma_1$, i.e. we have only two regions of making a decision. One of them is basic hypothesis acceptance region and another is alternative hypothesis acceptance region.

Theorem 2 *CBM 2 defined in Eqs. (2 and 3), for hypotheses (b) of Eq. (1b) and losses Eqs. (14 and 15) ensures a decision rule with the error rates Type-I (alpha) and Type-II (beta) restricted by the following inequalities.*

$$\alpha \le \sum_{i=1}^{S_0} \frac{r_2^i}{K_0 \cdot S_0 \cdot p(H_i)}, \beta \le \sum_{i=S_0+1}^{S} \frac{r_2^i}{K_0 \cdot (S - S_0) \cdot p(H_i)} \tag{19}$$

Proof Restrictions Eq. (17) for hypotheses acceptance regions Eq. (18) are transformed to the forms.

$$\begin{aligned} &K_0 \cdot p(H_i) \cdot S_0 \cdot (1 - p(x \in \Gamma_0|H_i)) \\ &\le r_2^i, i = 1, ..., S_0, \\ &K_0 \cdot p(H_i) \cdot (S - S_0) \cdot (1 - p(x \in \Gamma_1|H_i)) \\ &\le r_2^i, i = S_0 + 1, ..., S \end{aligned} \tag{20}$$

The use of these ratios for the Type-I and Type-II error rates Eqs. (11 and 12) respectively, gives

$$\begin{aligned} \alpha &\le \sum\nolimits_{i=1}^{S_0} p(x \in \Gamma_1|H_i) \\ &= \sum\nolimits_{i=1}^{S_0} (1 - p(x \in \Gamma_0|H_i)) \le \sum\nolimits_{i=1}^{S_0} \frac{r_2^i}{K_0 \cdot S_0 \cdot p(H_i)}, \end{aligned}$$

and

$$\begin{aligned} \beta &\le \sum\nolimits_{i=S_0+1}^{S} p(x \in \Gamma_0|H_i) \\ &= \sum\nolimits_{i=S_0+1}^{S} (1 - p(x \in \Gamma_1|H_i)) \\ &\le \sum\nolimits_{i=S_0+1}^{S} \frac{r_2^i}{K_0 \cdot (S - S_0) \cdot p(H_i)} \end{aligned}$$

Similarly, this Theorem can be proved for other combinations of hypotheses Eq. (1b).

4 Examples

Let's consider examples for illustrating the fact that well known cases of statistical hypotheses formulations are particular cases of hypotheses given by formula (1).

Example 1 Case of One-parameter H_0 (SenGupta 2007, p. 4).

Let a random variable X follow the distribution $f(x; \theta)$, where θ is scalar parameter. Let's consider testing

$$H_0 : \theta \le \theta_1 \text{ or } \theta \ge \theta_2 \text{ vs } H_1 : \theta_1 < \theta < \theta_2 \tag{21}$$

Let's denote: $H_{01} : \theta \le \theta_1$, $H_{02} : \theta \ge \theta_2$, $H_{11} : \theta > \theta_1$ and $H_{12} : \theta < \theta_2$. Then hypotheses Eq. (21) can be rewritten in the form

$$H_0 : H_{01} \bigcup H_{02} \text{ vs. } H_1 : H_{11} \bigcap H_{12} \tag{22}$$

i.e. we have case Eq. (1b) (d), where $S_0 = 2$, $S_1 = 2$ and $S = S_0 + S_1 = 4$.

Remark 3 We are forced to choose four hypotheses (H_{01}, H_{02}, H_{11} and H_{12}) instead of three H_{01}, H_{02} and H_1 (to which correspond disjoint parametrical subsets $\Theta_{01} = \{\mu : \mu \le \mu_1\}$, $\Theta_{02} = \{\mu : \mu \ge \mu_2\}$ and $\Theta_1 = \{\mu : \mu_1 < \mu < \mu_2\}$) because of the specificity of the example under consideration. Otherwise the suitable choice of the parameter μ of truncated normal distribution at H_1 is impossible and the quality of decision made depends on a chosen value of μ ($\mu \in (\mu_1, \mu_2)$). This note will be more evident when presenting concrete examples in Item 6.

(a) Let's consider the case of loss functions Eq. (6).
Restrictions Eq. (7) and decision regions Eq. (8) take the following forms in this case

$$\int_{\Gamma_{01}} p(x|H_{01})dx \geq 1 - \frac{r_2^{01}}{K_0 \cdot p(H_{01})},$$
$$\int_{\Gamma_{02}} p(x|H_{02})dx \geq 1 - \frac{r_2^{02}}{K_0 \cdot p(H_{02})},$$
$$\int_{\Gamma_{11}} p(x|H_{11})dx \geq 1 - \frac{r_2^{11}}{K_0 \cdot p(H_{11})},$$
$$\int_{\Gamma_{12}} p(x|H_{12})dx \geq 1 - \frac{r_2^{12}}{K_0 \cdot p(H_{12})} \quad (23)$$

$$\Gamma_{01}=\{x:K_1\cdot(p(H_{02}|x)+p(H_{11}|x)+p(H_{12}|x))<K_0\cdot\lambda_{01}\cdot p(H_{01}|x)\},$$
$$\Gamma_{02}=\{x:K_1\cdot(p(H_{01}|x)+p(H_{11}|x)+p(H_{12}|x))<K_0\cdot\lambda_{02}\cdot p(H_{02}|x)\},$$
$$\Gamma_{11}=\{x:K_1\cdot(p(H_{01}|x)+p(H_{02}|x)+p(H_{12}|x))<K_0\cdot\lambda_{11}\cdot p(H_{11}|x)\},$$
$$\Gamma_{12}=\{x:K_1\cdot(p(H_{01}|x)+p(H_{02}|x)+p(H_{11}|x))<K_0\cdot\lambda_{12}\cdot p(H_{12}|x)\} \quad (24)$$

The errors of the Type-I and the Type-II accordingly are:

$$\alpha = \int_{\Gamma_1} p(x|H_{01})dx + \int_{\Gamma_1} p(x|H_{02})dx = p(x\in\Gamma_1|H_{01}) + p(x\in\Gamma_1|H_{02}),$$
$$\beta = \int_{\Gamma_0} p(x|H_{11})dx + \int_{\Gamma_0} p(x|H_{12})dx = p(x\in\Gamma_0|H_{11}) + p(x\in\Gamma_0|H_{12}) \quad (25)$$

and, by Theorem 1, for their restriction on the desired levels at making decision, restriction levels in Eq. (23) must be chosen on the basis of the following conditions

$$\alpha \leq \frac{r_2^{01}}{K_0\cdot p(H_{01})} + \frac{r_2^{02}}{K_0\cdot p(H_{02})},$$
$$\beta \leq \frac{r_2^{11}}{K_0\cdot p(H_{11})} + \frac{r_2^{12}}{K_0\cdot p(H_{12})} \quad (26)$$

(b) Let's consider the case of the restriction functions Eqs. (14 and 15).

Risk function Eq. (16) and restriction conditions Eq. (17) take following forms in this case

$$r_\delta = \min_{\{\Gamma_j\}}\Big\{K_1\cdot\Big[\int_{\Gamma_{11}}(p(H_{01})p(x|H_{01})+p(H_{02})p(x|H_{02}))dx + \int_{\Gamma_{12}}(p(H_{01})p(x|H_{01})+p(H_{02})p(x|H_{02}))dx + \int_{\Gamma_{01}}(p(H_{11})p(x|H_{11})+p(H_{12})p(x|H_{12}))dx + \int_{\Gamma_{02}}(p(H_{11})p(x|H_{11})+p(H_{12})p(x|H_{12}))dx\Big]\Big\} \quad (27)$$

and

$$K_0\cdot p(H_{01})\cdot\Big[\int_{R^n-\Gamma_{01}}p(x|H_{01})dx + \int_{R^n-\Gamma_{02}}p(x|H_{01})dx\Big] \leq r_2^{01},$$
$$K_0\cdot p(H_{02})\cdot\Big[\int_{R^n-\Gamma_{01}}p(x|H_{02})dx + \int_{R^n-\Gamma_{02}}p(x|H_{02})dx\Big] \leq r_2^{02},$$
$$K_0\cdot p(H_{11})\cdot\Big[\int_{R^n-\Gamma_{11}}p(x|H_{11})dx + \int_{R^n-\Gamma_{12}}p(x|H_{11})dx\Big] \leq r_2^{11},$$
$$K_0\cdot p(H_{12})\cdot\Big[\int_{R^n-\Gamma_{11}}p(x|H_{12})dx + \int_{R^n-\Gamma_{12}}p(x|H_{12})dx\Big] \leq r_2^{12} \quad (28)$$

Solution of Eqs. (27 and 28) gives the hypotheses acceptance regions $\Gamma_{01}\equiv\Gamma_{02}\equiv\Gamma_0$ and $\Gamma_{11}\equiv\Gamma_{12}\equiv\Gamma_1$, where

$$\Gamma_0 = \{x : K_1\cdot(p(H_{11})p(x|H_{11}) + p(H_{12})p(x|H_{12})) < K_0\cdot(\lambda_{01}\cdot p(H_{01})p(x|H_{01}) + \lambda_{02}\cdot p(H_{02})p(x|H_{02}))\},$$
$$\Gamma_1 = \{x : K_1\cdot(p(H_{01})p(x|H_{01}) + p(H_{02})p(x|H_{02})) < K_0\cdot(\lambda_{11}\cdot p(H_{11})p(x|H_{11}) + \lambda_{12}\cdot p(H_{12})p(x|H_{12}))\} \quad (29)$$

Here, Lagrange multipliers λ_{01}, λ_{02}, λ_{11} and λ_{12} are determined so that in the conditions Eq. (28) equalities hold.

Taking into account Eq. (29), restriction conditions Eq. (28) take the forms

$$\int_{\Gamma_0} p(x|H_{01})dx \geq 1 - \frac{r_2^{01}}{2\cdot K_0\cdot p(H_{01})},$$
$$\int_{\Gamma_0} p(x|H_{02})dx \geq 1 - \frac{r_2^{02}}{2\cdot K_0\cdot p(H_{02})},$$
$$\int_{\Gamma_1} p(x|H_{11})dx \geq 1 - \frac{r_2^{11}}{2\cdot K_0\cdot p(H_{11})},$$
$$\int_{\Gamma_1} p(x|H_{12})dx \geq 1 - \frac{r_2^{12}}{2\cdot K_0\cdot p(H_{12})} \quad (30)$$

It should be noted that the determination of Lagrange multipliers is more difficult for Eqs. (14 and 15) than for Eq. (6), because in the first case the two-dimensional equations with respect to Lagrange multipliers must be solved, instead of one-dimensional in the second case.

For guaranteeing restrictions of Type-I and Type-II error rates at the desired levels, according to Theorem 2, restriction levels in conditions Eq. (30) must be chosen so that the following inequalities are fulfilled

$$\alpha \leq \frac{r_2^{01}}{2\cdot K_0\cdot p(H_{01})} + \frac{r_2^{02}}{2\cdot K_0\cdot p(H_{02})},$$
$$\beta \leq \frac{r_2^{11}}{2\cdot K_0\cdot p(H_{11})} + \frac{r_2^{12}}{2\cdot K_0\cdot p(H_{12})} \quad (31)$$

The comparison of Eq. (26 and 31) allows us to conclude that at identical K_0, $p(H_{01})$, $p(H_{02})$, $p(H_{11})$, $p(H_{12})$, r_2^{01}, r_2^{02}, r_2^{11} and r_2^{12}, Type I and Type II error rates for losses Eqs. (14 and 15), in general, are less than the same error rates for losses Eq. (6) at the solution of the stated condition of the problem (see, Remark 4 below). Therefore, the use of losses Eqs. (14 and 15) is not only more logical than the use of losses Eq. (6) but, also, it is preferable for minimization of Type I and Type II error rates.

Example 2 Case of Multi-parameter H_0 (SenGupta 2007, p. 13).

Let's consider the mixture model with density

$$g(x|p,\theta,\vartheta) = p\cdot f(x|\theta,\vartheta) + (1-p)\cdot f(x|\theta_0,\vartheta) \tag{32}$$

where $0 \leq p \leq 1$, $\theta \in \Theta$, an interval of the real line; both p, θ are unknown and θ_0 is a known point of Θ; and ϑ is an unknown parameter (possibly vector-valued), to be interpreted as a nuisance parameter. The density $f(x|\theta,\vartheta)$ is assumed to be sufficiently "regular". We want to test the null hypothesis H_0: "no contamination" against the alternative H_1: "there is contamination". Under the above setup, the null hypothesis of the contamination translates to the union of three parametric hypotheses: $[H_{01} : p = 0 \bigcup H_{02} : \theta = \theta_0 \bigcup H_{03} : p = 0 \; and \; \theta = \theta_0]$.

Taking into account Eq. (32), the null parametric sub-hypotheses are

$$\begin{aligned} H_{01} &: g_{01}(x|\theta,\vartheta) = f(x|\theta_0,\vartheta), \\ H_{02} &: g_{02}(x|p,\theta,\vartheta) = p\cdot f(x|\theta_0,\vartheta) \\ &+(1-p)\cdot f(x|\theta_0,\vartheta) = f(x|\theta_0,\vartheta), \\ H_{03} &: g_{03}(x|\theta,\vartheta) = f(x|\theta_0,\vartheta) \end{aligned} \tag{33}$$

Because of $g_{01} \equiv g_{02} \equiv g_{03}$, hypotheses H_{01}, H_{02} and H_{03} are the same.

It is obvious that to the alternative hypothesis correspond the following parametric hypothesis $H_1 : p \neq 0 \; and \; \theta \neq \theta_0$ with underlying density Eq. (32). Finally, we have the following set of hypotheses for testing

$$H_0 : X \sim f(x|\theta_0,\vartheta) \text{ vs. } H_1 \sim g(x|p,\theta,\vartheta) \tag{34}$$

Thus we have $S_0 = 1$ and $S_1 = 1$.

Let's introduce $p(H_i)$, $i = 0, 1$, a priori probabilities; $\pi(\omega|H_i)$, a priori density with support Ω_i $(\omega \equiv (p,\theta,\vartheta))$; and $p(x|H_i)$ the marginal density of x given H_i, i.e.

$$p(x|H_i) = \int_{\Omega_i} g_i(x|\omega)\pi(\omega|H_i)d\omega, \; i = 0, 1 \tag{35}$$

Taking into account Eq. (33), more specifically, for marginal densities we have

$$\begin{aligned} p(x|H_0) &= \int_Q f(x|\theta_0,\vartheta)\pi(\vartheta|H_0)d\vartheta, \\ p(x|H_1) &= \int_0^1 \int_{\Theta/\theta_0} \int_Q p\cdot f(x|\theta,\vartheta)\pi(p|H_1)\pi(\vartheta|H_1)dpd\theta d\vartheta \\ &+ \int_0^1 \int_Q (1-p)\cdot f(x|\theta_0,\vartheta)\pi(p|H_1)\pi(\vartheta|H_1)dpd\vartheta \end{aligned} \tag{36}$$

Here Q is the domain of support of ϑ. It is obvious that we have a particular situation of the previous case (see Example 1), where for testing we use again CBM 2. Therefore, the results obtained for Example 1, and, in particular, Theorem 2, are in force but conditions Eq. (26) are simplified and have the following forms

$$\begin{aligned} \alpha &\leq \frac{r_2^0}{K_0 \cdot p(H_0)}, \\ \beta &\leq \frac{r_2^1}{K_0 \cdot p(H_1)} \end{aligned} \tag{37}$$

5 Calculations for Concrete Examples

With the purpose to reinforce the theoretical results given above, and for investigation of their behavior depending on different parameters, let us consider the following examples.

Examples

Example 1 (a) For testing hypotheses Eq. (22), let us use a sample $X_1, X_2, ..., X_n$, obtained from $N(x|\mu,\sigma^2)$ with known σ^2. Because the sub-hypotheses introduced above are complex, with appropriate densities, let us use Stein's method for finding the uniformly most powerful invariant test (Wijsman 1967; Andersson 1982; Kachiashvili 2016). Let's introduce $\pi(\mu|H_i)$, a prior density with support Θ_i, $i \in (01, 02, 11, 12)$. Here $\Theta_{01} = \{\mu : \mu \leq \mu_1\}$, $\Theta_{02} = \{\mu : \mu \geq \mu_2\}$, $\Theta_{11} = \{\mu : \mu > \mu_1\}$ and $\Theta_{12} = \{\mu : \mu < \mu_2\}$. As densities $\pi(\mu|H_i)$, $i \in (01, 02, 11, 12)$, let us use truncated normal densities $f(\mu; \mu_i, \sigma_1, a, b)$ over $(-\infty, \mu_1)$, $(\mu_2, +\infty)$, $(\mu_1, +\infty)$ and $(-\infty, \mu_2)$, respectively. Here μ_{01}, μ_{02}, μ_{11} and μ_{12} belong to the appropriate regions. Because $\bar{x}$ is a sufficient statistic with normality of distribution, we use $\bar{x}$ as a test statistic. Then for the marginal densities of $\bar{x}$ given H_i, we have

$$p(\bar{x}|H_{01}) = \int_{-\infty}^{\mu_1} N(\bar{x}|\mu, \sigma_{\bar{x}}^2) f(\mu; \mu_{01}, \sigma_1, -\infty, \mu_1) d\mu,$$
$$p(\bar{x}|H_{02}) = \int_{\mu_2}^{+\infty} N(\bar{x}|\mu, \sigma_{\bar{x}}^2) f(\mu; \mu_{02}, \sigma_1, \mu_2, +\infty) d\mu,$$
$$p(\bar{x}|H_{11}) = \int_{\mu_1}^{+\infty} N(\bar{x}|\mu, \sigma_{\bar{x}}^2) f(\mu; \mu_{11}, \sigma_1, \mu_1, +\infty) d\mu,$$
$$p(\bar{x}|H_{12}) = \int_{-\infty}^{\mu_2} N(\bar{x}|\mu, \sigma_{\bar{x}}^2) f(\mu; \mu_{12}, \sigma_1, -\infty, \mu_2) d\mu \tag{38}$$

Probability density function of the truncated normal distribution is (Johnson et al. 2004, p. 156):

$$f(x; \mu, \sigma, a, b) = \sigma^{-1} \phi\left(\frac{x-\mu}{\sigma}\right) \left[\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)\right]^{-1} \tag{39}$$

where $\phi(\xi) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\xi^2\right)$ is the probability density function of the standard normal distribution and $\Phi(\cdot)$ is its cumulative distribution function.

Thus acceptance regions are given by Eq. (24), where a posterior probabilities $p(H_{01}|\bar{x})$, $p(H_{02}|\bar{x})$, $p(H_{11}|\bar{x})$ and $p(H_{12}|\bar{x})$ are computed using appropriate a priori probabilities of hypotheses and marginal densities Eq. (38), and truncated normal densities Eq. (39) for unknown parameters of initial distribution.

Remark 4 Because of the difficulties to find common (sufficient) statistics for all densities Eq. (38), we are forced to use these densities for determination of Lagrange multipliers and decision making regions. But at modeling, for making decision, we simulate test statistic $\bar{x}$ which is normally distributed (as is in the assumed situation). As a result, the condition of Theorem 1 is violated and it is expected that its result will not hold.

(b) Let us consider the same example when σ^2 is unknown. In this case we can consider two methods based on the maximum likelihood ratio and Stein's method (Anderson 1982; Wijsman 1967; Kachiashvili 2016). In the first case instead of σ^2 we use its estimator $s^2 = \frac{1}{n-1}\sum_i^n (x_i - \bar{x})^2$ in formulae Eq. (38). In the second case, for averaging influence of σ^2, we use a prior density $\pi(s^2) \equiv \chi_{n-1}^2(s^2)$, that is the chi-square densities with $n-1$ degree of freedom over $(0, \infty)$. Then the marginal densities of $\bar{x}$ given H_i are:

$$p(\bar{x}|H_{01}) = \int_{-\infty}^{\mu_1}\int_0^{\infty} N(\bar{x}; \mu, s^2/n) f(\mu; \mu_{01}, \sigma_1, -\infty, \mu_1) \chi_{n-1}^2(s^2) d\mu ds^2,$$
$$p(\bar{x}|H_{02}) = \int_{\mu_2}^{+\infty}\int_0^{\infty} N(\bar{x}; \mu, s^2/n) f(\mu; \mu_{02}, \sigma_1, \mu_2, +\infty) \chi_{n-1}^2(s^2) d\mu ds^2,$$
$$p(\bar{x}|H_{11}) = \int_{\mu_1}^{+\infty}\int_0^{\infty} N(\bar{x}; \mu, s^2/n) f(\mu; \mu_{11}, \sigma_1, \mu_1, +\infty) \chi_{n-1}^2(s^2) d\mu ds^2,$$
$$p(\bar{x}|H_{12}) = \int_{-\infty}^{\mu_2}\int_0^{\infty} N(\bar{x}; \mu, s^2/n) f(\mu; \mu_{12}, \sigma_1, -\infty, \mu_2) \chi_{n-1}^2(s^2) d\mu ds^2 \tag{40}$$

Example 2 In this example we can consider two methods too based on the maximum likelihood ratio and Stein's method.

(a) The maximum likelihood ratio:

at $H_0 : f(\bar{x}|\theta_0, \vartheta) \equiv N(\bar{x}; \theta_0, s_0^2/n)$, where $s_0^2 \approx \frac{1}{n}\sum_i^n (x_i - \theta_0)^2$ and

at H_1 :

$$g(\bar{x}|p, \theta, \vartheta) = \frac{1}{2}\int_{-\infty}^{\theta_0} N(\bar{x}|\mu, s_0^2/n) f(\mu; \mu_{01}, \sigma_1, -\infty, \theta_0) d\mu +$$

$+\frac{1}{2}\int_{\theta_0}^{+\infty} N(\bar{x}|\mu, s_0^2/n) f(\mu; \mu_{02}, \sigma_1, \theta_0, +\infty) d\mu$
$+\frac{1}{2} N(\bar{x}|\theta_0, s_0^2/n)$.

In this case, hypotheses acceptance regions are

$$\Gamma_0 = \{\bar{x} : K_1 \cdot p(H_1|\bar{x}) < K_0 \cdot \lambda_0 \cdot p(H_0|\bar{x})\},$$
$$\Gamma_1 = \{\bar{x} : K_1 \cdot p(H_0|\bar{x}) < K_0 \cdot \lambda_1 \cdot p(H_1|\bar{x})\} \tag{41}$$

where $p(\bar{x}|H_0) = \frac{\sqrt{n}}{\sqrt{2\pi} \cdot s_0} \exp\left\{-\frac{n(\bar{x}-\theta_0)^2}{2s_0^2}\right\}$,

$$\begin{aligned} p(\bar{x}|H_1) = & \frac{A_{21}}{\sqrt{2}}\sqrt{\frac{\pi}{a_2}} \cdot \exp\left\{\left(b_{21}^2(\bar{x}) + 4a_2 c_{21}(\bar{x})\right)/(4a_2)\right\} \\ & \cdot (2\Phi(d_{21}) - 1) \cdot I_{21} \\ & + \frac{A_{22}}{\sqrt{2}}\sqrt{\frac{\pi}{a_2}} \cdot \exp\left\{\left(b_{22}^2(\bar{x}) + 4a_2 c_{22}(\bar{x})\right)/(4a_2)\right\} \\ & \cdot I_{22} + \frac{\sqrt{n}}{2\sqrt{2\pi} s_0} \exp\left\{-\frac{n(\bar{x}-\theta_0)^2}{2s_0^2}\right\} \end{aligned} \tag{42}$$

where

$I_{21} = \begin{cases} 0, & if\ d_{21} \le 0, \\ 1, & if\ d_{21} > 0, \end{cases}$ and

$$I_{22} = \begin{cases} \sqrt{2}, & if\ d_{22} \le 0, \\ 2\sqrt{2}(1 - \Phi(d_{22})), & if\ d_{22} > 0. \end{cases}$$

Here

$$A_{21} = \frac{\sqrt{n}}{4\pi s_0^2 \Phi\left(\frac{\theta_0 - \mu_{01}}{s_0}\right)}, a_2 = \frac{n s_{01}^2 + s_0^2}{2 s_0^2 s_{01}^2}, b_{21}$$
$$= -\frac{2n\bar{x} s_{01}^2 + 2\mu_{01} s_0^2}{2 s_0^2 s_{01}^2}, c_{21} = -\frac{n\bar{x}^2 s_{01}^2 + \mu_{01}^2 s_0^2}{2 s_0^2 s_{01}^2},$$
$$A_{22} = \frac{\sqrt{n}}{4\pi s_0^2 \left[1 - \Phi\left(\frac{\theta_0 - \mu_{02}}{s_0}\right)\right]}, b_{22} = -\frac{2n\bar{x} s_{01}^2 + 2\mu_{02} s_0^2}{2 s_0^2 s_{01}^2}, c_{22}$$
$$= -\frac{n\bar{x} s_{01}^2 + \mu_{02}^2 s_0^2}{2 s_0^2 s_{01}^2},$$
$$d_{21} = \theta_0 \sqrt{a_2} + \frac{b_{21}}{2\sqrt{a_2}}, d_{22} = \theta_0 \sqrt{a_2} + \frac{b_{22}}{2\sqrt{a_2}}$$

(b) Stein's method

at $H_0 : f(\bar{x}|\theta_0, \vartheta) = \int_0^\infty N(\bar{x}; \theta_0, s^2/n) \chi_{n-1}^2(s^2) ds^2$, where $s^2 = \frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2$, $\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i$.

at

$$H_1 : g(\bar{x}|p, \theta, \vartheta)$$
$$= \frac{1}{2} \int_{-\infty}^{\theta_0} \int_0^\infty N(\bar{x}|\mu, s^2/n) f(\mu; \mu_{01}, \sigma_1, -\infty, \theta_0) \chi_{n-1}^2(s^2) d\mu ds^2$$
$$+ \frac{1}{2} \int_{\theta_0}^\infty \int_0^\infty N(\bar{x}|\mu, s^2/n) f(\mu; \mu_{02}, \sigma_1, \theta_0, +\infty) \chi_{n-1}^2(s^2) d\theta ds^2$$
$$+ \frac{1}{2} \int_0^1 \int_0^\infty (1 - p) N(\bar{x}|\theta_0, s^2/n) \chi_{n-1}^2(s^2) dp ds^2 \quad (43)$$

6 Computational Results

Let's compute example 1 with the following initial data: the values of the loss functions $K_0 = K_1 = 1$, variance $\sigma^2 = 0.0025$, $\mu_1 = -0.05$, $\mu_2 = 0.02$, $\mu_{01} = -0.05$, $\mu_{02} = 0.02$, $\mu_{11} = -0.0499$, $\mu_{12} = 0.0199$, $\sigma_1^2 = \omega_0^{-1} \cdot \sigma^2$, $p(H_{01}) = p(H_{02}) = p(H_{11}) = p(H_{12}) = 1/4$, $r_2^{01} = r_2^{02} = 0.00625$, $r_{11}^1 = r_{12}^1 = 0.0025$.

Let's consider the case of loss functions Eq. (6). For the determination of Lagrange multipliers, probability integrals in the suitable restriction conditions Eq. (23) were computed by Monte-Carlo method using appropriate samples (distributed with densities of Eq. (38)) with size 5000. For computation of Type-I and Type-II error rates by formulae Eq. (25) for acceptance regions Eq. (24), normally distributed samples of 10,000 were used. This sample size is used below in all Monte-Carlo computations for similar probabilities.

For the considered data, in accordance with Eq. (23), restriction levels in the restriction conditions are 0.975 and 0.99 under H_0 and H_1, respectively. Therefore, in accordance with Eq. (19), Type-I and Type-II error rates for testing of hypotheses must be restricted at the levels 0.05 and 0.02, respectively.

Lagrange multipliers computed for these data are: λ_{01} = 16.40625, λ_{02} = 12.451171875, λ_{11} = 16.40625 and λ_{12} = 15.625. The values of acceptance probabilities depending on μ (mathematical expectation of generated normally distributed random variables) are given in Table 1. Hypotheses acceptance probabilities depending on the variance of truncated normal distribution ($\sigma_1^2 = \omega_0^{-1} \cdot \sigma^2$), when expectation of generated normally distributed random variables $\mu = -0.049$, i.e. hypothesis H_1 is true but sample distribution is close to hypothesis H_0, are given in Table 2. And hypotheses acceptance probabilities depending on expectation of truncated normal distribution (μ_{01}), when expectation of generated normally distributed random variables $\mu = -0.049$, i.e. hypothesis H_1 is true but sample distribution is close to hypothesis H_0, are given in Table 3 (here $\omega_0 = 1$). Appropriate graphical illustrations of computed results are presented in Figs. 1, 2 and 3, respectively.

On the basis of these results the following conclusions follow:

- because of Remark 4 the computed values of Type-I and Type-II error probabilities exceed the values 0.05 and 0.02, respectively, determined in accordance with Theorem 1; discrepancies are greater, the closer is the expectation of sample's distribution to the borders of hypotheses domains of definitions;
- probabilities of correct decisions at hypothesis H_0 are higher than at hypothesis H_1 for small distances between mathematical expectation of sample's distribution and the borders of hypotheses;
- when distances between mathematical expectation of sample's distribution and the borders of hypotheses domains of definitions are increasing, Type-I and Type-II error probabilities are decreasing and condition of Theorem 1 is satisfied;
- the number of observations necessary for making a decision at hypothesis H_1 is greater than at hypothesis H_0 in average; at hypothesis H_0 it quickly decreases at increasing distances between mathematical expectation of sample's distribution and the borders of hypotheses domains of definitions;
- at decreasing variance of truncated normal distribution, true hypothesis acceptance probabilities increase at hypothesis H_1;
- at increasing variance of truncated normal distribution, sample size, necessary for making decision, increases at hypothesis H_1;
- at increase of distances between parameters μ_{01} and μ_{02} of truncated normal distributions at H_0 and the borders of tested hypotheses domains of definitions,

Table 1 Hypotheses acceptance probabilities depending on mathematical expectation of generated random variables at losses Eq. (6)

$N(x\|\mu,\sigma^2),\mu$	$p011$	$p021$	$p01$	$p111$	$p121$	$p11$	AN
− 0.049	0.4587	0	0.4587	0.5411	0.0002	0.5413	47.9315
− 0.045	0.1494	0	0.1494	0.8495	0.0011	0.8506	36.9096
− 0.03	0.0113	0.0002	0.0115	0.9302	0.0583	0.9885	24.6289
− 0.015 (MP)	0.001	0.0021	0.0031	0.3999	0.597	0.9969	23.2197
0	0.0004	0.0107	0.0111	0.0287	0.9602	0.9889	22.8407
0.015	0	0.1521	0.1521	0.0008	0.8471	0.8479	30.8568
0.019	0	0.4188	0.4188	0	0.5812	0.5812	38.6426
	$p011$	$p021$	$p01$	$p111$	$p121$	$p11$	AN
− 0.05	0.6312	0	0.6312	0.3687	0.0001	0.3688	47.616
− 0.051	0.7697	0	0.7697	0.2303	0	0.2303	45.5127
− 0.065	0.9998	0	0.9998	0.0002	0	0.0002	10.8521
− 0.08	1	0	1	0	0	0	5.2847
− 0.095	1	0	1	0	0	0	3.2542
	$p011$	$p021$	$p01$	$p111$	$p121$	$p11$	AN
0.02	0	0.5576	0.5576	0.0001	0.4423	0.4424	38.9823
0.021	0	0.7008	0.7008	0.0001	0.2991	0.2992	38.7634
0.035	0	0.9997	0.9997	0	0.0003	0.0003	10.4433
0.05	0	1	1	0	0	0	5.1322
0.065	0	1	1	0	0	0	3.2211

AN—average number of observations for making a decision; MP—middle point

$p011 \equiv P(\bar{x} \in \Gamma_{01}|H_1)$, $p021 \equiv P(\bar{x} \in \Gamma_{02}|H_1)$, $p01 \equiv P(\bar{x} \in \Gamma_0|H_1)$- Type II error rate, $p111 \equiv P(\bar{x} \in \Gamma_{11}|H_1)$, $p121 \equiv P(\bar{x} \in \Gamma_{12}|H_1)$, $p11 \equiv P(\bar{x} \in \Gamma_1|H_1)$- power

Table 2 Hypotheses acceptance probabilities depending on variance of truncated normal distribution at losses Eq. (6), when mathematical expectation of generated random variables $\mu = -0.049$

ω_0	$p011$	$p021$	$p01$	$p111$	$p121$	$p11$	AN
1	0.4587	0	0.4587	0.5411	0.0002	0.5413	47.9315
2	0.2555	0	0.2555	0.7441	0.0004	0.7445	145.3
3	0.115	0	0.115	0.8846	0.0004	0.885	376.1434
4	0.0388	0	0.0388	0.9612	0	0.9612	819.2266

Table 3 Hypotheses acceptance probabilities depending on mathematical expectation of truncated normal distribution (μ_{01}) at H_{01} and losses Eq. (6), when mathematical expectation of generated random variables $\mu = -0.049$

μ_{01}	$p011$	$p021$	$p01$	$p111$	$p121$	$p11$	AN
− 0.05	0.4587	0	0.4587	0.5411	0.0002	0.5413	47.9315
− 0.051	0.7697	0	0.7697	0.2303	0	0.2303	45.5127
− 0.065	0.3827	0	0.3827	0.6172	0.0001	0.6173	22.3593
− 0.08	0.2736	0	0.2736	0.7264	0	0.7264	13.4365
− 0.095	0.2006	0.0001	0.2007	0.7993	0	0.7993	9.9875
− 0.11	0.1447	0	0.1447	0.8553	0	0.8553	8.274
− 0.125	0.1042	0	0.1042	0.8958	0	0.8958	7.2969
− 0.15	0.0594	0	0.0594	0.9406	0	0.9406	6.4671
− 0.2	0.0173	0	0.0173	0.9827	0	0.9827	6.0286

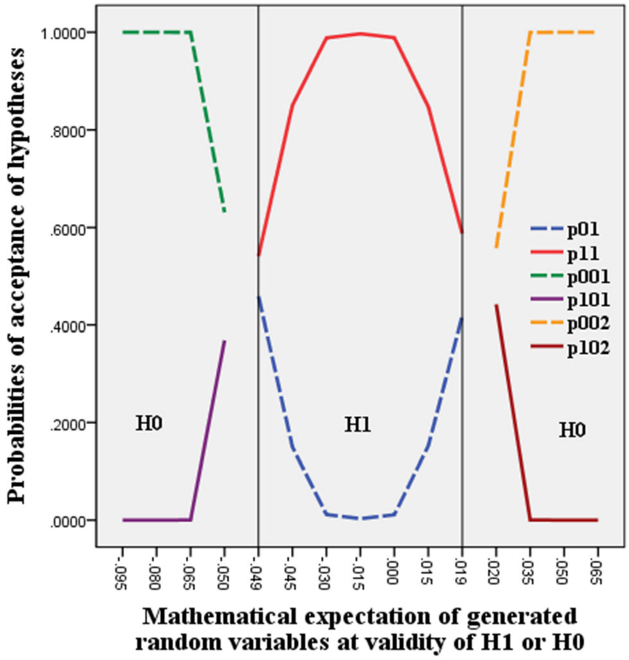

Fig. 1 Dependence of hypotheses acceptance probabilities on expectation of generated random variables at losses Eq. (6) when $\mu'_{11} = -0.05$ and $\mu'_{12} = 0.02$

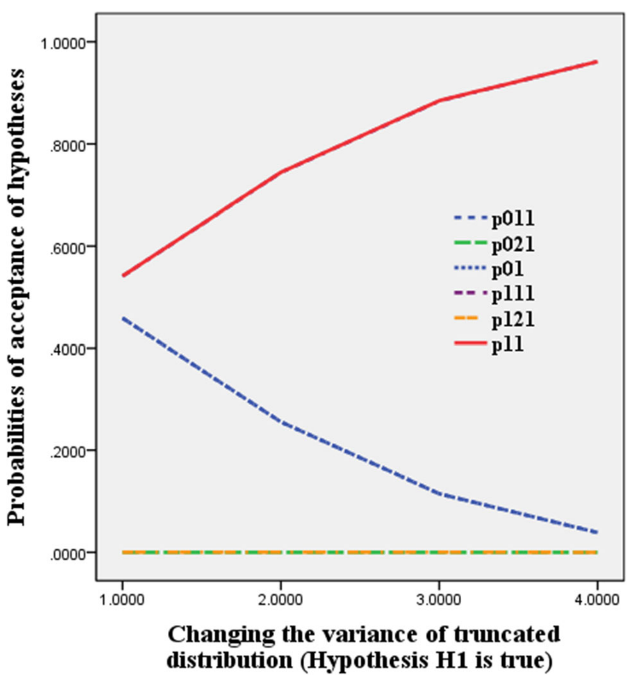

Fig. 2 Dependence of hypotheses acceptance probabilities on variance of truncated normal distributions at losses Eq. (6) ($\sigma_1^2 = \omega_0^{-1} \cdot \sigma^2$, $\omega_0 = 1, 2, 3, 4$), when expectation of generated random variables $\mu = -0.049$

probabilities of acceptance of true H_1 hypotheses improves, and sample size, necessary for making decision, decreases.

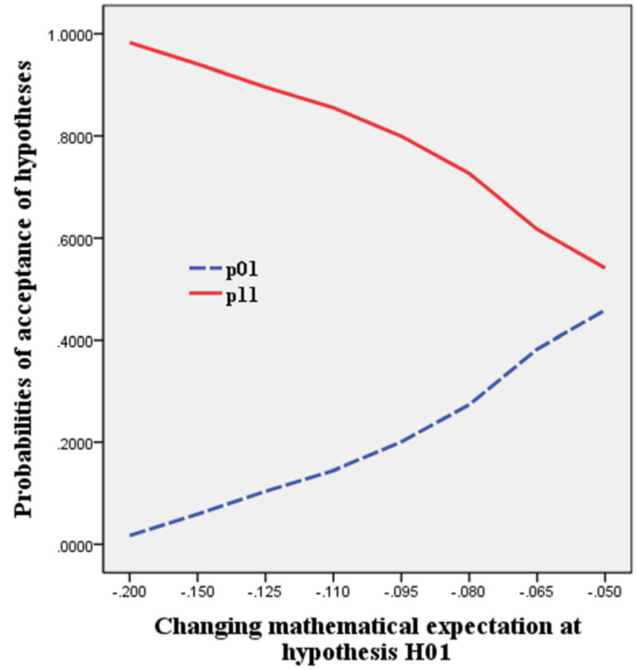

Fig. 3 Dependence of hypotheses acceptance probabilities on expectation (μ_{01}) of truncated normal distribution at H_{01} and losses Eq. (6), when expectation of generated random variables $\mu = -0.049$

Table 4 Dependence of hypotheses acceptance probabilities on expectation of generated random variables at losses Eqs. (26 and 27)

$N(x\|\mu, \sigma^2)$, μ	$p01$	$p11$	AN
− 0.049	0.0612/0.3069	0.9388/0.6931	3.7334/2.9333
− 0.045	0.0409/0.244	0.9591/0.756	3.4617/2.8923
− 0.03	0.008/0.0989	0.992/0.9011	2.8928/2.6634
− 0.015 (MP)	0.0031/0.0598	0.9969/0.9402	2.725/2.5945
0	0.0086/0.0974	0.9914/0.9026	2.8954/2.6708
0.015	0.0411/0.2332	0.9589/0.7668	3.4736/2.8694
0.019	0.0673/0.3087	0.9327/0.6913	3.7763/2.9215
	$p01$	$p11$	AN
− 0.05	0.076/0.323	0.924/0.677	3.8517/2.9552
− 0.051	0.0894/0.3527	0.9106/0.6473	3.9212/2.9655
− 0.065	0.4264/0.6217	0.5736/0.3783	4.8223/2.8564
− 0.08	0.7611/0.8371	0.2389/0.1629	4.2269/2.5113
− 0.095	0.9087/0.9405	0.0913/0.0595	3.5008/2.1674
− 0.11	0.9703/0.9760	0.0297/0.0240	2.8979/1.8609
	$p01$	$p11$	AN
0.02	0.0812/0.3255	0.9188/0.6745	3.8245/2.945
0.021	0.0894/0.3474	0.9106/0.6526	3.9141/2.9412
0.035	0.4360/0.6234	0.5640/0.3766	4.8511/2.8493
0.05	0.7658/0.8396	0.2342/0.1604	4.2833/2.5103
0.065	0.9099/0.9385	0.0901/0.0615	3.5095/2.1603
0.08	0.9707/0.9730	0.0293/0.0270	2.8937/1.8599

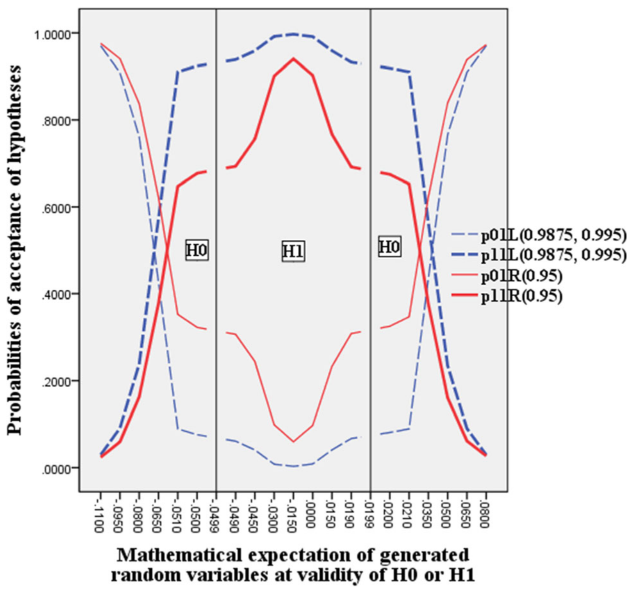

Fig. 4 Dependence of hypotheses acceptance probabilities on expectation of generated random variables when $\mu'_{11} = -0.05$ and $\mu'_{12} = 0.02$ at losses Eqs. (26 and 27)

Now let's compute for the same initial data by decision rule Eq. (29) when Loss functions are determined by Eq. (14 and 15). Initial data are the same as for the previous case. Therefore, restriction levels in conditions Eq. (30) are 0.9875 and 0.995 under H_0 and H_1, respectively. Lagrange multipliers determined by solution of Eq. (30) are: $\lambda_{01} = \lambda_{02} = 2.2198486328125$ and $\lambda_{11} = \lambda_{12} = 8.59375$. Computed results of hypotheses acceptance probabilities, depending on mathematical expectation of generated random variables, are given in Table 4 (see left side of the oblique line). Computed results for restriction levels $r_2^{01} = r_2^{02} = r_{11}^1 = r_{12}^1 = 0.025$, i.e. when restriction probabilities in Eq. (30) are equal to 0.95, are given in Table 4 too (see right side of the oblique line). In this case Lagrange multipliers are equal to: $\lambda_{01} = \lambda_{02} = 2.2137451171875$ and $\lambda_{11} = \lambda_{12} = 2.65380859375$. Dependences of hypotheses acceptance probabilities on mathematical expectation of generated random variables for the considered restriction levels are graphically presented in Fig. 4.

From the obtained results, it is seen that:

- sample size for making decision is quite small;
- despite such small sample size, probabilities of acceptance of true hypotheses are quite great and by choosing restriction levels they can be adjusted in favor of one of the hypotheses H_0 or H_1.

For the elimination of the situation when probability distribution law of a sample differs from the probability distribution laws used for determination of Lagrange multipliers and for making decision, let us consider the following. It is clear that for hypotheses $H_{01}: \theta \le \theta_1$, $H_{02}: \theta \ge \theta_2$, $H_{11}: \theta > \theta_1$ and $H_{12}: \theta < \theta_2$, more logical is, instead of losses Eq. (6), the use of the following losses

$$L_1(H_i, \delta_j(x) = 1) = \begin{cases} 0, & at\ i = j \& H_i \bigcap H_j \ne \emptyset, \\ K_1, & at\ i \ne j \end{cases} \text{ and}$$

$$L_2(H_i, \delta_j(x) = 0) = \begin{cases} K_0, & at\ i = j, \\ 0, & at\ i \ne j \end{cases} \tag{44}$$

These losses are not fully correct because $L_1(H_i, \delta_j(x) = 1)$ must be equal to zero not only when $i = j \& H_i \bigcap H_j \ne \emptyset$, but also when $i = 01$ and $j = 02$, or for inverse values of indices. We ignore this situation because the sub-hypotheses, forming the null hypotheses, are distinct from each other. Generally speaking, loss $L_2(H_i, \delta_j(x) = 0)$ requires to add some additional conditions too.

In this case, hypotheses acceptance regions Eq. (8) transform to the forms

$$\begin{aligned} \Gamma_{01} &= \{x : K_1 \cdot p(H_{11}|x) < K_0 \cdot \lambda_{01} \cdot p(H_{01}|x)\}, \\ \Gamma_{02} &= \{x : K_1 \cdot p(H_{12}|x) < K_0 \cdot \lambda_{02} \cdot p(H_{02}|x)\}, \\ \Gamma_{11} &= \{x : K_1 \cdot p(H_{01}|x) < K_0 \cdot \lambda_{11} \cdot p(H_{11}|x)\}, \\ \Gamma_{12} &= \{x : K_1 \cdot p(H_{02}|x) < K_0 \cdot \lambda_{12} \cdot p(H_{12}|x)\} \end{aligned} \tag{45}$$

where λ_{01}, λ_{02}, λ_{11} and λ_{12} are determined so that in conditions Eq. (7) the equalities take place.

Taking into account the fact that risk of incorrect decision in CBM, as well as in other hypotheses testing methods, decreases when information distances between tested hypotheses increase, instead of composite ones, we consider the simple hypotheses: $H_{01}: X \sim N(x|\mu_{01}, \sigma^2)$, $H_{02}: X \sim N(x|\mu_{02}, \sigma^2)$, $H_{11}: X \sim N(x|\mu_{11}, \sigma^2)$ and $H_{12}: X \sim N(x|\mu_{12}, \sigma^2)$ two by two, i.e. H_{01} vs. H_{11} and H_{02} vs. H_{12}, where μ_{01} =-0.05, $\mu_{02} =$ 0.02, μ_{11} =-0.0499 and $\mu_{12} =$ 0.0199. The hypotheses acceptance regions are: $\Gamma_0 = \Gamma_{01} \bigcup \Gamma_{02}$ and $\Gamma_1 = \Gamma_{11} \bigcap \Gamma_{12}$.

In this case, computations can be made analytically with the following Lagrange multipliers and decision regions:

Table 5 Hypotheses acceptance probabilities depending on mathematical expectation of generated random variables at losses Eq. (44)

$N(x\|\mu,\sigma^2)$, μ	$p011$	$p021$	$p01$	$p11$	AN
− 0.049	0.5606/0.4010/0.3897	0.0002/0/0	0.5608/0.4010/0.3897	0.4392/0.599/0.6103	13.49/52.94/53.53
− 0.045	0.3201/0.1055/0.1003	0.0003/0/0	0.3204/0.1055/0.1003	0.6796/0.8945/0.8997	12.92/35.36/34.94
− 0.03	0.0475/0.0068/0.0064	0.0023/0.0005/0	0.0498/0.0073/0.0064	0.9502/0.9927/0.9936	9.70/17.12/17.11
− 0.0225	0.0207/0.0024/0.0026	0.0048/0.0003/0.003	0.0255/0.0027/0.0029	0.9745/0.9973/0.9971	9.14/15.65/15.61
− 0.015 (MP)	0.0079/0.0006/0.0007	0.0096/0.001/0.0005	0.0175/0.0016/0.0012	0.9825/0.9984/0.9988	8.96/15.22/15.23
− 0.0075	0.0057/0.0006/0.0005	0.0222/0.0034/0.0023	0.0279/0.0040/0.0028	0.9721/0.9960/0.9972	9.14/15.64/15.63
0	0.0017/0.0003/0.0002	0.0502/0.006/0.0065	0.0519/0.0063/0.0067	0.9481/0.9937/0.9933	9.66/17.16/17.11
0.015	0.0003/0.0001/0	0.3254/0.1028/0.1055	0.3257/0.1029/0.1055	0.6743/0.8971/0.8945	12.84/35.11/35.15
0.019	0.0002/0.0001/0	0.5515/0.3842/0.4014	0.5517/0.3843/0.4014	0.4483/0.6157/0.5986	13.54/53.31/53.34
	$p011$	$p021$	$p01$	$p11$	AN
− 0.05	0.6265/0.5684/0.5645	0.0003/0/0.0001	0.6268/0.5684/0.5646	0.3732/0.4316/0.4354	13.45/56.67/55.18
− 0.051	0.7014/0.7277/0.7277	0/0/0	0.7014/0.7277/0.7277	0.2986/0.2723/0.2723	13.04/51.77/51.08
− 0.065	0.9892/0.9993/0.9993	0.0002/0/0	0.9894/0.9993/0.9993	0.0106/0.0007/0.0007	6.67/12.96/13.07
− 0.08	0.9997/1/1	0/0/0	0.9997/1/1	0.0003/0/0	4.27/6.77/6.82
− 0.095	1/1/1	0/0/0	1/1/1	0/0/0	3.20/4.52/4.55
	$p011$	$p021$	$p01$	$p11$	AN
0.02	0.0001/0/0	0.6267/0.5646/0.5657	0.6268/0.5646/0.5657	0.3732/0.4354/0.4343	13.40/55.55/54.86
0.021	0.0003/0/0	0.6986/0.7235/0.7300	0.6989/0.7235/0.7300	0.3011/0.2765/0.2700	12.89/52.51/52.11
0.035	0/0/0	0.9901/0.9995/0.9996	0.9901/0.9995/0.9996	0.0099/0.0005/0.0004	6.67/13.13/13.06
0.05	0/0/0	0.9997/1/1	0.9997/1/1	0.0003/0/0	4.20/6.71/6.76
0.065	0/0/0	1/1/1	1/1/1	0/0/0	3.22/4.53/4.57

$$\lambda_{01}=\left(\frac{K_1\cdot p(H_{11})}{K_0\cdot p(H_{01})}\right)\exp\left\{\left[2\cdot\sigma\cdot(\mu_{11}-\mu_1)\cdot\Phi^{-1}\left(1-\frac{r_2^{01}}{K_0\cdot p(H_{01})}\right)+(\mu_1^2-\mu_{11}^2)+2(\mu_{11}-\mu_1)\mu_1\right]/(2\sigma^2)\right\},$$

$$\lambda_{02}=\left(\frac{K_1\cdot p(H_{12})}{K_0\cdot p(H_{02})}\right)\exp\left\{\left[2\cdot\sigma\cdot(\mu_{12}-\mu_2)\cdot\Phi^{-1}\left(\frac{r_2^{02}}{K_0\cdot p(H_{02})}\right)+(\mu_2^2-\mu_{12}^2)+2(\mu_{12}-\mu_2)\mu_2\right]/(2\sigma^2)\right\},$$

$$\frac{1}{\lambda_{11}}=\left(\frac{K_0\cdot p(H_{11})}{K_1\cdot p(H_{01})}\right)\exp\left\{\left[2\cdot\sigma\cdot(\mu_{11}-\mu_1)\cdot\Phi^{-1}\left(\frac{r_2^{11}}{K_0\cdot p(H_{11})}\right)+(\mu_1^2-\mu_{11}^2)+2(\mu_{11}-\mu_1)\mu_{11}\right]/(2\sigma^2)\right\},$$

$$\frac{1}{\lambda_{12}}=\left(\frac{K_0\cdot p(H_{12})}{K_1\cdot p(H_{02})}\right)\exp\left\{\left[2\cdot\sigma\cdot(\mu_{12}-\mu_2)\cdot\Phi^{-1}\left(1-\frac{r_2^{12}}{K_0\cdot p(H_{12})}\right)+(\mu_2^2-\mu_{12}^2)+2(\mu_{12}-\mu_2)\mu_{12}\right]/(2\sigma^2)\right\},$$

$$\Gamma_{01}=\left\{\bar{x}:\exp\left\{\frac{2(\mu_{11}-\mu_1)\bar{x}+(\mu_1^2-\mu_{11}^2)}{2\sigma_{\bar{x}}^2}\right\}<\lambda_{01}\frac{K_0\cdot p(H_{01})}{K_1\cdot p(H_{11})}\right\},$$

$$\Gamma_{02}=\left\{\bar{x}:\exp\left\{\frac{2(\mu_{12}-\mu_2)\bar{x}+(\mu_2^2-\mu_{12}^2)}{2\sigma_{\bar{x}}^2}\right\}<\lambda_{02}\frac{K_0\cdot p(H_{02})}{K_1\cdot p(H_{12})}\right\},$$

$$\Gamma_{11}=\left\{\bar{x}:\exp\left\{\frac{2(\mu_{11}-\mu_1)\bar{x}+(\mu_1^2-\mu_{11}^2)}{2\sigma_{\bar{x}}^2}\right\}>\frac{1}{\lambda_{11}}\frac{K_1\cdot p(H_{01})}{K_0\cdot p(H_{11})}\right\},$$

$$\Gamma_{12}=\left\{\bar{x}:\exp\left\{\frac{2(\mu_{12}-\mu_2)\bar{x}+(\mu_2^2-\mu_{12}^2)}{2\sigma_{\bar{x}}^2}\right\}>\frac{1}{\lambda_{12}}\frac{K_1\cdot p(H_{02})}{K_0\cdot p(H_{12})}\right\}$$

Computational results are given in Table 5. Three computed values of the appropriate probabilities are given in each cell. They are separated by oblique lines. Computed results for $r_2^{01}=r_2^{02}=r_2^{11}=r_2^{12}=0.0125$ are given from the left side, i.e. for the restriction levels in Eq. (7) equal to 0.95, for $r_2^{01}=r_2^{02}=r_2^{11}=r_2^{12}=0.01$—in the middle, i.e. for restriction levels in Eq. (7) equal to 0.96 and for $r_2^{01}=r_2^{02}=r_2^{11}=r_2^{12}=$ 0.0025—from the right side, i.e. for restriction levels in Eq. (7) equal to 0.99. Dependences of hypotheses acceptance probabilities on mathematical expectation of generated random variables at losses Eq. (44), for different restriction levels, are given in Fig. 5.

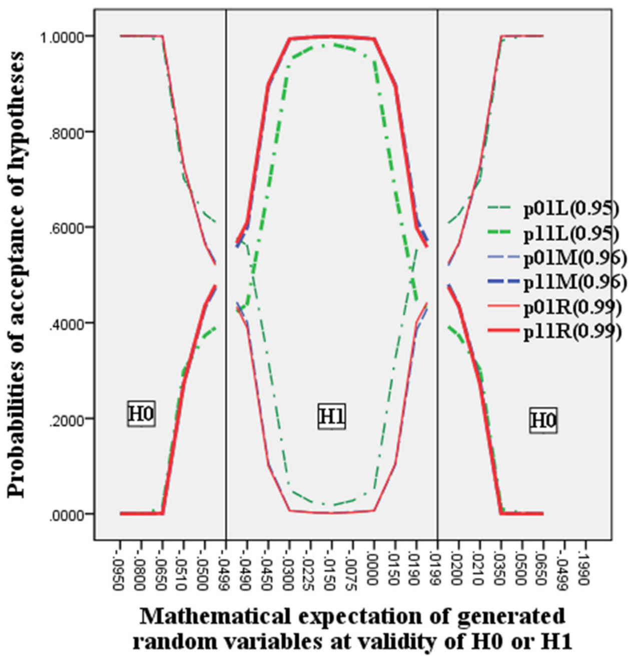

Fig. 5 Dependence of hypotheses acceptance probabilities on mathematical expectation of generated random variables at losses Eq. (44)

It is clear that the increase of restriction levels in Eq. (7) entails the increase of qualities of decisions made for both hypotheses.

Finally, for the same aim that was expressed in previous case and for more complete use of existing information, let's consider hypotheses acceptance regions Eq. (8) when loss functions Eq. (6) are used. Let's again use the fact that probabilities of errors at testing hypotheses decrease when information distance between hypotheses increase. For determination of Lagrange multipliers, let's consider hypotheses H_{01}, H_{02}, H_{11} and H_{12} pairwise, similar to the previous case (H_{01} vs. H_{11} and H_{02} vs. H_{12}), using restriction conditions Eq. (23) and hypotheses acceptance regions Eq. (24). Normal distributions correspond to the hypotheses with the worst expectations among all possible values in the appropriate sets of composite hypotheses, i.e. we consider the cases when $\mu_{01} = -0.05$; $\mu_{02} = 0.02$; μ_{11} =-0.0499 and $\mu_{12} = 0.0199$. The hypotheses acceptance regions are: $\Gamma_0 = \Gamma_{01} \bigcup \Gamma_{02}$ and $\Gamma_1 = \Gamma_{11} \bigcap \Gamma_{12}$ where sub-regions are determined by Eq. (24).

Further computational results are given in Table 6. Lagrange Multipliers, computed for $r_2^{01} = r_2^{02} = r_2^{11} = r_2^{12} = 0.0125$, i.e. for restriction levels in Eq. (23) equal to 0.95, are $\lambda_{01} = 8.125$, $\lambda_{02} = 8.4375$, $\lambda_{11} = 8.4375$ and $\lambda_{12} = 8.4375$. Samples with sizes 10,000 observations are used in computations, as at determination of Lagrange multipliers, so at computation of error probabilities.

Despite of ideal results at making decision (all decisions are correct), its practical value is limited because for making decisions a huge number of observations are necessary, which is less likely to be possible when solving many practical problems.

But still, the results of making decisions are so impressive that we consider it more appropriate to discuss this case in more details in cases where a large number of observational results can be obtained, i.e. for big data. Therefore, computation results of this case for different restriction levels are presented in next Table 7. Here computation results for three different cases are divided by

Table 6 Hypotheses acceptance probabilities depending on mathematical expectation of generated random variables at losses Eq. (6) and at the worst versions of hypotheses, i.e. when $\mu_{11} = -0.0499$ and $\mu_{12} = 0.0199$, for restriction level equal to 0.95

$N(x\|\mu, \sigma^2)$ μ	$p011$	$p021$	$p01$	$p111$	$p121$	$p11$	AN
At H_1: $-$ 0.049	0	0	0	1	0	1	26606.3153
$-$ 0.045	0	0	0	1	0	1	8052.9062
$-$ 0.03	0	0	0	1	0	1	2361.1116
$-$ 0.015 (MP)	0	0	0	0.7694	0.2306	1	1484.1451
0	0	0	0	0	1	1	2410.1896
0.015	0	0	0	0	1	1	8227.5673
0.019	0	0	0	0	1	1	27292.1561
At H_{01}	$p011$	$p021$	$p01$	$p111$	$p121$	$p11$	AN
$-$ 0.051	1	0	1	0	0	0	25417.4772
$-$ 0.065	1	0	1	0	0	0	3095.4674
$-$ 0.08	1	0	1	0	0	0	1631.9951
$-$ 0.095	1	0	1	0	0	0	1112.812
At H_{02}	$p011$	$p021$	$p01$	$p111$	$p121$	$p11$	AN
0.021	0	1	1	0	0	0	25625.7289
0.035	0	1	1	0	0	0	3112.2344
0.05	0	1	1	0	0	0	1642.3756
0.065	0	1	1	0	0	0	1118.9968

Table 7 Hypotheses acceptance probabilities depending on expectation of generated random variables at losses Eq. (6) and at the worst versions of hypotheses when $\mu_{11} = -0.0499$ and $\mu_{12} = 0.0199$, for different restriction levels

$N(x\|\mu,\sigma^2)\mu$	$p011$	$p021$	$p01$	$p111$	$p121$	$p11$	AN
At H_1: − 0.049	0/0/0.0005	0.0008/0.0016/0	0.0008/0.0016/0.0005	0.9974/0.9970/0.9958	0.0018/0.0014/0.0037	0.9992/0.9984/0.9995	9165.39/5858.21/4304.59
− 0.045	0/0/0.0006	0.0015/0.0023/0	0.0015/0.0023/0.0006	0.9957/0.9954/0.9935	0.0028/0.0023/0.0059	0.9985/0.9977/0.9994	3242.24/2210.27/1716.65
− 0.03	0/0/0.0002	0.0018/0.0041/0	0.0018/0.0041/0.0002	0.9420/0.9416/0.9159	0.0562/0.0543/0.0839	0.9982/0.9959/0.9998	897.61/5794.11/465.10
− 0.015 (MP)	0/0/0.0003	0.0016/0.0059/0	0.0016/0.0059/0.0003	0.5149/0.6193/0.5048	0.4835/0.3748/0.4949	0.9984/0.9941/0.9997	83.6327/61.5289/61.00
0	0/0/0.0002	0.0027/0.0044/0	0.0027/0.0044/0.0002	0.0652/0.1468/0.0829	0.9321/0.8488/0.9169	0.9973/0.9956/0.9998	893.77/615.71/477.58
0.015	0/0/0	0.0022/0.004/0	0.0022/0.004/0	0.0071/0.0301/0.0072	0.9907/0.9659/0.9928	0.9978/0.9960/1	3214.92/2329.36/1755.67
0.019	0/0/0.0001	0.0009/0.0047/0.0001	0.0009/0.0047/0.0002	0.0044/0.0178/0.0026	0.9947/0.9775/0.9972	0.9991/0.9953/0.9998	9080.27/6280.02/4427.65
At H_{01}	$p011$	$p021$	$p01$	$p111$	$p121$	$p11$	AN
− 0.051	0.9940/0.9627/0.9907	0.0013/0.0020/0	0.9953/0.9647/0.9907	0.0033/0.0348/0.007	0.0014/0.0005/0.0023	0.0047/0.0353/0.0093	8700.52/5959.13/4120.99
− 0.065	0.9985/0.9865/0.9981	0.0006/0.0004/0	0.9991/0.9869/0.9981	0.0007/0.0130/0.0016	0.0002/0.0001/0.0003	0.0009/0.0131/0.0019	1312.68/989.83/734.88
− 0.08	0.9990/0.9927/0.9995	0.0006/0.0005/0	0.9996/0.9932/0.9995	0.0004/0.0068/0.0003	0/0/0.0002	0.0004/0.0068/0.0005	708.38/541.25/404.41
− 0.095	0.9997/0.9978/1	0.0002/0.0001/0	0.9999/0.9979/1	0.0001/0.0021/0	0/0/0	0.0001/0.0021/0	487.62/375.57/280.52
At H_{02}	$p011$	$p021$	$p01$	$p111$	$p121$	$p11$	AN
0.021	0/0/0.0001	0.9952/0.9830/0.9884	0.9952/0.9830/0.9885	0.0033/0.0148/0.0028	0.0015/0.0022/0.0087	0.0048/0.0170/0.0115	8566.59/5924.53/4236.15
0.035	0/0/0	0.9992/0.9943/0.9976	0.9992/0.9943/0.9976	0.0006/0.0053/0.0002	0.0002/0.0004/0.0022	0.0008/0.0057/0.0024	1296.07/975.92/751.73
0.05	0/0/0.0001	0.9998/0.9983/0.9996	0.9998/0.9983/0.9997	0.0002/0.0017/0	0/0/0.0003	0.0002/0.0017/0.0003	699.57/533.93/413.95
0.065	0/0/0	1/0.9993/0.9997	1/0.9993/0.9997	0/0.0007/0	0/0/0.0003	0/0.0007/0.0003	481.61/369.24/286.82

slanting lines. The sequence of calculation results corresponds to the next sequence of data: (1) $r_2^{01} = r_2^{02} = r_2^{11} = r_2^{12} = 0.075$ (restriction levels in Eq. (23) are equal to 0.7), $\lambda_{01} = 2.584228515625$, $\lambda_{02} = 2.568359375$, $\lambda_{11} = 2.578125$, $\lambda_{12} = 2.548599243164063$; (2) $r_2^{01} = r_2^{02} = r_2^{11} = r_2^{12} = 0.1$ (restriction levels in Eq. (23) are equal to 0.6), $\lambda_{01} = 2.03125$; $\lambda_{02} = 2.08038330078125$; $\lambda_{11} = 2.086811065673828$; $\lambda_{12} = 2.060518189682625$ and (3) $r_2^{01} = r_2^{02} = r_2^{11} = r_2^{12} = 0.125$ (restriction levels in Eq. (23) are equal to 0.5), $\lambda_{01} = 1.7431640625$; $\lambda_{02} = 1.7626953125$; $\lambda_{11} = 1.73828125$; $\lambda_{12} = 1.75933837890625$, are divided.

Here, as well as in the previous case, computations are realized very fast because normal distributions are used.

It is obvious that the obtained results are excellent for all considered restriction levels but the number of observations necessary for making decision are quite big especially when restriction levels in Eq. (23) are high. The necessary number for making decision increases at increasing restriction levels in Eq. (23).

The results, given in Tables 6 and 7, confirm Theorem 1, because the conditions therein are satisfied (see Remark 4).

7 Discussions

– It is clear that the changes of the values of Type-I and Type-II error rates in all the considered cases is possible not only by changing restriction levels in the appropriate restriction conditions but, also by changing a prior probabilities and by choosing the values of loss functions.

– Probabilities of acceptance of hypothesis H_1 when it is true at losses Eqs. (14 and 15) are greater than at losses Eq. (6) and, on the contrary, probabilities of acceptance of hypothesis H_0 when it is true at losses Eqs. (14 and 15) are lower than at losses Eq. (6).
– The sample size for making decision at losses Eqs. (14 and 15) is significantly less than for losses Eq. (6).
– Probabilities of acceptance of hypotheses H_0 and H_1, when these are valid, are higher in Table 5 (i.e. at losses Eq. (44)) than in Table 1 (i.e. at losses Eq. (6)); at the same time, necessary sample size at losses Eq. (44) is significantly less than at losses Eq. (6) and computations are realized analytically in the first case.
– Probabilities of acceptance of hypothesis H_1 when it is true at losses Eqs. (14 and 15) (Table 4) are a little better than at losses Eq. (44) (Table 5); on the other hand, probabilities of acceptance of hypothesis H_0 when it is true, at losses Eq. (44), are significantly better than at losses Eqs. (14 and 15); sample size at Eq. (44) is no much higher than at Eqs. (14 and 15).
– On the basis of above findings, we conclude that losses Eq. (44) are preferable than losses Eq. (6) and losses Eqs. (14 and 15) as well; moreover, the computations at losses Eq. (44) are realized analytically and, therefore, are very fast.
– The best results are obtained at losses Eq. (6) when the densities of the worst simple hypotheses are used instead of averaged ones over the appropriate parameter subsets (see Tables 6 and 7).

Author's Contribution Both authors contributed to the study conception and design. Material preparation, data collection and analysis were performed by both authors. The first draft of the manuscript was written by both authors and they commented on previous versions of the manuscript. Both authors read and approved the final manuscript.

Funding K. Kachiashvili's research was partially supported by European Commission HORIZON EUROPE WIDERA-2021-ACCESS-03 Grant Project GAIN (grant agreement no.101078950).

Declarations

Conflict of interest Authors Kachiashvili KJ and SenGupta A declare they have no financial interests.

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