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RESEARCH PAPER

Existence of Positive Solutions of a Fractional Dynamic Equation Involving Integral Boundary Conditions on Time Scales

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Abstract

The existence and uniqueness of positive solutions to a fractional dynamic equation involving integral boundary conditions on time scale are examined using the Banach fixed point theorem and Schauder's fixed point theorem. The existence of the proposed dynamic equation has been determined using the Caputo nabla derivative operator (Caputo derivative on time scale in the nabla sense), the upper and lower solution approach, and the characteristics of the Green's function on time scales. Further, some appropriate examples has been given to demonstrate the implementation of theoretical results.

Keywords Caputo nabla derivative · Riemann–Liouville fractional integro-differential equation · Time scales · Schauder's fixed point theorem · Green's function · Upper and lower method of solutions

Mathematics Subject Classification Primary 26A33; Secondary 26E70

1 Introduction

This manuscript focuses on the existence and uniqueness of the positive solutions for the fractional boundary value problem (FBVP) involving integral boundary conditions of the type

$$\begin{cases} {}^{C}D^{\beta}k(\zeta) = \mathscr{K}(\zeta, k(\zeta), {}^{C}D^{\beta}k(\zeta)), \ \zeta \in \mathcal{J} \\ k(T) = \mu \int_{o}^{T} k(\theta)\nabla\theta, \ \mu \in \mathcal{R}^{+} \\ k(0) = 0, \end{cases} \tag{1.1}$$

where $\mathcal{J} = [0, T] \cap \mathbb{T}_{\mathcal{K}}$ for $T \in \mathbb{T}$. ${}^{C}D^{\beta}k(\zeta)$ is a Caputo nabla derivative of ld (left dense) continuous function $k(\zeta)$ of order $\beta \in (0, 1]$ on the time scale interval $\mathcal{J}$. The mapping $\mathscr{K} : \mathcal{J} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a ld continuous which is discussed in the paper.

Fractional calculus is one of the oldest branch of mathematics similar to ordinary calculus. In short one can say that ordinary calculus is a generalization of a fractional calculus. In real world applications we prefer fractional calculus over ordinary calculus due to the accuracy and advantages in the practical field. There have been a lot of work done on the topic of fractional differentail equation, fractional integro-differential equation, qualitative study of the solution of fractional differential equations with the Caputo fractional derivative operator and study of the existence of solution of a integral equations one can see Alabedalhadi et al. (2020), Al-Smadi and Arqub (2019), Al-Smadi (2021), Bohner et al. (2021), Chauhan et al. (2022), Tunç et al. (2021) and Tunç and Tunç (2023).

Fractional dynamic equation is used to solve a dynamic model in a common domain which is a unification of both discrete and continuous cases called time scale $\mathbb{T}$, which generally takes the form $\mathbb{T} = \bigcup_{m=0}^{\infty}[2m, 2m+1]$ for $m \in \mathbb{N} \cup \{0\}$. In such cases, solution of the dynamic equation can give the required data of the dynamic model under consideration. For convenience, one can see the

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model presented in the articles (Agarwal et al. 2002; Agarwal and O'Regan 2001) and the books (Bohner and Peterson 2003, 2001). Due to the wide application of the dynamic equation in the field of mathematics, engineering, economics, etc, many researchers are attracted to this topic in recent times. In the literature an ample amount of work exist in various fractional dynamic equations with initial and boundary conditions on time scale (Bai and Lü 2005; Benkhettou et al. 2016; Kumar and Malik 2019; Slavík 2012; Torres 2021; Zhao and You 2016; Zhao et al. 2016; Zhu and Wu 2015; Wu and Zhu 2013). But as compared to fractional dynamic equation involving initial conditions, less number of work can be seen with boundary conditions. However, the idea of the topic arises from the manuscripts (Agarwal et al. 2002; Yan et al. 2016b) and the books (Miller and Ross 1993; Podlubny 1999).

The discussion of the positive solution of the dynamic equation has been a very impectfull research from the inception of the topic. There have several manuscripts published in order to investigate the existence and uniqueness of the positive solution of a dynamic equation by employing various fixed point theorem in time scale (Feng et al. 2009; Kaufmann and Raffoul 2005; Dogan 2020; Yan et al. 2016a, b). Kaufmann et. al. Kaufmann and Raffoul (2005) gave the sufficient conditions for the existence of positive solution to a nonlocal eigen value problems for a class of nonlinear functional dynamic equation on time scale by emplyoing a cone theoretic fixed point theorem. The necessary and sufficient criteria for the existence of positive solution for singular boundary value problems on time scales were obtained by Feng et al. Feng et al. (2009). Goodrich (2011) studied the existence of a positive solution to a system of discrete fractional boundary value problems. Yan et al. (2016a) investigated the existence and uniqueness of solution of the boundary value problem of fractional order dynamic equation on time scales,

$$\begin{aligned} &{}^{C}\Delta^{\alpha}u(t) = f(t,u(t)),\ t\in[0,1]_{\mathbb{T}^{\kappa^2}},\ 1<\alpha<2,\\ &u(0)+u^{\Delta}(0)=0,\ u(1)+u^{\Delta}(1)=0, \end{aligned} \tag{1.2}$$

where $\mathbb{T}$ is a general time scale with $0,1\in\mathbb{T}$, ${}^{C}\Delta^{\alpha}$ is the Caputo Δ-fractional derivatives. Then they have discussed the existence of the positive solution of the problem (1.2) by using the Krasonoseleskii theorem. But best of our knowledge no work has been done on implicit type fractional dynamic equation with periodic integral boundary conditions involving Caputo nabla fractional derivative on time scale. However, Abdo et al. (2018) discussed the existence and uniqueness of a positive solution of similar type problem by using the method of upper and lower control functions in fractional calculus.

The rest of the manuscript is presented as follows. In Sect. 2, we provide the auxiliary results related to the fractional dynamic equation on time scales. In Sect. 3, we highlight the existence and uniqueness result of the FBVP (1.1). In Sect. 4, we give certain examples to demonstrate the implementation of theoretical results. Finally, the conclusion of the paper is presented in Sect. 5.

2 Preliminaries

Time scale $\mathbb{T}$ is a closed subset of $\mathbb{R}$, inherited from the standard topology of $\mathbb{R}$ with the properties $\rho(\zeta)=\sup\{\theta\in\mathbb{T}:\theta<\zeta\}$ and $\sigma(\zeta)=\inf\{\theta\in\mathbb{T}:\theta>\zeta\}$ which is used for connectedness of $\mathbb{T}$.

Throughout the paper we assume $\mathcal{J}=\{\zeta\in\mathbb{T}:0\leq\zeta\leq T,\quad T\in\mathcal{R}^{+}\}$.

In Anastassiou (2010), Anastassiou presented the nabla fractional integration in the following way

$$\mathcal{I}^{\beta}_{0^+}g(\zeta)=\int_0^{\zeta}h_{\beta-1}(\zeta,\rho(\theta))g(\theta)\nabla\theta,\quad \theta\in U, \tag{2.1}$$

where U is a neihbourhood of ζ and g is a Lebsesgue ∇-integrable function on the time scale interval $[0,T]\cap\mathbb{T}=\mathcal{J}$ and

$$h_{\beta-1}(\zeta,\rho(\theta))=\frac{(\zeta-\rho(\theta))^{\beta-1}}{\Gamma(\beta)}$$

varies with respect to different time scales.

If $\mathbb{T}=\mathbb{R}$, then $\rho(\theta)=\theta$, hence

$$h_{\beta-1}(\zeta,\rho(\theta))=\frac{(\zeta-\theta)^{\beta-1}}{\Gamma(\beta)},$$

therefore the Eq. (2.1) become

$$\mathcal{I}^{\beta}_{0^+}g(\zeta)=\int_0^{\zeta}\frac{(\zeta-\theta)^{\beta-1}}{\Gamma(\beta)}g(\theta)\mathrm{d}\theta.$$

If $\mathbb{T}=\mathbb{Z}$, then $h_{\beta-1}(\zeta,\rho(\theta))=\frac{(\zeta-\rho(\theta))^{\overline{\beta-1}}}{\Gamma(\beta)}$, so from the Eq. (2.1), we get

$$\begin{aligned} \mathcal{I}^{\beta}_{0^+}g(\zeta)&=\int_0^{\zeta}h_{\beta-1}(\zeta,\rho(\theta))g(\zeta)\nabla\theta\\ &=\frac{1}{\Gamma(\beta)}\int_0^{\zeta}(\zeta-\rho(\theta))^{\overline{\beta-1}}g(\theta)\nabla\theta\\ &=\frac{1}{\Gamma(\beta)}\sum_{\zeta=0}^{\zeta-1}(\zeta-(\theta-1))^{\overline{\beta-1}}g(\theta). \end{aligned}$$

If $\mathbb{T}=q^{\mathbb{N}_0}$, then $h_{\beta-1}(\zeta,\rho(\theta))=\Gamma_q(\beta)\frac{q^{\beta}-1}{q-1}(\zeta-q\theta)_q^{\beta-1}$, where Γ_q is a q-gamma function. For detailed of the Eq. (2.1), we prefer the reader to see the book (Georgiev

2018). Later, in the literature, we have seen that a lot of work has already been done based on (Benkhettou et al. 2016, Definition 10) which is

$$\mathcal{I}_{0^+}^{\beta} g(\zeta) = \frac{1}{\Gamma} \int_0^{\zeta} (\zeta - \theta)^{\beta-1} g(\theta) \nabla\theta. \tag{2.2}$$

As an improvement of the Eq. (2.2), in (Torres 2021, Definition 4) Torres proposed the most natural definition of fractional integral in a pure sense of Riemann-Liouville, which is

$$\mathcal{I}_{0^+}^{\beta} g(\zeta) = \frac{1}{\Gamma(\beta)} \int_0^{\zeta} (\zeta - \rho(\theta))^{\beta-1} g(\theta) \nabla\theta. \tag{2.3}$$

For checking the existence of the dynamic Eq. (1.1), we use the Eq. (2.3). On the basis of the definition given in Torres (2021), we introduce the Caputo nabla fractional derivative as follows

$$^{C}D^{\beta} g(\zeta) = \frac{1}{\Gamma(n-\beta)} \int_0^{\zeta} (\zeta - \rho(\theta))^{n-\beta-1} g_{\nabla^n}(\theta) \nabla\theta, \tag{2.4}$$

where $n = [\beta] + 1$. If $\beta \in (0, 1)$, then

$$^{C}D^{\beta} g(\zeta) = \frac{1}{\Gamma(1-\beta)} \int_0^{\zeta} (\zeta - \rho(\theta))^{-\beta} g_{\nabla}(\theta) \nabla\theta,$$

where $g(\zeta) \in \mathbb{T}_{\mathcal{K}^m}$, $n < m$. $\mathbb{T}_{\mathcal{K}^m}$ is attained by cutting out 'm' right scattered minimum left end points of $\mathbb{T}$.

Definition 2.1 (Tikare and Tisdell 2020) A function $\mathscr{K} : \mathcal{J} \to \mathbb{R}$ is said to be a left-dense (ld) continuous if, at all left dense point of $\mathbb{T}$, the function is continuous, and at the right dense point of $\mathbb{T}$ the right sided limit exists.

The set of all function from $\mathcal{J}$ to $\mathbb{R}$ is said to be a space of ld continuous function which is denoted by $\mathscr{C}(\mathcal{J}, \mathbb{R})$.

Remark 2.2 $\mathscr{C}(\mathcal{J}, \mathbb{R})$ form a Banach space endowed with the supremum norm, for $k \in \mathscr{C}(\mathcal{J}, \mathbb{R})$ such that

$$\|k\|_{\mathscr{C}} = \sup_{\zeta \in \mathcal{J}} |k(\zeta)|. \tag{2.5}$$

Definition 2.3 (Gogoi et al. 2021) For a ld continuous function $g(\zeta) \in \mathbb{T}$, the nabla derivative does not exist. Define $\mathbb{T}_{\mathcal{K}} = \mathbb{T} \backslash \{t\}$, else $\mathbb{T}_{\mathcal{K}} = \mathbb{T}$, where t is the right scattered minimum left end point of $\mathbb{T}$.

Definition 2.4 (Agarwal et al. 2021) Let $\mathcal{A} \subset \mathscr{C}(\mathcal{J}, \mathbb{R})$ be nonempty, closed and convex set. We say that $\mathcal{A}$ is a cone in $\mathscr{C}(\mathcal{J}, \mathbb{R})$, if

(1) $\lambda\mathcal{A} \subset \mathcal{A}$ for all $\lambda \geq 0$
(2) $-\mathcal{A} \cap \mathcal{A} = \{0_c\}$, where 0_c is the zero vector of $\mathscr{C}(\mathcal{J}, \mathbb{R})$.

The cone $\mathcal{A}$ induces a partial ordering $\leq$ in $\mathscr{C}(\mathcal{J}, \mathbb{R})$ defined by

$$k \leq \mathcal{A} g \Longleftrightarrow g - k \in \mathcal{A}.$$

The cone $\mathcal{A}$ is said to be normal or solid cone, if there exists $\rho \geq 1$ such that

$$0_c \leq \mathcal{A} k \leq \mathcal{A} g \Rightarrow \|k\|_{\mathscr{C}} \leq \rho \|g\|_{\mathscr{C}}$$

for all $k, g \in \mathscr{C}(\mathcal{J}, \mathbb{R})$.

Definition 2.5 (Abbas 2022) A mapping Ψ is a non negative, continuous concave functional on a cone $\mathcal{A}$, if it satisfies the conditions:

(1) $\Psi : \mathcal{A} \to [0, \infty)$ is continuous.
(2) $\Psi(\zeta k + (1-\zeta) g) \geq \zeta \Psi(k) + (1-\zeta)\Psi(g)$ for all $k, g \in \mathcal{A}$ and $0 \leq \zeta \leq 1$.

Definition 2.6 (Agarwal and O'Regan 2001) If $K^{\nabla}(\zeta) = k(\zeta)$, then the nabla integral is defined by

$$\int_{z_1}^{z_2} k(\zeta) \nabla\zeta = K(z_2) - K(z_1).$$

Theorem 2.7 *(Tikare and Tisdell 2020) Let $D \subset \mathscr{C}(\mathcal{J}, \mathbb{R})$ be a non empty set. D is a relatively compact, if it is bounded and equicontinuous simultaneously.*

Definition 2.8 (Tikare and Tisdell 2020) A mapping $\mathscr{K} : A \to B$ is completely continuous, if for a bounded subset $\mathscr{B} \subseteq A$, $\mathscr{K}(\mathcal{B})$ is relatively compact in A.

Proposition 2.9 *(Benkhettou et al. 2016) For a non decreasing ld continuous function $k(\zeta)$, defined on a time scale interval $[0, T]_{\mathbb{T}}$, and if $\mathscr{K}$ is the extension of k to the real line interval [0, T] such that*

$$\mathscr{K}(\zeta) = \begin{cases} k(\zeta) & if, \zeta \in \mathcal{T} \\ k(\theta) & if, \zeta \in (\beta(\theta), \theta) \notin \mathbb{T}, \end{cases}$$

then

$$\int_0^T k(\zeta) \nabla\zeta \leq \int_0^T \mathscr{K}(\zeta) \mathrm{d}\zeta.$$

3 Existence and Uniqueness of Positive Solutions

Definition 3.1 A ld continuous function $k \in \mathscr{C}(\mathcal{J}, \mathbb{R}) \cap \mathscr{C}^1(\mathcal{J}, \mathbb{R})$ is a solution of the FPBVP (1.1), if k satisfies the equation $^{C}\mathcal{D}^{\beta} k(\zeta) = \mathscr{K}(\zeta, k(\zeta), {}^{C}D^{\beta} k(\zeta))$ for $\zeta \in \mathcal{J}$ along with the boundary condition.

$\mathscr{C}^1(\mathcal{J},\mathbb{R})$ is used to denote the set of all continuous functions, whose first order nabla derivatives are ld continuous.

The following lemma helps us to transform the FBVP (1.1) into integral equation, which is key to apply fixed point theorem.

Lemma 3.2 *Let* $1<\beta\leq 2$ *and* $k\in\mathscr{C}(\mathcal{J},\mathbb{R})\cap\mathscr{C}^1(\mathcal{J},\mathbb{R})$ *for* $\zeta\in\mathcal{J}$. *Then the FBVP* (1.1) *has a unique solution given by*

$$k(\zeta)=\int_0^T G(\zeta,\theta)\mathscr{K}(\theta,k(\theta),{}^C D^\beta k(\theta))\nabla\theta, \tag{3.1}$$

where $G(\zeta,\theta)$ is the Green function defined by

$$G(\zeta,\theta)=\begin{cases}\dfrac{(\zeta-\rho(\theta))^{\beta-1}}{\Gamma(\beta)}+\dfrac{2\zeta(T-\rho(\theta))^{\beta-1}}{(\mu T^2-2T)\Gamma(\beta)}+\dfrac{-2\zeta\mu(T-\rho(\theta))^{\beta}}{(\mu T^2-2T)\Gamma(\beta+1)}, & \text{if } 0\leq\theta<\zeta\\[2ex] \dfrac{2\zeta(T-\rho(\theta))^{\beta-1}}{(\mu T^2-2T)\Gamma(\beta)}+\dfrac{-2\zeta\mu(T-\rho(\theta))^{\beta}}{(\mu T^2-2T)\Gamma(\beta+1)}, & \text{if } \zeta\leq\theta<T.\end{cases} \tag{3.2}$$

Proof For $1<\beta<2$, in view of the Eq. (2.4) we have ${}^C D^\beta k(\zeta)=\mathcal{I}^{2-\beta}g^2_\nabla(\zeta),\ \zeta\in\mathcal{J}$

Next, from the Lemma 2.7 (Yan et al. 2016b), we obtain

$$\begin{aligned}\mathcal{I}^{\beta C}D^\beta k(\zeta)=&\mathcal{I}^\beta\mathcal{I}^{2-\beta}k^2_\nabla(\zeta)\\=&\mathcal{I}^2k^2_\nabla(\zeta)\\=&k(\zeta)+p_0+p_1\zeta.\end{aligned}$$

For $p_0,p_1\in\mathbb{R}$. Again assuming, ${}^C D^\beta k(\zeta)=g(\zeta),\ \zeta\in\mathcal{J}$, then we get

$$\begin{aligned}k(\zeta)=&\mathcal{I}^\beta g(\zeta)-p_0-p_1\zeta\\=&\frac{1}{\Gamma(\beta)}\int_0^\zeta(\zeta-\rho(\theta))^{\beta-1}g(\theta)\nabla\theta-p_0-p_1\zeta.\end{aligned} \tag{3.3}$$

Now using the boundary conditions of the FPBVP (1.1) we obtain $p_0=0$, hence

$$k(T)=\frac{1}{\Gamma(\beta)}\int_0^T(T-\rho(\theta))^{\beta-1}g(\theta)\nabla\theta-p_1T. \tag{3.4}$$

Using the Fubini theorem on time scale (Benchohra and Ouaar 2010, Lemma 3.2) in the Eq. (3.28), then

$$\begin{aligned}\int_0^T k(\theta)\nabla\theta&=\int_0^T\Big(\int_0^\zeta\frac{(\zeta-\rho(\eta))^{\beta-1}}{\Gamma(\beta)}g(\eta)\nabla\eta-p_1\zeta\Big)\nabla\theta\\&=\int_0^T\Big(\int_\eta^T\frac{(\theta-\rho(\eta))^{\beta-1}}{\Gamma(\beta)}\nabla\theta\Big)g(\eta)\nabla\eta-\frac{p_1}{2}T^2\\&=\int_0^T\frac{(T-\rho(\eta))^\beta}{\Gamma(\beta+1)}g(\eta)\nabla\eta-\frac{p_1}{2}T^2.\end{aligned} \tag{3.5}$$

Applying the Eqs. (3.4) and (3.5) in the boundary conditions of the FPBVP (1.1), we get

$$p_1=\int_0^T\Big[\frac{2\mu(T-\rho(\theta))^\beta}{(\mu T^2-2T)\Gamma(\beta+1)}+\frac{-2(T-\rho(\theta))^{\beta-1}}{(\mu T^2-2T)\Gamma(\beta)}\Big]g(\theta)\nabla\theta. \tag{3.6}$$

Now using the Eqs. (3.4), (3.5) and (3.6) in Eq. (3.28) we obtain

$$\begin{aligned}k(\zeta)=&\int_0^\zeta\frac{(\zeta-\rho(\theta))^{\beta-1}}{\Gamma(\beta)}g(\theta)\nabla\theta\\&-\int_0^T\Big[\frac{2\zeta\mu(T-\rho(\theta))^\beta}{(\mu T^2-2T)\Gamma(\beta+1)}\\&+\frac{-2\zeta(T-\rho(\theta))^{\beta-1}}{(\mu T^2-2T)\Gamma(\beta)}\Big]g(\theta)\nabla\theta\\=&\int_0^\zeta\frac{(\zeta-\rho(\theta))^{\beta-1}}{\Gamma(\beta)}g(\theta)\nabla\theta\\&-\Big(\int_0^\zeta\Big[\frac{2\zeta\mu(T-\rho(\theta))^\beta}{(\mu T^2-2T)\Gamma(\beta+1)}\\&+\frac{-2\zeta(T-\rho(\theta))^{\beta-1}}{(\mu T^2-2T)\Gamma(\beta)}\Big]g(\theta)\nabla\theta\\&+\int_\zeta^T\Big[\frac{2\zeta\mu(T-\rho(\theta))^\beta}{(\mu T^2-2T)\Gamma(\beta+1)}\\&+\frac{-2\zeta(T-\rho(\theta))^{\beta-1}}{(\mu T^2-2T)\Gamma(\beta)}\Big]g(\theta)\nabla\theta\Big)\\=&\int_0^\zeta\Big[\frac{(\zeta-\rho(\theta))^{\beta-1}}{\Gamma(\beta)}+\frac{2\zeta(T-\rho(\theta))^{\beta-1}}{(\mu T^2-2T)\Gamma(\beta)}\\&+\frac{-2\zeta\mu(T-\rho(\theta))^\beta}{(\mu T^2-2T)\Gamma(\beta+1)}\Big]g(\theta)\nabla\theta\\&+\int_\zeta^T\Big[\frac{2\zeta(T-\rho(\theta))^{\beta-1}}{(\mu T^2-2T)\Gamma(\beta)}\\&+\frac{-2\zeta\mu(T-\rho(\theta))^\beta}{(\mu T^2-2T)\Gamma(\beta+1)}\Big]g(\theta)\nabla\theta\\=&\int_0^T G(\zeta,\theta)g(\theta)\nabla\theta.\end{aligned}$$

For the existence and uniqueness result, we assume the following:

(A1) The mapping $\mathscr{K} = \mathcal{J} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a ld continuous in its first variable and continuous in its second and third variable separately.

(A2) For a function $\mathscr{K}$ in (A1), there exists a function $\mathscr{A} \in \mathscr{C}(\mathcal{J}, \mathbb{R})$, and two constants $\mathscr{B} > 0, 0 < \mathcal{C} < 1$ such that

$$|\mathscr{K}(\zeta, \eta_1, \eta_2)| \le |\mathscr{A}(\zeta)| + \mathscr{B}|\eta_1| + \mathcal{C}|\eta_2|,$$

for $(\zeta, \eta_1, \eta_2) \in \mathcal{J} \times \mathbb{R} \times \mathbb{R}$.

(A3) For a function $\mathscr{K}$ in (A1), there exist two constants $\mathcal{G}, \mathcal{H} > 0$ such that

$$|\mathscr{K}(\zeta, \eta_1, \eta_2) - \mathscr{K}(\zeta, \theta_1, \theta_2)| \le \mathcal{G}|\eta_1 - \theta_1| + \mathcal{H}|\eta_2 - \theta_2|,$$

for $(\zeta, \theta_i, \eta_i) \in \mathcal{J} \times \mathbb{R} \times \mathbb{R}$, for $i = 1, 2$.

(A4) The Green function $G(\zeta, \theta)$ is bounded piece wise continuous on [0, T]. Moreover, G is non negative increasing, such that

$$\mathscr{G} = \sup_{\zeta \in \mathcal{J}} \int_0^T G(\zeta, \theta) \nabla\theta.$$

Later, to prove the existence and uniqueness of the positive solution of the FPBVP (1.1), we shall use fixed point theorems. For this, first we consider the following essential notations.

Consider a set

$$\mathcal{A}_{\mathcal{C}} = \left\{k \in \mathscr{C}(\mathcal{J}, \mathbb{R}) : \|k\|_{\mathscr{C}} \le \alpha, k(\zeta) \ge 0, \zeta \in \mathcal{J}\right\} \tag{3.7}$$

and an operator $\mathcal{F} : \mathcal{A}_{\mathcal{C}} \to \mathcal{A}_{\mathcal{C}}$ defined by

$$\mathcal{F}(k(\zeta)) = \int_0^T G(\zeta, \theta) \mathscr{K}(\theta, k(\theta), {}^C D^\beta k(\theta)) \nabla\theta. \tag{3.8}$$

It is obvious that $\mathcal{A}_{\mathcal{C}}$ is a normal cone in $\mathscr{C}(\mathcal{J}, \mathbb{R})$. □

Definition 3.3 (Abbas 2022, Method of upper and lower solutions) Consider that $\mathscr{K}(\zeta, \cdot, \cdot)$ be a ld continuous functions for each $\zeta \in \mathcal{J}$. Let $c, d \in \mathscr{J}$ satisfy $0 \le c < d \le T$, and a function k such that $k \in [c, d]$. Define, the upper and lower control function as $\overline{\mathscr{K}}(\zeta, k, h) = \sup_{c \le g \le k, h \le r} \mathscr{K}(\zeta, g, r)$ and $\underline{\mathscr{K}}(\zeta, k, h) = \inf_{k \le g \le b, h \le r} \mathscr{K}(\zeta, g, r)$, respectively, where r is a function of k. Clearly the functions $\overline{\mathscr{K}}(\zeta, k, h)$ and $\underline{\mathscr{K}}(\zeta, k, h)$ is non-decreasing on k and satisfies the following condition:

$$\overline{\mathscr{K}}(\zeta, k, h) \le \mathscr{K}(\zeta, k, h) \le \underline{\mathscr{K}}(\zeta, k, h) \tag{3.9}$$

Definition 3.4 let $\overline{k}, \underline{k} \in \mathcal{A}_{\mathcal{C}}$, for $\zeta \in \mathcal{J} = [0, T]$ such that $0 \le \underline{k}(\zeta) \le \overline{k}(\zeta) \le T$ conform to

$$\overline{k}(\zeta) = \int_0^T G(\zeta, \theta) \overline{\mathscr{K}}(\theta, \overline{k}(\theta), {}^C D^\beta \overline{k}(\theta)) \nabla\theta \tag{3.10}$$

$$\underline{k}(\zeta) = \int_0^T G(\zeta, \theta) \underline{\mathscr{K}}(\theta, \underline{k}(\theta), {}^C D^\beta \underline{k}(\theta)) \nabla\theta. \tag{3.11}$$

Then the function $\overline{k}(\zeta)$ is a upper solution and $\underline{k}(\zeta)$ is a lower solution of the FBVP (1.1).

Next theorem is based on the Schauder's fixed point theorem (see Tikare and Tisdell 2020).

Theorem 3.5 *If the assumptions* (A1)–(A4) hold and

$$0 < \mathscr{G} = \sup_{\zeta \in \mathcal{J}} \int_0^T G(\zeta, \theta) \nabla\theta \le T$$

for $T \in \mathbb{R}^+$, then the Eq. (1.1) has at least one solution.

Proof Assume that ${}^C D^\beta k_n(\zeta) = g_n(\zeta),\ n \in \mathcal{N}$ and ${}^C D^\beta k(\zeta) = g(\zeta),\ \zeta \in \mathcal{J}$. Consider $\mathcal{F} : \mathcal{A}_{\mathcal{C}} \to \mathcal{A}_{\mathcal{C}}$, the mapping defined in (3.8). We divide the proof into the following steps:

Step 1: The operator $\mathcal{F} : \mathcal{A}_{\mathcal{C}} \to \mathcal{A}_{\mathcal{C}}$ is continuous.

Let $\{k_n\}_{n \in \mathcal{N}}$ be a sequence in $\mathcal{A}_{\mathcal{C}}$ which is converges to k in $\mathcal{A}_{\mathcal{C}}$. Now, for $\zeta \in \mathcal{J}$ we have

$$\begin{aligned} &|\mathcal{F}[k_n](\zeta) - \mathcal{F}[k](\zeta)| \le \\ &\int_0^T |G(\zeta, \theta)\big(\mathscr{K}(\theta, k_n(\theta), g_n(\theta) - \mathscr{K}(\theta, k(\theta), g(\theta)\big)| \nabla\theta \\ &\le \int_0^T |G(\zeta, \theta)||g_n(\theta) - g(\theta)| \nabla\theta. \end{aligned} \tag{3.12}$$

For, $g_n, g \in \mathcal{A}_{\mathcal{C}}$. In view of (1.1) for $\theta \in \mathcal{J}$ we get

$$\begin{aligned} |g_n(\theta) - g(\theta)| &= |\mathscr{K}(\theta, k_n(\theta), g_n(\theta)) - \mathscr{K}(\theta, k(\theta), g(\theta))| \\ &\overset{(A3)}{\le} \mathcal{G}|k_n(\theta) - k(\theta)| + \mathcal{H}|g_n(\theta) - g(\theta)|. \end{aligned}$$

This gives

$$|g_n(\theta) - g(\theta)| \le \frac{\mathcal{G}}{1 - \mathcal{H}} |k_n(\theta) - k(\theta)|. \tag{3.13}$$

Now using the Eqs. (3.13) in (3.12), and taking the norm of $\mathscr{C}(\mathcal{J}, \mathbb{R})$, we get

$$\|\mathcal{F}[k_n] - \mathcal{F}[k]\|_{\mathscr{C}} \overset{(A4)}{\le} \frac{\mathscr{G}\mathcal{G}}{1 - \mathcal{H}} ||k_n - k||_{\mathscr{C}}, \tag{3.14}$$

This yields that the right side of (3.14) approaches to 0 as k_n approaches k. Hence, $\mathcal{F} : \mathcal{A}_{\mathcal{C}} \to \mathcal{A}_{\mathcal{C}}$ is continuous.

Step 2: The operator $\mathcal{F} : \mathcal{A}_{\mathcal{C}} \to \mathcal{A}_{\mathcal{C}}$ is bounded. From (3.7), one can write for $\zeta \in \mathcal{J}$

$$\begin{aligned}\|\mathcal{F}[k](\zeta)\| &\leq \int_0^T |G(\zeta,\theta)\mathscr{K}(\theta,k(\theta),{}^C\mathcal{D}^{\beta}k(\theta))|\nabla\theta \\ &\leq \int_0^T |G(\zeta,\theta)\mathscr{K}(\theta,k(\theta),g(\theta))|\nabla\theta \\ &\leq \int_0^T |G(\zeta,\theta)||g(\theta)|\nabla\theta.\end{aligned} \tag{3.15}$$

For $g \in \mathcal{A}_{\mathcal{C}},\ \theta \in \mathcal{J}$. In view of Eq. (1.1) for $\theta \in \mathcal{J}$ we have

$$\begin{aligned}|g(\theta)| &= |\mathscr{K}(\theta,k(\theta),g(\theta))| \\ &\overset{(A2)}{\leq} |\mathscr{A}(\theta)| + B|k(\theta)| + C|g(\theta)|.\end{aligned} \tag{3.16}$$

Using the Eq. (3.7) we have

$$|g(\theta)| \leq \frac{|\mathscr{A}(\theta)| + B|k(\theta)|}{1-C}. \tag{3.17}$$

Using (3.17) in (3.16) and applying the norm of $\mathscr{C}(\mathcal{J},\mathbb{R})$, we obtain

$$\begin{aligned}\|\mathcal{F}[g](\zeta)\| &\leq \int_0^T |G(\zeta,\theta)|\frac{\|\mathscr{A}\| + B\|k\|}{1-C}\nabla\theta. \\ &\overset{(A4)}{\leq} \frac{\mathscr{G}(\|\mathscr{A}\| + B\alpha)}{1-C}.\end{aligned}$$

That is

$$\|\mathcal{F}[g](\zeta)\| \leq \frac{T(\|\mathscr{A}\| + B\alpha)}{1-C}. \tag{3.18}$$

Thus, the operator $\mathcal{F} : \mathcal{A}_{\mathcal{C}} \to \mathcal{A}_{\mathcal{C}}$ is bounded.

Step 3: The operator $\mathcal{F} : \mathcal{A}_{\mathcal{C}} \to \mathcal{A}_{\mathcal{C}}$ is equicontinuous.

Let $\zeta_1, \zeta_2 \in \mathcal{J}$ such that $\zeta_1 < \zeta_2$, then for $g \in \mathcal{A}_{\mathcal{C}}$, we get

$$\begin{aligned}&\|\mathcal{F}[g](\zeta_1) - \mathcal{F}[g](\zeta_2)\|_{\mathscr{C}} \\ &\leq \int_0^T \Big|G(\zeta_1,\theta)\mathscr{K}(\theta,k(\theta),{}^C\mathcal{D}^{\beta}k(\theta))\nabla\theta \\ &- G(\zeta_2,\theta)\mathscr{K}(\theta,k(\theta),{}^C\mathcal{D}^{\beta}k(\theta))\nabla\theta\Big| \\ &= \int_0^T \Big|G(\zeta_1,\theta)\nabla\theta - G(\zeta_2,\theta)\nabla\theta\Big||\mathscr{K}(\theta,k(\theta),{}^C\mathcal{D}^{\beta}k(\theta))| \\ &= \int_0^T \Big|G(\zeta_1,\theta)\nabla\theta - G(\zeta_2,\theta)\nabla\theta\Big||g(\theta)| \\ &\overset{(3.1)}{\leq} \int_0^T \Big|G(\zeta_1,\theta)\nabla\theta - G(\zeta_2,\theta)\nabla\theta\Big|\Big(\frac{|\mathscr{A}(\theta)| + B|k(\theta)|}{1-C}\Big).\end{aligned}$$

That is

$$\begin{aligned}&\|\mathcal{F}[g](\zeta_1) - \mathcal{F}[g](\zeta_2)\| \leq \frac{\mathscr{G}(\|\mathscr{A}\| + B\alpha)}{1-C} \\ &\int_0^T \Big|G(\zeta_1,\theta)\nabla\theta - G(\zeta_2,\theta)\nabla\theta\Big|.\end{aligned} \tag{3.19}$$

Since, the Green's function is continuous so $\zeta_1 \to \zeta_2$, then $\|\mathcal{F}[g](\zeta_1) - \mathcal{F}[g](\zeta_2)\| \to 0$. Thus the operator $\mathcal{F} : \mathcal{A}_{\mathcal{C}} \to \mathcal{A}_{\mathcal{C}}$ is equicontinuous. Now, since $\mathcal{F}(\mathcal{A}_{\mathcal{C}})$ is bounded and equicontinuous, then by the Theorem 2.7 the operator $\mathcal{F} : \mathcal{A}_{\mathcal{C}} \to \mathcal{A}_{\mathcal{C}}$ is relatively compact. So, by virtue of Schauder's fixed point theorem the operator has a fixed point, which is the solution of the FBVP (1.1). □

The existence of positive solution of the FBVP (1.1) is based on the Schauder's fixed point theorem.

Theorem 3.6 *If the assumptions* (A1) - (A4) hold and if $\overline{k}(\zeta)$ and $\underline{k}(\zeta)$ be a pair of upper and lower solutions, then the FBVP (1.1) possess at least one postive solution. Additionally

$$\underline{k}(\zeta) \leq k(\zeta) \leq \overline{k}(\zeta),\ \zeta \in \mathcal{J}.$$

Proof Consider a set

$$\mathcal{D} = \{k(\zeta) \in \mathcal{A}_{\mathcal{C}} : \underline{k}(\zeta) \leq k(\zeta) \leq \overline{k}(\zeta),\ \zeta \in \mathcal{J}\}.$$

Clearly, the set $\mathcal{D}$ is convex, bounded, and closed subset of $\mathscr{C}(\mathcal{J},\mathbb{R})$. Taking into account of Theorem 3.5, we have that the opeartor $\mathcal{F} : \mathcal{D} \to \mathcal{D}$ is relatively compact. Now for $k \in \mathcal{D},\ \zeta \in \mathcal{J}$ we have

$\underline{k}(\zeta) \leq k(\zeta) \leq \overline{k}(\zeta)$.

By using (3.9) we obtain

$$\begin{aligned}\mathcal{F}[k](\zeta) &= \int_0^T G(\zeta,\theta)\mathscr{K}(\theta,k(\theta),{}^C\mathcal{D}^{\beta}k(\theta))\nabla\theta \\ &\overset{(3.9)}{\leq} \int_0^T G(\zeta,\theta)\overline{K}(\theta,k(\theta),{}^C\mathcal{D}^{\beta}k(\theta))\nabla\theta \\ &\leq \int_0^T G(\zeta,\theta)\overline{K}(\theta,\overline{k}(\theta),{}^C\mathcal{D}^{\beta}\overline{k}(\theta))\nabla\theta \\ &\leq \overline{k}(\zeta).\end{aligned} \tag{3.20}$$

Similarly

$$\begin{aligned}\mathcal{F}[k](\zeta) &= \int_0^T G(\zeta,\theta)\mathscr{K}(\theta,k(\theta),{}^C\mathcal{D}^{\beta}k(\theta))\nabla\theta \\ &\overset{(3.9)}{\geq} \int_0^T G(\zeta,\theta)\underline{K}(\theta,k(\theta),{}^C\mathcal{D}^{\beta}k(\theta))\nabla\theta \\ &\geq \int_0^T G(\zeta,\theta)\underline{K}(\theta,\underline{k}(\theta),{}^C\mathcal{D}^{\beta}\underline{k}(\theta))\nabla\theta \\ &\geq \underline{k}(\zeta).\end{aligned} \tag{3.21}$$

From (3.20) and (3.21) we get

$$\underline{k}(\zeta) \leq \mathcal{F}[k](\zeta) \leq \overline{k}(\zeta),\ \zeta \in \mathcal{J}.$$

Thus $\mathcal{F}[k] \in \mathcal{D}$, hence the operator $\mathcal{F} : \mathcal{D} \to \mathcal{D}$ is relatively compact. So by virtue of Schauder's fixed point theorem,

$\mathcal{F}$ has a fixed point in $\mathcal{D}$, which is a positive solution of the FBVP (1.1) in $\mathcal{A}_{\mathcal{C}}$. □

Corollary 3.7 *Suppose the assumption* (*A*1) hold. If there exist $M_1, M_2 \in \mathbb{R}^+$, $0 < M_1 \le M_2$ such that

$$M_1 \le \mathscr{K}(\zeta, \psi, \phi) \le M_2 \tag{3.22}$$

for $(\zeta, \psi, \phi) \in \mathcal{J} \times \mathbb{R} \times \mathbb{R}$, then the FBVP (1.1) has at least one positive solution $k \in \mathcal{A}_{\mathcal{C}}$. Moreover $k(\zeta)$ satisfies

$$\int_0^T G(\zeta, \theta) M_1 \nabla\theta \le \; k(\zeta) \; \le \int_0^T G(\zeta, \theta) M_2 \nabla\theta. \tag{3.23}$$

Proof Using (3.9) we have

$$M_1 \le \underline{\mathscr{K}}(\zeta, \psi, \phi) \le \overline{\mathscr{K}}(\zeta, \psi, \phi) \le M_2. \tag{3.24}$$

Let us consider the equation

$$\begin{cases} {}^C\mathcal{D}^\beta \overline{k}(\zeta) = M_2, \; \zeta \in \mathcal{J} \\ \overline{k}(T) = \mu \int_o^T \overline{k}(\theta)\nabla\theta, \;\; \mu \in \mathbb{R} \\ \overline{k}(0) = 0, \end{cases} \tag{3.25}$$

which has a positive solution given by

$$\overline{k}(\zeta) = \int_0^T G(\zeta, \theta) M_2 \nabla\theta.$$

In view of (3.24), we obtain

$$\overline{k}(\zeta) \ge \int_0^T G(\zeta, \theta) \overline{\mathscr{K}}(\theta, \overline{k}(\theta), {}^C D^\beta \overline{k}(\theta) \nabla\theta. \tag{3.26}$$

Similarly, for the dynamic equation

$$\begin{cases} {}^C D^\beta \underline{k}(\zeta) = M_1, \; \zeta \in \mathcal{J} \\ \underline{k}(T) = \mu \int_o^T \underline{k}(\theta)\nabla\theta, \;\; \mu \in \mathbb{R} \\ \underline{k}(0) = 0, \end{cases}$$

has a positive solution given by

$$\begin{aligned} \underline{k}(\zeta) &= \int_0^T G(\zeta, \theta) M_1 \nabla\theta \\ &\overset{3.24)}{\le} \int_0^T G(\zeta, \theta) \underline{\mathscr{K}}(\theta, \underline{k}(\theta), {}^C D^\beta \underline{k}(\theta)) \nabla\theta \end{aligned} \tag{3.27}$$

Thus, from (3.26) and (3.27), we obtain the solutions $\overline{k}(\zeta)$, $\underline{k}(\zeta)$ are the upper and lower solution of the equation (1.1), respectively. Hence, an implementation of Theorem 3.6 we conclude that the FBVP (1.1) has atleast one positive solution $k(\zeta) \in \mathcal{A}_{\mathcal{C}}$, $\zeta \in \mathcal{J}$ which satisfies the inequality (3.23). □

For uniqueness of the positive solution, we use Banach fixed point theorem (Tikare and Tisdell 2020).

Theorem 3.8 *Let the assumptions* (*A*3), (*A*4) hold. If

$$\frac{\mathscr{G}\mathcal{G}}{1 - \mathcal{H}} < 1,$$

then the equation (1.1) has a unique solution.

Proof Let ${}^C D^\beta k_i(\zeta) = g_i(\zeta)$, $\zeta \in \mathcal{J}$ and $i = 1, 2$. Then for $k_1, k_2 \in \mathcal{A}_{\mathcal{C}}$ we have

$$\begin{aligned} &\|\mathcal{F}[k_1](\zeta) - \mathcal{F}[k_2](\zeta)\| \\ &\le \int_0^T \Big| G(\zeta, \theta) \mathscr{K}(\theta, k_1(\theta), {}^C D^\beta k_1(\theta)) \nabla\theta \\ &- G(\zeta, \theta) \mathscr{K}(\theta, k_2(\theta), {}^C D^\beta k_2(\theta)) \nabla\theta \Big| \\ &\le \int_0^T \big| G(\zeta, \theta) \nabla\theta \big| \Big| \mathscr{K}(\theta, k_1(\theta), g_1(\theta) - \mathscr{K}(\theta, k_1(\theta), g_2(\theta)) \Big| \\ &\le \int_0^T \big| G(\zeta, \theta) \nabla\theta \big| |g_1(\theta) - g_2(\theta)|. \end{aligned} \tag{3.28}$$

For $g_1, g_2 \in \mathcal{A}_{\mathcal{C}}$. But in view of (1.1) for $\theta \in \mathcal{J}$

$$\begin{aligned} |g_1(\theta) - g_2(\theta)| &= |\mathscr{K}(\theta, k_1(\theta), g_1(\theta)) - \mathscr{K}(\theta, k_2(\theta), g_2(\theta))| \\ &\overset{(A3)}{\le} \mathcal{G}|k_1(\theta) - k_2(\theta)| + \mathcal{H}|g_1(\theta) - g_2(\theta)| \\ &\le \frac{\mathcal{G}}{1 - \mathcal{H}} |k_1(\theta) - k_2(\theta)|. \end{aligned}$$

This gives

$$|g_1(\theta) - g_2(\theta)| \le \frac{\mathcal{G}}{1 - \mathcal{H}} |k_1(\theta) - k_2(\theta)|. \tag{3.29}$$

Now using (3.29) in (3.28), we obtain

$$\begin{aligned} \|\mathcal{F}[k_1](\zeta) - \mathcal{F}[k_2](\zeta)\| &\le \frac{\mathcal{G}}{1 - \mathcal{H}} \int_0^T G(\zeta, \theta) \|k_1 - k_2\| \nabla\theta \\ &\overset{(A4)}{\le} \frac{\mathscr{G}\mathcal{G}}{1 - \mathcal{H}} \|k_1 - k_2\| \end{aligned} \tag{3.30}$$

Since, $\frac{\mathscr{G}\mathcal{G}}{1-\mathcal{H}} < 1$, the mapping $\mathcal{F} : \mathcal{A}_{\mathcal{C}} \to \mathcal{A}_{\mathcal{C}}$ is contractive. Hence, by Banach contraction theorem the operator has a unique fixed point, which is a solution of the FBVP (1.1). □

4 Illustrative Example

Example 4.1 Consider a time scale $\mathbb{T} = [0, 1] \cup [2, 3]$ and T = 2, then $\mathcal{J} = [0, 2] \cap \{2\} = [0, 1] \cup \{2\}$. Consider the FPBVP

$$\begin{cases} {}^{C}D^{1.5}k(\zeta) = \dfrac{e^{-2\zeta}}{5(2+e^{3\zeta})}\left(1 + k(\zeta) + {}^{C}D^{\beta}k(\zeta)\right), & \zeta \in \mathcal{J} \\ k(2) = \int_0^2 k(\theta)\nabla \\ k(0) = 0. \end{cases} \tag{4.1}$$

Here

$$\mathscr{K}(\zeta, k(\zeta), {}^{C}D^{1.5}k(\zeta)) = \frac{e^{-2\zeta}}{5(2+e^{3\zeta})}\left(1 + k(\zeta), {}^{C}D^{1.5}k(\zeta)\right) \tag{4.2}$$

which satisfies condition (A1).

Again,let

$${}^{C}D^{1.5}k(\zeta) = g(\zeta),\ \zeta \in \mathcal{J},\ \text{for } g \in \mathcal{A}_{\mathcal{C}}$$

we have

$$\begin{aligned} |\mathscr{K}(\zeta, k(\zeta), g(\zeta))| &= \left|\frac{e^{-2\zeta}}{5(2+e^{3\zeta})}\left(1 + k(\zeta) + {}^{C}D^{1.5}k(\zeta)\right)\right| \\ &\le \left|\frac{1}{15}\right| + \frac{1}{15}|k(\zeta)| + \frac{1}{15}|g(\zeta)|. \end{aligned} \tag{4.3}$$

Here (A2) is satisfy with $\mathscr{A} = \frac{1}{15},\ \mathscr{B} = \frac{1}{15},\ \mathcal{C} = \frac{1}{15}$. Similarly, assume that

$${}^{C}D^{1.5}k_i(\zeta) = g_i(\zeta) \text{ for } g_i(\zeta) \in \mathcal{A}_{\mathcal{C}},\ \zeta \in \mathcal{J}.$$

Then

$$\begin{aligned} &|\mathscr{K}(\zeta, k_1(\zeta), g_1(\zeta)) - \mathscr{K}(\zeta, k_2(\zeta), g_2(\zeta))| \\ &= \left|\frac{e^{-2\zeta}}{5(2+e^{3\zeta})}\left(1 + k_1(\zeta) + g_1(\zeta)\right) \right. \\ &\left. - \frac{e^{-2\zeta}}{5(2+e^{3\zeta})}\left(1 + k_2(\zeta) + g_2(\zeta)\right)\right| \\ &\le \frac{1}{15}|k_1(\zeta) - k_2(\zeta)| + \frac{1}{15}|g_1(\zeta) - g_2(\zeta)|. \end{aligned}$$

That is,

$$\begin{aligned} &|\mathscr{K}(\zeta, k_1(\zeta), g_1(\zeta)) - \mathscr{K}(\zeta, k_2(\zeta), g_2(\zeta))| \\ &\le \frac{1}{15}|k_1 - k_2| + \frac{1}{15}|g_1 - g_2| \end{aligned} \tag{4.4}$$

which satisfies the condition (A3), with $\mathcal{G} = \frac{1}{15},\ \mathcal{H} = \frac{1}{15}$.

Further, from the boundary conditions $k(0) = 0,\ k(2) = \int_0^2 k(\theta)\nabla\theta$, using the Eq. (3.2) of Green function for $T = 2$ and the Proposition 2.9, we have

$$\begin{aligned} \left|\int_0^2 G(\zeta,\theta)\nabla\theta\right| &\le \left|\int_0^2 G(\zeta,\theta)\mathrm{d}\theta\right| \\ &\le \left|\frac{1}{\Gamma(0.5)}\int_0^{\zeta}(\zeta-\theta)^{0.5}\mathrm{d}\theta\right| \\ &\le 2. \end{aligned}$$

That is, for $\zeta \in \mathcal{J}$ we can assume

$$\mathscr{G} = \sup_{\zeta \in \mathcal{J}} \int_0^2 G(\zeta,\theta)\nabla\theta \le 2. \tag{4.5}$$

This yields that (A4) is satisfied. Thus in view of Eqs. (4.2), (4.3), (4.4) and (4.5) we have, all conditions of the Theorem 3.5 is satisfied. Hence, the FBVP (4.2) has a solution.

Putting the above data from (4.4) and (4.5), the inequality

$$\frac{\mathscr{G}\mathcal{G}}{1-\mathcal{H}} \le \frac{2}{14} < 1, \text{ is satisfied.} \tag{4.6}$$

Thus, in view of (4.4), (4.5) and (4.6), all the condition of the Theorem 3.8 are satisfied which implies the unique solution of the FBVP (4.1). Again, for any $k \in \mathcal{A}_{\mathcal{C}}$, from Lemma 3.7, the solution is given by

$$k(\zeta) = \int_0^2 G(\zeta,\theta)\left[\frac{e^{-2\zeta}}{5(2+e^{3\zeta})}\left(1 + k(\zeta) + {}^{C}D^{1.5}k(\zeta)\right)\right]\nabla\theta.$$

Setting $k(\zeta) = \zeta$, for $\zeta \in [0,1] \cup \{2\}$ then we obtain

$$\frac{1}{15} \le k(\zeta) \le \frac{2}{15}.$$

In view of the Corollary 3.7, $M_1 = \frac{1}{5},\ M_2 = \frac{2}{5}$. Hence, we say that the positive solution $k(\zeta)$ of the FBVP (4.1) satisfy the following condition:

$$\frac{1}{5}\int_0^2 G(\zeta,\theta)\nabla\theta \le k(\zeta) \le \frac{2}{5}\int_0^2 G(\zeta,\theta)\nabla\theta.$$

Thus, as an implimentation of the Theorem 3.6, the upper and lower solutions are given by

$$\overline{k(\zeta)} = \tfrac{2}{5}\int_0^2 G(\zeta,\theta)\nabla\theta \ \text{ and } \underline{k(\zeta)} = \tfrac{1}{5}\int_0^2 G(\zeta,\theta)\nabla\theta.$$

.

5 Conclusion

In this paper, we have discussed the existence and uniqueness of the positive solution of a fractional dynamic equation involving integral boundary condition on time scales with the newly developed Caputo derivative (see Gogoi et al. 2021) in the sense of nabla (∇) derivative on time scales. Our approach is based on the Schauder's fixed point theorem which allows to prove the existence of the required solution. For the gurantee of the unique solution we apply Banach fixed point theorem. One illustrative example is also given for better understanding the results. We believe the results presented here are employable in the mathematical modelling of hybrid continuous and discrete phenomena. Further, the involvment of nabla (∇) derivative gives significantly better accuracy in the modelling procss. Apart from this, we can say that the topic has potential application in population dynamics, Engineering sciences, Economics, etc.

The discussion of the qualitative properties of the solutions such as stability analysis, continuous dependency, etc. of nonlinear fractional dynamic equation with different types of boundary conditions will be our future work.

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