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RESEARCH PAPER

A Novel Operational Matrix Method for Solving the Fractional Delay Integro-Differential Equations with a Weakly Singular Kernel

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Abstract
In this work, we introduce a feasible and efficient method for solving weakly singular fractional pantograph delay integro-differential equations. To implement the proposed method, we get the operational matrices based on the shifted fractional-order fifth-kind Chebyshev polynomials. These matrices, together with the collocation method, are applied to convert the main equation to a system of algebraic equations. We consider the existence and uniqueness of solutions and then give an upper error bound for this method. At last, several numerical tests are carried out to demonstrate the usefulness and capability of the suggested algorithm.

Keywords Delay inetgro-differential equation · Collocation method · Operational matrix · Shifted fractional-order fifth-kind Chebyshev polynomials · Caputo fractional derivative · Riemann–Liouville integral · Weakly singular

1 Introduction

Fractional calculus is a part of mathematics that investigates derivative and integral operators of arbitrary orders. It is an attractive field of applied analysis having the aim of simulating biological issues in science (Ata and Kıymaz 2023; Jafari et al. 2023; Bhatter et al. 2024; Singh et al. 2023). The numerous properties of fractional operators have generated considerable interest in fractional calculus in recent years. As well as, it has provided a powerful tool for describing many physical phenomena so that nowadays, many researchers widely utilize fractional differential equations for modeling in engineering science and mathematics problems, you can see Podlubny (1998); Hilfer (2000); Sweilam et al. (2007); Khan and Atangana (2020); Atanackovi et al. (2014); Baleanu and Agarwal (2021). The fractional integro-differential equations have good applications for describing the physical phenomena in the real world system. The fractional delay integro-differential equations are a category of these equations which are of interest to scientists due to the better clarification of behavior of the real processes. Therefore, finding the solutions of the fractional delay integro-differential equations is very important, but most of them do not have analytical solutions or the calculations of the analytic solutions of these equations are hard and even impossible. Due to the practical application of these equations, getting a numerical solution is essential. In recent years, a good deal of the attempt has been devoted to the numerical solutions of the delay integro-differential equations. For example, In Rezabeyk et al. (2020), authors have used the operational matrices based on the fractional-order Euler polynomials to solve fractional-order delay integro-

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diferential equations. Doha and Ezz-Eldien applied a collocation spectral approach based on shifted Chebyshev polynomials to solve a general form of PVIDEs (Ezz-Eldien and Doha 2019). In Yang and Huang (2013), a spectral Jacobi-collocation approximation was proposed for fractional-order PVIDEs. The Legendre spectral collocation methods were used to approximate smooth solution PVIDEs (Yunxia and Yanping 2012). The Sinc collocation method was considered to obtain the numerical solution of PVIDEs (Zhao et al. 2017). A collocation method based on the Laguerre polynomials was presented in Yüzbaşı (2014) to solve PVIDEs under the initial conditions. In Bellour et al. (2020), an algorithm based on the Taylor polynomials was presented for approximating the solution of second-order linear delay differential and integro-differential equations. A numerical technique based on the Dickson polynomials was investigated for solving generalized delay integro-differential equations with functional bounds (Kürkçü et al. 2018). In this paper, we consider the following weakly singular fractional pantograph delay integro-differential equations (FPDIDEs) with proportional delay:

$$\begin{aligned}
\mathfrak{D}^{\gamma}y(x) &= \lambda_1 \int_0^x \frac{K_1(x,z)y(z)}{(x-z)^{\mu}}\,dz + \lambda_2 \int_0^x \frac{K_2(x,z)y(qz)}{(x-z)^{\nu}}\,dz \\
&\quad + \lambda_3 \int_0^{qx} K_3(x,z)y(z)\,dz \\
&\quad + g(x)y(qx) + p(x)y(x) + f(x), \qquad x \in [0,1], \\
&\quad y^{(j)}(0) = y_0^j, \qquad j = 0,1,\ldots,m-1, \\
&\quad m-1<\gamma \le m,
\end{aligned} \tag{1}$$

where $m = \lceil \gamma \rceil$ is the ceiling function of γ, $K_1(x,z), K_2(x,z), K_3(x,z)$ are continuous known functions defined on $[0,1]\times[0,1]$, $0<\mu,\nu<1$, $\lambda_1,\lambda_2,\lambda_3 \in \mathbb{R}$ are constants, $q \in (0,1)$, $\mathfrak{D}^{\gamma}$ is the Caputo fractional derivative operator, and $y(x)$ is an unknown function.

Since the fractional derivative is essentially a global differential operator, so using a global method is normal. Therefore, the spectral method for its global feature and high order accuracy is natural. For some of the important applications of these methods, see Singh et al. (2020); Babolian and Shamloo (2008); Sabermahani et al. (2020); Boyd (2001); Doha et al. (2019); Zheng and Chen (2021). The spectral collocation method presents the approximate solutions as a finite series of basic functions that are usually orthogonal polynomials. One of these orthogonal polynomials is the Chebyshev polynomials that have been extensively applied to solve various problems (Azevedo et al. 2020; Sahlan and Feyzollahzadeh 2017; Abd-Elhameed and Bassuony 2015; Abd-Elhameed and Youssri 2019). In Masjed-Jamei (2006), Masjed-Jamei introduced the new categories of the Chebyshev polynomials that recently have been used by a few authors Babaei et al. (2020); Abd-Elhameed and Youssri (2018); Atta et al. (2021); Abd-Elhameed and Youssri (2019). Less being considered the fifth-kind Chebyshev polynomials as basis functions and also the importance of the pantograph equations for modelling many phenomena motivate us to present a new algorithm for solving equation (1). Therefore, by applying the shifted fractional-order fifth-kind Chebyshev polynomials, we demonstrate their efficiency as a basis function. For this aim, first, by utilizing the combination of the collocation method with operational matrices, the main equation is converted to an algebraic equation. Then, by substituting roots of the $(N+1)$-th shifted fifth-kind Chebyshev polynomials as the collocation points, we get the algebraic system that can be solved by Newton's iterative method. The rest of the article is arranged as follows: In Sect. 2, we review some necessary definitions and properties of the fractional calculus. Then, we consider the existence and uniqueness of the problem under-study in Sect. 3. The shifted fractional-order fifth-kind Chebyshev polynomials are introduced in Sect. 4, and we get the operational matrices in Sect. 5. In Sect. 6, we present the numerical method and get the error bound in Sect. 7. In Sect. 8, we solve some numerical test examples for indicating the efficiency of the proposed method. At last, we present the main conclusions in Sect. 9.

2 Fractional Operators

In this section, we present some definitions and properties of fractional integral and derivative operators, which will be used later.

Definition 2.1 Let $\alpha > 0$, the operator $\mathfrak{J}^{\alpha}$, defined on $L^1[0,\infty)$ by Podlubny (1998); Nemati et al. (2016):

$$\mathfrak{J}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-z)^{\alpha-1}f(z)\,dz, \qquad \alpha>0,\ x>0.$$

is called the Riemann–Liouville fractional integral operator of order α.

Definition 2.2 Let $\alpha \in \mathbb{R}$, $n-1<\alpha\le n$, $n \in \mathbb{N}$, and $f(x) \in C^n[0,\infty)$, then the Caputo fractional derivative of order $\alpha > 0$ is defined by Podlubny (1998); Nemati et al. (2016) as:

$$\begin{cases} {}_0\mathfrak{D}_x^{\alpha}f(x) = \dfrac{1}{\Gamma(n-\alpha)}\displaystyle\int_0^x \frac{f^n(z)}{(x-z)^{\alpha+1-n}}\,dz, \\ f^{(n)}(z), \qquad \alpha = n, \end{cases}$$

where $\Gamma(x)$ is the Gamma function:

$$\Gamma(x) = \int_0^\infty e^{-z} z^{x-1}\, dz, \qquad \mathrm{Re}(z) > 0,$$
$$\Gamma(x+1) = x\Gamma(x),$$
$$B(u,v) = \int_0^1 z^{u-1}(1-z)^{v-1}\, dz = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}, \quad \mathrm{Re}(u), \mathrm{Re}(v) > 0.$$

The last integral is often called the Beta integral.

The Riemann–Liouville integral operator $\mathfrak{J}^\alpha$ and the Caputo fractional derivative operator $\mathfrak{D}^\alpha$ satisfy the following properties:

1. $\mathfrak{J}^{\alpha_1}(\mathfrak{J}^{\alpha_2} f(x)) = \mathfrak{J}^{\alpha_2}(\mathfrak{J}^{\alpha_1} f(x)) = \mathfrak{J}^{\alpha_1+\alpha_2} f(x)$,
2. $\mathfrak{J}^\alpha(c_1 f(x) + c_2 g(x)) = c_1 \mathfrak{J}^\alpha f(x) + c_2 \mathfrak{J}^\alpha g(x)$,
3. $\mathfrak{J}^\alpha(\mathfrak{D}^\alpha f(x)) = f(x) - \sum_{i=0}^{n-1} f^{(i)}(0) \frac{x^i}{i!}, \qquad n-1 < \alpha \le n, \ x > 0,$
4. $\mathfrak{D}^\alpha x^\gamma = \begin{cases} 0, & \alpha > \gamma, \\ \dfrac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} x^{\gamma-\alpha}, & \text{otherwise}, \end{cases}$
5. $\mathfrak{J}^\alpha x^v = \frac{\Gamma(v+1)}{\Gamma(v+\alpha+1)} x^{v+\alpha}, \qquad v > -1,$

where $\alpha, \alpha_1, \alpha_2, \gamma \in \mathbb{R}^+$ and $c_1, c_2 \in \mathbb{R}$.

3 Existence and Uniqueness of the Solution of Fractional-Order Delay Integro-Differential Equation

The main aim of the current section is to prove the uniqueness of the solution of Eq. (1). For this aim, we use the fixed point theorem. We denote Banach continuous functions by $C(J), J = [0,1]$ with the maximum norm as $\|g\|_\infty = \max_{x \in J} |g(x)|$ for $g \in C(J)$. Moreover, $C^{m,\sigma}(J), m \ge 0, \sigma \in [0,1]$ is the space of all functions whose $m-$th derivatives are Holder continuous with the exponent σ and equipped with the following norm:

$$\|y\|_{C^{m,\sigma}} = \max_{0 \le k \le m} \max_{x \in J} |y^{(k)}(x)| + \max_{0 \le k \le m} \sup_{x \ne t} \frac{|y^{(k)}(x) - y^{(k)}(t)|}{|x-t|^\sigma}.$$

Furthermore, suppose that B_r is a closed ball defined as $B_r = \{y(x) \in C^{m,\sigma}(J) \mid \|y\|_{C^{m,\sigma}} \le r\}$.

Now, we apply the Riemann–Liouville fractional integral operator of the order γ on Eq. (1) and get the following integral equation:

$$\begin{aligned} y(x) = G(x) &+ \frac{\lambda_1 \Gamma(1-\mu)}{\Gamma(\gamma-\mu+1)} \int_0^x (x-z)^{\gamma-\mu} K_1(x,z) y(z)\, dz \\ &+ \frac{\lambda_2 \Gamma(1-v)}{\Gamma(\gamma-v+1)} \int_0^x (x-z)^{\gamma-v} K_2(x,z) y(qz)\, dz \\ &+ \lambda_3 \mathfrak{J}^\gamma \Big(\int_0^{qx} K_3(x,z) y(z)\, dz \Big) \\ &+ \frac{1}{\Gamma(\gamma)} \int_0^x (x-z)^{\gamma-1} g(z) y(qz)\, dz \\ &+ \frac{1}{\Gamma(\gamma)} \int_0^x (x-z)^{\gamma-1} p(z) y(z)\, dz \\ &+ \frac{1}{\Gamma(\gamma)} \int_0^x (x-z)^{\gamma-1} f(z)\, dz, \end{aligned}$$

where $G(x) = \sum_{k=0}^{m-1} \frac{y^{(k)}(0)}{k!} x^k$.

Theorem 3.1 *If* $\frac{|\lambda_1|\Gamma(1-\mu)M_1}{\Gamma(\gamma-\mu+2)} + \frac{|\lambda_2|\Gamma(1-v)M_2}{\Gamma(\gamma-v+2)} + \frac{|\lambda_3| q M_3}{\Gamma(\gamma+2)} + \frac{(N_1+N_2)}{\Gamma(\gamma+1)} \le \eta_1 < 1$ *then fractional-order delay integro-differential Eq.* (1) *has a unique solution* (*Biazar and Sadri* 2019) .

Proof Suppose that $V = C^{m,\sigma}(J)$ and we define the mapping $\Xi y(x) : V \longrightarrow V$ as follows:

$$\begin{aligned} \Xi y(x) = G(x) &+ \frac{\lambda_1 \Gamma(1-\mu)}{\Gamma(\gamma-\mu+1)} \int_0^x (x-z)^{\gamma-\mu} K_1(x,z) y(z)\, dz \\ &+ \frac{\lambda_2 \Gamma(1-v)}{\Gamma(\gamma-v+1)} \int_0^x (x-z)^{\gamma-v} K_2(x,z) y(qz)\, dz \\ &+ \lambda_3 \mathfrak{J}^\gamma \Big(\int_0^{qx} K_3(x,z) y(z)\, dz \Big) \\ &+ \frac{1}{\Gamma(\gamma)} \int_0^x (x-z)^{\gamma-1} g(z) y(qz)\, dz \\ &+ \frac{1}{\Gamma(\gamma)} \int_0^x (x-z)^{\gamma-1} p(z) y(z)\, dz \\ &+ \frac{1}{\Gamma(\gamma)} \int_0^x (x-z)^{\gamma-1} f(z)\, dz. \end{aligned}$$

To use the fixed point theorem, we must show that Ξ has a fixed point. For this purpose, we must prove that $\Xi B_r \subseteq B_r$ while $r \ge \eta_2(\|G\|_\infty + \frac{F}{\Gamma(\gamma+1)})$ where $\eta_2 \ne 0, \frac{1}{\eta_2} + \eta_1 < 1, F = \max_{x \in J} \|f(x)\|_\infty$ and set $\|K_1\|_\infty \le M_1$, $\|K_2\|_\infty \le M_2$, $\|K_3\|_\infty \le M_3$, $\|g\|_\infty \le N_1$, $\|p\|_\infty \le N_2$. So we have:

$$\begin{aligned}
\|\Xi y\|_{C^{m,\sigma}} &\le \|G\|_\infty + \frac{1}{\Gamma(\gamma)}\int_0^x |x-z|^{\gamma-1}\|f\|_\infty\,dz \\
&\quad + \frac{|\lambda_1|\Gamma(1-\mu)}{\Gamma(1-\mu+\gamma)}\int_0^x |x-z|^{\gamma-\mu}\|K_1\|_\infty\|y\|_{C^{m,\sigma}}\,dz \\
&\quad + \frac{|\lambda_2|\Gamma(1-v)}{\Gamma(1-v+\gamma)}\int_0^x |x-z|^{\gamma-v}\|K_2\|_\infty\|y\|_{C^{m,\sigma}}\,dz \\
&\quad + |\lambda_3|\Im^\gamma\Big(\int_0^{qx}\|K_3\|_\infty\|y\|_{C^{m,\sigma}}\,dz\Big) \\
&\quad + \frac{1}{\Gamma(\gamma)}\int_0^x |x-z|^{\gamma-1}\|g\|_\infty\|y\|_{C^{m,\sigma}}\,dz \\
&\quad + \frac{1}{\Gamma(\gamma)}\int_0^x |x-z|^{\gamma-1}\|p\|_\infty\|y\|_{C^{m,\sigma}}\,dz \\
&\le \Big(\|G\|_\infty + \frac{F}{\Gamma(\gamma+1)}\Big) \\
&\quad + \Big(\frac{|\lambda_1|\Gamma(1-\mu)M_1}{\Gamma(\gamma-\mu+2)} + \frac{|\lambda_2|\Gamma(1-v)M_2}{\Gamma(\gamma-v+2)} \\
&\quad + \frac{|\lambda_3|M_3 q}{\Gamma(\gamma+2)} + \frac{(N_1+N_2)}{\Gamma(\gamma+1)}\Big)r \\
&\le \frac{r}{\eta_2} + \eta_1 r \le r.
\end{aligned}$$

Thus, Ξ maps B_r into itself, so we show that this map has a fixed point. For $y_1(x), y_2(x) \in V$, we have:

$$\begin{aligned}
&\|\Xi y_1 - \Xi y_2\|_{C^{m,\sigma}} \\
&\quad\le \frac{|\lambda_1|\Gamma(1-\mu)}{\Gamma(\gamma-\mu+1)}\int_0^x |x-z|^{\gamma-\mu}\|K_1\|_\infty\|y_1-y_2\|_{C^{m,\sigma}}\,dz \\
&\quad + \frac{|\lambda_2|\Gamma(1-v)}{\Gamma(\gamma-v+1)}\int_0^x |x-z|^{\gamma-v}\|K_2\|_\infty\|y_1-y_2\|_{C^{m,\sigma}}\,dz \\
&\quad + |\lambda_3|\Im^\gamma\Big(\int_0^{qx}\|K_3\|_\infty\|y_1-y_2\|_{C^{m,\sigma}}\,dz\Big)\Big) \\
&\quad + \frac{1}{\Gamma(\gamma)}\int_0^x |x-z|^{\gamma-1}\|g\|_\infty\|y_1-y_2\|_{C^{m,\sigma}}\,dz \\
&\quad + \frac{1}{\Gamma(\gamma)}\int_0^x |x-z|^{\gamma-1}\|p\|_\infty\|y_1-y_2\|_{C^{m,\sigma}}\,dz \\
&\quad\le \Big(\frac{|\lambda_1|\Gamma(1-\mu)M_1}{\Gamma(\gamma-\mu+2)} + \frac{|\lambda_2|\Gamma(1-v)M_2}{\Gamma(\gamma-v+2)} \\
&\quad + \frac{|\lambda_3|M_3 q}{\Gamma(\gamma+2)} + \frac{(N_1+N_2)}{\Gamma(\gamma+1)}\Big)\|y_1-y_2\|_{C^{m,\sigma}}.
\end{aligned}$$

According to the assumption of the theorem we have $\frac{|\lambda_1|\Gamma(1-\mu)M_1}{\Gamma(\gamma-\mu+2)} + \frac{|\lambda_2|\Gamma(1-v)M_2}{\Gamma(\gamma-v+2)} + \frac{|\lambda_3|qM_3}{\Gamma(\gamma+2)} + \frac{(N_1+N_2)}{\Gamma(\gamma+1)} \le \eta_1 < 1$, the operator Ξ is a contraction mapping, therefor a unique fixed point $y(x) \in B_r$ exists such that $\Xi y(x) = y(x)$. □

4 Fractional-Order Fifth-Kind Chebyshev Polynomials

In this section, we introduce the shifted fifth-kind Chebyshev polynomials (SFKCP). Then, we present relations associated with the shifted fifth-kind Chebyshev polynomials of the fractional order.

4.1 Shifted Fifth-Kind Chebyshev Polynomials

The shifted fifth-kind Chebyshev polynomials are defined on the interval [0, 1]. These polynomials are orthogonal with the weight function $w(t) = (2t-1)^2/\sqrt{t-t^2}$ which these polynomials are determined by the recurrence relation as follows:

$$\begin{aligned}
F^*_{j+1}(t) &= (2t-1)F^*_j(t) \\
&\quad - \frac{(j-1)^2 + j + (-1)^j(2j-1)}{4j(j-1)}F^*_{j-1}(t), \qquad (2) \\
&\quad j \ge 1, \quad t \in [0,1], \\
F^*_0(t) &= 1, \qquad F^*_1(t) = 2t-1,
\end{aligned}$$

And the orthogonality relation of Chebyshev polynomials is as:

$$\int_0^1 F^*_i(t)F^*_j(t)w(t)\,dt = \mathfrak{h}_i\delta_{ij},$$

where

$$\mathfrak{h}_i = \begin{cases} \dfrac{\pi}{2^{2i+1}}, & i \text{ even}, \\ \dfrac{\pi(i+2)}{i2^{2i+1}}, & i \text{ odd}. \end{cases} \qquad (3)$$

The analytic form of the fifth-kind Chebyshev polynomials is shown as the following series:

$$F^*_j(t) = \sum_{r=0}^{j}\xi_{r,j}t^r, \qquad (4)$$

$$\xi_{r,j} = \frac{2^{2r-j}}{(2r)!}\begin{cases} 2\displaystyle\sum_{i=\lfloor\frac{r+1}{2}\rfloor}^{\frac{j}{2}} \frac{(-1)^{\frac{j}{2}+m-r}m\delta_m\,(2m+r-1)!}{(2m-r)!}, & j \text{ even}, \\ \dfrac{1}{j}\displaystyle\sum_{i=\lfloor\frac{r}{2}\rfloor}^{\frac{j-1}{2}} \frac{(-1)^{\frac{j+1}{2}+m-r}(2m+1)^2(2m+r)!}{(2m-r+1)!}, & j \text{ odd}, \end{cases} \qquad (5)$$

where

$$\delta_m = \begin{cases} \dfrac{1}{2}, & m = 0, \\ 1, & m > 0, \end{cases}$$

And $\xi_{0,2j} = \frac{1}{2^j}$ for $j = 0, 1, 2....$

4.2 Shifted Fractional-Order Fifth-Kind Chebyshev Polynomials

The shifted fractional-order fifth-kind Chebyshev polynomials (SFFKCP) are obtained by changing t to $x^{\sigma}(\sigma > 0)$ dependent on the shifted fifth-kind Chebyshev polynomials. We denote the fractional-order fifth-kind Chebyshev polynomials, $F_i^*(x^{\sigma})$ by $F_i^{(\sigma)}(x)$. It is clear that the shifted fifth-kind Chebyshev polynomials are got for $\sigma = 1$. The weight function is $w^{(\sigma)}(x) = \sigma x^{\sigma-1}(2x^{\sigma} - 1)^2/\sqrt{x^{\sigma} - x^{2\sigma}}$. Additionally, the relations (2)–(5) are changed over to:

$$\begin{aligned} &F_{j+1}^{(\sigma)}(x) \\ &\quad = (2x^{\sigma} - 1)F_j^{(\sigma)}(x) \\ &\qquad - \frac{(j-1)^2 + j + (-1)^j(2j-1)}{4j(j-1)} F_{j-1}^{(\sigma)}(x), \\ &\qquad j \geq 1, \quad x \in [0, 1], \\ &F_0^{(\sigma)}(x) = 1, \qquad F_1^{(\sigma)}(x) = 2x^{\sigma} - 1, \\ &\int_0^1 F_i^{(\sigma)}(x)F_j^{(\sigma)}(x)w^{(\sigma)}(x)\,dx = \mathfrak{h}_i\delta_{ij}, \end{aligned} \tag{6}$$

and the analytic form of these equations is:

$$F_j^{(\sigma)}(x) = \sum_{r=0}^{j} \xi_{r,j}\, x^{\sigma r},$$

where $\mathfrak{h}_i$ and $\xi_{r,j}$ are the same in (3) and (5), respectively. We can approximate a continuous function y over the interval [0, 1] by a finite series of the shifted fractional-order fifth-kind Chebyshev polynomials as follows:

$$y(x) \approx \sum_{j=0}^{N} C_j F_j^{(\sigma)}(x) = \mathbf{C}^T\mathbf{F}^{(\sigma)}(x) = \mathbf{F}^{(\sigma)T}(x)\mathbf{C},$$

where

$$\mathbf{F}^{(\sigma)}(x) = \left[F_0^{(\sigma)}(x), F_1^{(\sigma)}(x), \ldots, F_N^{(\sigma)}(x)\right]^T, \quad \mathbf{C} = \left[C_0, C_1, \ldots, C_N\right]^T,$$

Such thatthe coefficients C_j are obtained by:

$$C_j = \frac{1}{\mathfrak{h}_j}\int_0^1 y(x)F_j^{(\sigma)}(x)w^{(\sigma)}(x)\,dx, \tag{7}$$

And $\mathfrak{h}_j$ is defined in Eq. (3). Also, we can express any continuous two-variable functions, say $K(x, z)$, defined on the domain $[0, 1] \times [0, 1]$ in the finite series of the shifted fractional-order fifth-kind Chebyshev polynomials as follows:

$$K_N(x, z) \approx \sum_{i=0}^{N}\sum_{j=0}^{N} \mathbf{K}_{ij}F_i^{(\sigma)}(x)F_j^{(\sigma)}(z) = \mathbf{F}^{(\sigma)T}(x)\mathbf{K}\mathbf{F}^{(\sigma)}(z),$$

where $\mathbf{K}$ is a $(N+1) \times (N+1)$ matrix and its entries are obtained as follows:

$$\mathbf{K}_{ij} = \frac{1}{\mathfrak{h}_i\mathfrak{h}_j}\int_0^1\int_0^1 K(x, z)F_i^{(\sigma)}(x)F_j^{(\sigma)}(z)w^{(\sigma)}(x)w^{(\sigma)}(z)\,dxdz, \quad i, j = 0, 1, 2, \ldots, N.$$

5 Operational Matrices

In this section, we present how to obtain the operational matrices for all of the terms in (1).

5.1 The Integral Operational Matrix

To derive the integral operational matrix, we first apply the Riemann–Liouville integral operator to SFFKCPs analytic form as follows:

$$\begin{aligned} \mathfrak{I}^{\gamma}(F_i^{(\sigma)}(x)) &= \mathfrak{I}^{\gamma}\left(\sum_{l=0}^{i}\xi_{l,i}x^{\sigma l}\right) \\ &= \sum_{l=0}^{i}\xi_{l,i}\frac{\Gamma(\sigma l + 1)}{\Gamma(\sigma l + \gamma + 1)}x^{\sigma l+\gamma}, \end{aligned} \tag{8}$$

Then, we expand $x^{\sigma l+\gamma}$ in terms of the shifted fractional-order fifth-kind Chebyshev polynomials:

$$x^{\sigma l+\gamma} \approx \sum_{j=0}^{N} C_{l,j}F_j^{(\sigma)}(x),$$

Using orthogonality properties of SFFKCPs, the coefficients $C_{l,j}$ are gotten by Eq. (7) as below:

$$C_{l,j} = \frac{1}{\mathfrak{h}_j}\int_0^1 x^{\sigma l+\gamma}F_j^{(\sigma)}(x)w^{(\sigma)}\,dx,$$

By calculating coefficients $C_{l,j}$ and replacing the approximate value of $x^{\sigma l+\gamma}$ in Eq. (8), we have:

$$\mathfrak{I}^{\gamma}(F_i^{(\sigma)}(x)) \approx \sum_{j=0}^{N}\Bigg\{\sum_{l=0}^{i}\xi_{l,i}\frac{\Gamma(\sigma l+1)\sqrt{\pi}}{\Gamma(\sigma l+\gamma+1)\mathfrak{h}_j} \times \sum_{k=0}^{j}\xi_{k,j}\Bigg[\frac{4\Gamma(l+k+\gamma/\sigma+5/2)}{\Gamma(l+k+\gamma/\sigma+3)} - \frac{4\Gamma(l+k+\gamma/\sigma+3/2)}{\Gamma(l+k+\gamma/\sigma+2)} + \frac{\Gamma(k+l+\gamma/\sigma+1/2)}{\Gamma(l+k+\gamma/\sigma+1)}\Bigg]\Bigg\}F_j^{(\sigma)}(x),$$

$$= \sum_{j=0}^{N} p(i,j)F_j^{(\sigma)}(x).$$

Let $\mathbf{F}^{(\sigma)}(x)$ be the SFFKCPs vector, we can rewrite the last relation in the matrix form as follows:

$$\mathfrak{I}^{\gamma}(\mathbf{F}^{(\sigma)}(x)) \approx P^{(\gamma)}\mathbf{F}^{(\sigma)}(x), \tag{9}$$

where $P^{(\gamma)}$ is the integral operational matrix with the following entries:

$$P_{i,j}^{(\gamma)} = \sum_{l=0}^{i}\xi_{l,i}\frac{\Gamma(\sigma l+1)\sqrt{\pi}}{\Gamma(\sigma l+\gamma+1)\mathfrak{h}_j} \sum_{k=0}^{j}\xi_{k,j}\Bigg[\frac{4\Gamma(l+k+\gamma/\sigma+5/2)}{\Gamma(l+k+\gamma/\sigma+3)} - \frac{4\Gamma(l+k+\gamma/\sigma+3/2)}{\Gamma(l+k+\gamma/\sigma+2)} + \frac{\Gamma(k+l+\gamma/\sigma+1/2)}{\Gamma(l+k+\gamma/\sigma+1)}\Bigg].$$

5.2 The Product Operational Matrix

To obtain the product operational matrix, consider $F_i^{(\sigma)}(x), F_j^{(\sigma)}(x), F_k^{(\sigma)}(x)$ that are ith, jth, kth shifted fractional-order fifth-kind Chebyshev polynomials, respectively. We can obtain the product of $F_j^{(\sigma)}(x)$ and $F_k^{(\sigma)}(x)$ as follows:

$$\mathcal{Q}_{j+k}^{(\sigma)}(x) = F_j^{(\sigma)}(x)F_k^{(\sigma)}(x) = \sum_{r=0}^{j+k}\lambda_r^{(j,k)}x^{\sigma r}, \tag{10}$$

where the coefficients $\lambda_r^{(j,k)}$ are determined by the following algorithm:

$If\ j \geq k:$
$r = 0,1,\ldots,j+k,$
if $r > j$ **then**
$\quad \lambda_r^{(j,k)} = \sum_{l=r-j}^{k}\xi_{r-l,j}\,\xi_{l,k},$
else
$\quad \hat{r} = \min\{r,k\}$
$\quad \lambda_r^{(j,k)} = \sum_{l=0}^{\hat{r}}\xi_{r-l,j}\,\xi_{l,k},$
end if.
$If\ j < k:$
$r = 0,1,\ldots,j+k,$
if $r \leq j$ **then**
$\quad \hat{r} = \min\{r,j\},$
$\quad \lambda_r^{(j,k)} = \sum_{l=0}^{\hat{r}}\xi_{r-l,j}\,\xi_{l,k},$
else
$\quad \tilde{r} = \min\{r,k\}$
$\quad \lambda_r^{(j,k)} = \sum_{l=k-j}^{\tilde{r}}\xi_{r-l,j}\,\xi_{l,k},$
end if.

The quantities $\xi_{l,k}$ and $\xi_{r-l,j}$ are the coefficients in the analytic form of $F_k^{(\sigma)}(x)$ and $F_j^{(\sigma)}(x)$ that were introduced in Eq. (5). Now, suppose V is a $(N+1)-$vector and let $\mathbf{F}^{(\sigma)}(x)\mathbf{F}^{(\sigma)T}(x)V \approx \widetilde{V}\mathbf{F}^{(\sigma)}(x)$ where $\widetilde{V}$ is the $(N+1)\times(N+1)$ product operational matrix of SFFKCPs and its entries are obtained by

$$\widetilde{V}_{jk} = \frac{1}{\mathfrak{h}_k}\sum_{i=0}^{N}V_i\,s_{ijk},$$

where V_i is the element of the vector V and s_{ijk} is got as follows:

$$s_{ijk} = \int_0^1 F_i^{(\sigma)}(x)F_j^{(\sigma)}(x)F_k^{(\sigma)}(x)w^{(\sigma)}(x)\,dx.$$

Using Eq. (10), we have:

$$s_{ijk} = \sum_{r=0}^{j+k}\lambda_r^{(j,k)}\int_0^1 x^{\sigma r}F_i^{\sigma}(x)w^{(\sigma)}(x)\,dx,$$

By substituting the analytic form of SFFKCPs in the above integral and calculating the integral, we have:

$$s_{ijk} = \sum_{r=0}^{j+k}\lambda_r^{(j,k)}\sum_{l=0}^{i}\xi_{l,i}\sqrt{\pi}\Bigg[\frac{4\Gamma(r+l+5/2)}{\Gamma(r+l+3)} - \frac{4\Gamma(r+l+3/2)}{\Gamma(r+l+2)} + \frac{\Gamma(r+l+1/2)}{\Gamma(r+l+1)}\Bigg].$$

5.3 Approximating the Basis Functions Including the Delay

Here, we consider delay function $h:[0,1]\longrightarrow[0,1]$ as $h(x)=qx$. To find the relation between basic functions including the delay and the desired basic functions, we assume:

$$\mathbf{F}^{(\sigma)}(qx)=\left[F_0^{(\sigma)}(qx),F_1^{(\sigma)}(qx),\ldots,F_N^{(\sigma)}(qx)\right]^T,$$

where

$$F_j^{(\sigma)}(qx)=\sum_{r=0}^{j}\xi_{r,j}(qx)^{\sigma r}.$$

Now, we approximate $h(x)=qx$ using SFFKCPs as follows:

$$(qx)^{\sigma r}\approx\sum_{j=0}^{N}a_{pj}F_j^{(\sigma)}(x),$$

where a_{pj} are gotten by applying Eq. (7), thus we have:

$$a_{pj}(x)=\frac{q^{\sigma r}}{\mathfrak{h}_j}\int_0^1 x^{\sigma r}F_j^{(\sigma)}(x)w^{(\sigma)}(x)\,dx. \tag{11}$$

By substituting the analytic form of SFFKCPs and performing integral (11), we obtain:

$$\begin{aligned}F_i^{(\sigma)}(qx)&\approx\sum_{j=0}^{N}\Bigg\{\sum_{r=0}^{i}\xi_{r,i}\frac{q^{\sigma r}\sqrt{\pi}}{\mathfrak{h}_j}\\&\sum_{m=0}^{j}\xi_{m,j}\Bigg[\frac{4\Gamma(r+m+5/2)}{\Gamma(r+m+3)}\\&-\frac{4\Gamma(r+m+3/2)}{\Gamma(r+m+2)}+\frac{\Gamma(r+m+1/2)}{\Gamma(r+m+1)}\Bigg]\Bigg\}F_j^{(\sigma)}(x)\\&=\sum_{j=0}^{N}d_{ij}F_j^{(\sigma)}(x),\qquad i=0,1,\ldots,N.\end{aligned}$$

Also, the last relation can be rewritten as a matrix form as follows:

$$\mathbf{F}^{(\sigma)}(qx)\approx D\mathbf{F}^{(\sigma)}(x), \tag{12}$$

where

$$D=\begin{bmatrix}d_{00}&d_{01}&\ldots&d_{0N}\\d_{10}&d_{11}&\ldots&d_{1N}\\\vdots&\vdots&\ddots&\vdots\\d_{N0}&d_{N1}&\ldots&d_{NN}\end{bmatrix},$$

And its entries are as below:

$$\begin{aligned}D_{i,j}=\sum_{r=0}^{i}\xi_{r,i}\frac{q^{\sigma r}\sqrt{\pi}}{\mathfrak{h}_j}\sum_{m=0}^{j}\xi_{m,j}\Bigg[\frac{4\Gamma(r+m+5/2)}{\Gamma(r+m+3)}\\-\frac{4\Gamma(r+m+3/2)}{\Gamma(r+m+2)}+\frac{\Gamma(r+m+1/2)}{\Gamma(r+m+1)}\Bigg],\quad i,j=0,1,\ldots,N.\end{aligned}$$

5.4 Approximating the Integral Part with a Singular Kernel

Now, we present a matrix form of the integral with the singular kernel in Eq. (1). For this aim, suppose that $\mathbf{F}^{(\sigma)}(x)=\left[F_0^{(\sigma)}(x),F_1^{(\sigma)}(x),\ldots,F_N^{(\sigma)}(x)\right]^T$ is the SFFKCPs vector and $0\le\mu<1$. Then, we have:

$$\begin{aligned}\int_0^x\frac{\mathbf{F}^{(\sigma)T}(z)}{(x-z)^{\mu}}\,dz&=\Bigg[\sum_{m=0}^{0}\xi_{m,0}\int_0^x\frac{z^{\sigma m}}{(x-z)^{\mu}}\,dz,\\&\ldots,\sum_{m=0}^{N}\xi_{m,N}\int_{m=0}^{x}\frac{z^{\sigma m}}{(x-z)^{\mu}}\,dz\Bigg]\\&=\Bigg[\sum_{m=0}^{0}\xi_{m,0}\frac{\Gamma(\sigma m+1)\Gamma(1-\mu)}{\Gamma(\sigma m-\mu+2)}x^{\sigma m-\mu+1},\\&\ldots,\sum_{m=0}^{N}\xi_{m,N}\frac{\Gamma(\sigma m+1)\Gamma(1-\mu)}{\Gamma(\sigma m-\mu+2)}x^{\sigma m-\mu+1}\Bigg].\end{aligned}$$

We approximate $x^{\sigma m-\mu+1}$ in terms of SFFKCPs:

$$x^{\sigma m-\mu+1}\approx\sum_{j=0}^{N}a_{lj}F_j^{(\sigma)}(x),$$

where

$$a_{lj}=\frac{1}{\mathfrak{h}_j}\int_0^1 x^{\sigma m-\mu+1}F_j^{(\sigma)}(x)w^{(\sigma)}(x)\,dx.$$

According to the analytic form of SFFKCPs and $w^{(\sigma)}(x)=\sigma x^{\sigma-1}(2x^{\sigma}-1)^2/\sqrt{x^{\sigma}-x^{2\sigma}}$, we have:

$$a_{lj}=\frac{1}{\mathfrak{h}_j}\sum_{k=0}^{j}\xi_{k,j}\int_0^1 x^{\sigma m-\mu+1}\sigma x^{\sigma-1}\frac{(2x^{\sigma}-1)^2}{\sqrt{x^{\sigma}-x^{2\sigma}}}\,dx,$$

Using the definition of the Beta function to calculate integral, we obtain:

$$\begin{aligned}&\sum_{m=0}^{i}\xi_{m,i}\frac{\Gamma(\sigma m+1)\Gamma(1-\mu)}{\Gamma(\sigma m-\mu+2)}x^{\sigma m-\mu+1}\\&\approx\sum_{j=0}^{N}\Bigg\{\sum_{m=0}^{i}\xi_{m,i}\frac{\Gamma(\sigma m+1)\Gamma(1-\mu)\sqrt{\pi}}{\Gamma(\sigma m-\mu+2)\mathfrak{h}_j}\\&\times\sum_{k=0}^{j}\xi_{k,j}\Bigg[\frac{4\Gamma(m+k+5/2+\frac{1-\mu}{\sigma})}{\Gamma(m+k+3+\frac{1-\mu}{\sigma})}\\&-\frac{4\Gamma(m+k+3/2+\frac{1-\mu}{\sigma})}{\Gamma(m+k+2+\frac{1-\mu}{\sigma})}+\frac{\Gamma(m+k+1/2+\frac{1-\mu}{\sigma})}{\Gamma(m+k+1+\frac{1-\mu}{\sigma})}\Bigg]\Bigg\}F_j^{(\sigma)}(x)\\&=\sum_{j=0}^{N}b_{ij}F_j^{(\sigma)}(x),\qquad i=0,1,\ldots,N,\end{aligned}$$

Thus, we get:

$$\int_0^x \frac{\mathbf{F}^{(\sigma)T}(z)}{(x-z)^\mu}\,dz \approx B^{(\mu)}\mathbf{F}^{(\sigma)}(x), \tag{13}$$

where $B^{(\mu)}$ is an $(N+1)\times(N+1)$ operational matrix with the following entries:

$$B_{i,j}^{(\mu)} = \sum_{m=0}^{i} \xi_{m,i} \frac{\Gamma(\sigma m+1)\Gamma(1-\mu)\sqrt{\pi}}{\Gamma(\sigma m-\mu+2)\mathfrak{h}_j} \times \sum_{k=0}^{j} \xi_{k,j} \Big[\frac{4\Gamma(m+k+5/2+\frac{1-\mu}{\sigma})}{\Gamma(m+k+3+\frac{1-\mu}{\sigma})} - \frac{4\Gamma(m+k+3/2+\frac{1-\mu}{\sigma})}{\Gamma(m+k+2+\frac{1-\mu}{\sigma})} + \frac{\Gamma(m+k+1/2+\frac{1-\mu}{\sigma})}{\Gamma(m+k+1+\frac{1-\mu}{\sigma})} \Big].$$

6 Application of the Operational Matrices

In this section, we explain how the fractional-order delay integro-differential equations are solved using the obtained operational matrices and presenting appropriate approximations. To implement the method, we first consider Eq. (1) under the given initial conditions. We approximate the function $\mathfrak{D}^\gamma y(x)$ in a matrix form as follows:

$$\mathfrak{D}^\gamma y(x) \approx \mathbf{F}^{(\sigma)T}(x)\mathbf{C}, \tag{14}$$

Also, we approximate the known function $G(x)=\sum_{k=0}^{m-1}\frac{y^{(k)}(0)}{k!}x^k$, we have:

$$G(x) \approx \mathbf{F}^{(\sigma)T}(x)\mathfrak{E}, \qquad \mathfrak{E}_i = \frac{1}{\mathfrak{h}_i}\int_0^1 G(x)\,F_i^{(\sigma)}(x)w^{(\sigma)}(x)\,dx. \tag{15}$$

Then, we apply properties of the Caputo fractional derivative operator and using the integral operational matrix $P^{(\gamma)}$ that has been obtained in the previous section. Thus, we get:

$$y(x) \approx \mathbf{F}^{(\sigma)T}(x)P^{(\gamma)T}\mathbf{C} + \mathbf{F}^{(\sigma)T}(x)\mathfrak{E} = \mathbf{F}^{(\sigma)T}(x)U. \tag{16}$$

By utilizing approximations obtained in Sect. 5, we have for the remaining terms:

$$\begin{aligned} K_1(x,z) &\approx \mathbf{F}^{(\sigma)T}(x)K_1\mathbf{F}^{(\sigma)}(z),\\ K_2(x,z) &\approx \mathbf{F}^{(\sigma)T}(x)K_2\mathbf{F}^{(\sigma)}(z),\\ K_3(x,z) &\approx \mathbf{F}^{(\sigma)T}(x)K_3\mathbf{F}^{(\sigma)}(z),\\ g(x)y(qx) &\approx g(x)\mathbf{F}^{(\sigma)T}(x)DU\\ p(x)y(x) &\approx p(x)\mathbf{F}^{(\sigma)T}(x)U. \end{aligned} \tag{17}$$

Now, using approximations (17), we get:

$$\begin{aligned} \int_0^x \frac{K_1(x,z)y(z)}{(x-z)^\mu}\,dz &\approx \int_0^x \frac{\mathbf{F}^{(\sigma)T}(x)K_1\mathbf{F}^{(\sigma)}(z)\mathbf{F}^{(\sigma)T}(z)U}{(x-z)^\mu}\,dz\\ &\approx \mathbf{F}^{(\sigma)}(x)K_1\widetilde{U}\int_0^x \frac{\mathbf{F}^{(\sigma)}(z)}{(x-z)^\mu}\,dz\\ &\approx \mathbf{F}^{(\sigma)T}(x)K_1\widetilde{U}B^{(\mu)}\mathbf{F}^{(\sigma)}(x), \end{aligned} \tag{18}$$

Such that $\widetilde{U}$ is the operational matrix of the product corresponding to the vector U and $B^{(\mu)}$ is the matrix introduced in (13). Similarly, using the approximations in (12) and (17), we have:

$$\begin{aligned} \int_0^x \frac{K_2(x,z)y(qz)}{(x-z)^v}\,dz &\approx \int_0^x \frac{\mathbf{F}^{(\sigma)T}(x)K_2\mathbf{F}^{(\sigma)}(z)\mathbf{F}^{(\sigma)T}(z)U_1}{(x-z)^v}\,dz\\ &\approx \mathbf{F}^{(\sigma)}(x)K_2\widetilde{U}_1\int_0^x \frac{\mathbf{F}^{(\sigma)}(z)}{(x-z)^v}\,dz\\ &\approx \mathbf{F}^{(\sigma)T}(x)K_2\widetilde{U}_1B^{(v)}\mathbf{F}^{(\sigma)}(x),\\ U_1 &= D^TU. \end{aligned} \tag{19}$$

Also, we can approximate the integral $\int_0^{qx}K_3(x,z)y(z)\,dz$ as follows:

$$\int_0^{qx} K_3(x,z)y(z)\,dz \approx \mathbf{F}^{(\sigma)T}(x)K_3\widetilde{U}M\mathbf{F}^{(\sigma)}(x), \tag{20}$$

where M is a $(N+1)\times(N+1)$ matrix and its entries are obtained from the following relation:

$$\begin{aligned} M_{ij} = \sum_{k=0}^{i}\frac{\xi_{k,i}\sqrt{\pi}}{(\sigma k+1)}\frac{q^{\sigma k+1}}{\mathfrak{h}_j}\sum_{m=0}^{j}\xi_{m,j} &\times \Big[\frac{4\Gamma(k+m+5/2+1/\sigma)}{\Gamma(k+m+3+1/\sigma)} - \frac{4\Gamma(k+m+3/2+1/\sigma)}{\Gamma(k+m+2+1/\sigma)} + \frac{\Gamma(k+m+1/2+1/\sigma)}{\Gamma(k+m+1+1/\sigma)}\Big],\\ & i,j=0,1,\ldots,N. \end{aligned}$$

By substituting approximations (18)–(20) into Eq. (1), we calculate the residual function as follows:

$$\begin{aligned} &\mathbf{F}^{(\sigma)T}(x)\mathbf{C} - \lambda_1\mathbf{F}^{(\sigma)T}(x)K_1\widetilde{U}B^{(\mu)}\mathbf{F}^{(\sigma)}(x)\\ &- \lambda_2\mathbf{F}^{(\sigma)T}(x)K_2\widetilde{U}_1B^{(v)}\mathbf{F}^{(\sigma)}(x)\\ &- \lambda_3\mathbf{F}^{(\sigma)T}(x)K_3\widetilde{U}M\mathbf{F}^{(\sigma)}(x)\\ &- g(x)\mathbf{F}^{(\sigma)T}(x)DU - p(x)\mathbf{F}^{(\sigma)T}(x)U - f(x) \approx 0. \end{aligned} \tag{21}$$

We choose roots of the $(N+1)$-th shifted fifth-kind Chebyshev polynomials as the collocation points and by solving the resultant algebraic system by Newton's iterative method, therefore, we can determine the vector $\mathbf{C}$. Finally, we substitute the vector $\mathbf{C}$ into Eq. (16) and get an approximation for $y(x)$.

7 Error Analysis

In this section, we first present some theorems and afterward we get an upper error bound for the proposed method. For this aim, we use the following norm:

$$\|f\|_{L^2} = \left(\int_0^1 |f(x)|^2 w^{(\sigma)}(x)\, dx \right)^{1/2}.$$

Theorem 7.1 *If $f(x) \in C[0,1]$ is any continuous function, then we can expand it as SFFKCPs on the interval* [0, 1], *i.e.* $f_N(x) = \sum_{i=0}^{N} C_i F_i^{(\sigma)}(x)$. *In this case, the error bound for coefficients C_i, $i = 0, 1, \ldots, N$, are determined as follows:* (*Szego* 1975)

$$|C_i| \le \frac{M_f \sqrt{\pi}}{\mathfrak{h}_i} \sum_{l=0}^{i} \xi_{l,i} \frac{\Gamma(l+1/2)}{\Gamma(l+3)} (l^2 + l + 1), \tag{22}$$

where M_f denotes the maximum value of $f(x)$ on the interval [0, 1].

Proof According to the orthogonality properties of the shifted fractional-order fifth-kind Chebyshev polynomials and the analytic form of these polynomials, we have:

$$\begin{aligned} C_i &= \frac{1}{\mathfrak{h}_i} \int_0^1 f(x) F_i^{(\sigma)}(x) w^{(\sigma)}(x)\, dx \\ &= \frac{1}{\mathfrak{h}_i} \sum_{l=0}^{i} \xi_{l,i} \int_0^1 f(x) x^{\sigma l} w^{(\sigma)}(x)\, dx. \end{aligned} \tag{23}$$

Since $f(x)$ is a continuous function on the interval [0, 1], so there is a constant M_f such that:

$$\forall x \in [0,1], \qquad |f(x)| \le M_f, \tag{24}$$

Using (24) and calculating the integral in Eq. (23), we deduce the inequality (22). □

Theorem 7.2 *Let $f_N(x)$ be an approximation based on SFFKCPs for the continuous function $f(x)$ on the interval* [0, 1]. *Then, we can derive a bound for the approximation error as follows:*

$$\|f(x) - f_N(x)\|_{L^2} \le \left(\sum_{i=N+1}^{\infty} \Upsilon_i \right)^{\frac{1}{2}} = \boldsymbol{\Upsilon}_N, \tag{25}$$

where

$$\Upsilon_i = \frac{M_f^2 \pi}{\mathfrak{h}_i} \left(\sum_{l=0}^{i} \frac{\xi_{l,i} \Gamma(l+1/2)}{\Gamma(l+3)} (l^2 + l + 1) \right)^2$$

Proof Consider the arbitrary function $f(x)$ and $f_N(x)$ as series of SFFCKPs as follows:

$$f(x) = \sum_{i=0}^{\infty} C_i F_i^{(\sigma)}(x), \qquad f_N(x) = \sum_{i=0}^{N} C_i F_i^{(\sigma)}(x),$$

Then,

$$f(x) - f_N(x) = \sum_{i=N+1}^{\infty} C_i F^{(\sigma)}(x), \tag{26}$$

Using Eq. (26) and Theorem 7.1, we have:

$$\begin{aligned} &\|f(x) - f_N(x)\|_{L^2}^2 \\ &= \int_0^1 |f(x) - f_N(x)|^2\, dx \\ &= \int_0^1 \left(\sum_{i=N+1}^{\infty} C_i F_i^{(\sigma)}(x) \right)^2 w^{(\sigma)}(x)\, dx \\ &= \int_0^1 \sum_{j=N+1}^{\infty} \sum_{i=N+1}^{\infty} C_i C_j F_i^{(\sigma)}(x) F_j^{(\sigma)}(x) w^{(\sigma)}(x)\, dx \\ &= \sum_{i=N+1}^{\infty} C_i^2 \mathfrak{h}_i \\ &\le \sum_{i=N+1}^{\infty} \Upsilon_i \end{aligned}$$

□

Theorem 7.3 *Suppose that $K(x, z)$ is any continuous function with two variables on the interval $[0,1] \times [0,1]$ and $K_N(x,z) = \sum_{i=0}^{N} \sum_{j=0}^{N} K_{ij} F_i^{(\sigma)}(x) F_j^{(\sigma)}(x)$ is an SFFKCPs approximation of $K(x, z)$ then coefficients $\mathbf{K}_{ij}$ can be bounded as follows:*

$$\begin{aligned} |\mathbf{K}_{ij}| \le \frac{M_K \pi}{\mathfrak{h}_i \mathfrak{h}_j} &\sum_{l=0}^{N} \frac{\xi_{l,i} \Gamma(l+1/2)}{\Gamma(l+3)} (l^2 + l + 1) \\ &\sum_{k=0}^{N} \frac{\xi_{k,j} \Gamma(k+1/2)}{\Gamma(k+3)} (k^2 + k + 1), \end{aligned}$$

where M_K indicates the maximum value of $K(x, z)$ on the interval $[0,1] \times [0,1]$.

Proof Using orthogonality properties and analytic form of the shifted fractional-order fifth-kind Chebyshev polynomials, we do the following:

$$\mathbf{K}_{ij} = \frac{1}{\mathfrak{h}_i\mathfrak{h}_j}\int_0^1\int_0^1 K(x,z)F_i^{(\sigma)}(x)F_j^{(\sigma)}(z)w^{(\sigma)}(x)w^{(\sigma)}(z)\,dxdz$$
$$= \frac{1}{\mathfrak{h}_i\mathfrak{h}_j}\int_0^1 \sum_{l=0}^{i}\xi_{l,i}x^{\sigma l}w^{(\sigma)(x)}\Big(\int_0^1 K(x,z)\sum_{k=0}^{j}\xi_{k,j}z^{\sigma k}w^{(\sigma)}(z)\Big)\,dxdz$$
$$= \frac{1}{\mathfrak{h}_i\mathfrak{h}_j}\sum_{l=0}^{i}\xi_{l,i}\sum_{k=0}^{j}\xi_{k,j}\int_0^1\int_0^1 x^{\sigma l}k(x,z)z^{\sigma k}w^{(\sigma)}(x)w^{(\sigma)}(z)\,dxdz. \tag{27}$$

Since $K(x, z)$ is a continuous function on the interval $[0, 1]\times[0, 1]$, so there is a constant M_K such that:

$$\forall(x,z)\in[0,1]\times[0,1], \qquad |K(x,z)|\leq M_K, \tag{28}$$

Using Eqs. (27) and (28), we achieve the desired result. □

Theorem 7.4 *Let $K(x, z)$ be a continuous function of two variables, such that $K_N(x, z)$ is the SFFKCPs approximation to $K(x, z)$. Then, an error bound can be obtained as follows*:

$$\|K(x,z)-K_N(x,z)\|_{L^2}$$
$$\leq\Big(\sum_{i=0}^{N}\sum_{j=N+1}^{\infty}\Lambda_K^2\mathfrak{h}_i\mathfrak{h}_j\Big)^{\frac{1}{2}} + \Big(\sum_{i=N+1}^{\infty}\sum_{j=0}^{\infty}\Lambda_K^2\mathfrak{h}_i\mathfrak{h}_j\Big)^{\frac{1}{2}} = \mathbf{\Theta}_N$$

$$\Lambda_K = \frac{M_K\pi}{\mathfrak{h}_i\mathfrak{h}_j}\sum_{l=0}^{N}\frac{\xi_{l,i}\Gamma(l+1/2)}{\Gamma(l+3)}(l^2+l+1)\sum_{k=0}^{N}\frac{\xi_{k,j}\Gamma(k+1/2)}{\Gamma(k+3)}(k^2+k+1)$$

Proof Assume that $K(x, z)$ and its approximation have the following forms in terms of SFFKCPs:

$$K(x,z) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\mathbf{K}_{ij}F_i^{(\sigma)}(x)F_j^{(\sigma)}(z),$$
$$K_N(x,z) = \sum_{i=0}^{N}\sum_{j=0}^{N}\mathbf{K}_{ij}F_i^{(\sigma)}(x)F_j^{(\sigma)}(z),$$

thus,

$$K(x,z)-K_N(x,z) = \sum_{i=0}^{N}\sum_{j=N+1}^{\infty}\mathbf{K}_{ij}F_i^{(\sigma)}(x)F_j^{(\sigma)}(z) + \sum_{i=N+1}^{\infty}\sum_{j=0}^{\infty}\mathbf{K}_{ij}F_i^{(\sigma)}(x)F_j^{(\sigma)}(z). \tag{29}$$

Using orthogonality properties of the shifted fractional-order fifth-kind Chebyshev polynomials, Eq. (29), and Theorem 7.3, we get:

$$\|K(x,z)-K_N(x,z)\|_{L^2}$$
$$\leq\Big\|\sum_{i=0}^{N}\sum_{j=N+1}^{\infty}\mathbf{K}_{ij}F_i^{(\sigma)}(x)F_j^{(\sigma)}(z)\Big\|_{L^2} + \Big\|\sum_{i=N+1}^{\infty}\sum_{j=0}^{\infty}\mathbf{K}_{ij}F_i^{(\sigma)}(x)F_j^{(\sigma)}(z)\Big\|_{L^2}$$
$$=\Big(\int_0^1\int_0^1\Big(\sum_{i=0}^{N}\sum_{j=N+1}^{\infty}\mathbf{K}_{ij}F_i^{(\sigma)}(x)F_j^{(\sigma)}(z)\Big)^2 w^{(\sigma)}(x)w^{(\sigma)}(z)\,dxdz\Big)^{\frac{1}{2}}$$
$$+\Big(\int_0^1\int_0^1\Big(\sum_{i=N+1}^{\infty}\sum_{j=0}^{\infty}\mathbf{K}_{ij}F_i^{(\sigma)}(x)F_j^{(\sigma)}(z)\Big)^2 w^{(\sigma)}(x)w^{(\sigma)}(z)\,dxdz\Big)^{\frac{1}{2}}$$
$$=\Big(\sum_{i=0}^{N}\sum_{j=N+1}^{\infty}\mathbf{K}_{ij}^2\mathfrak{h}_i\mathfrak{h}_j\Big)^{\frac{1}{2}} + \Big(\sum_{i=N+1}^{\infty}\sum_{j=0}^{\infty}\mathbf{K}_{ij}^2\mathfrak{h}_i\mathfrak{h}_j\Big)^{\frac{1}{2}}$$
$$\leq\Big(\sum_{i=0}^{N}\sum_{j=N+1}^{\infty}\Lambda_K^2\mathfrak{h}_i\mathfrak{h}_j\Big)^{\frac{1}{2}} + \Big(\sum_{i=N+1}^{\infty}\sum_{j=0}^{\infty}\Lambda_K^2\mathfrak{h}_i\mathfrak{h}_j\Big)^{\frac{1}{2}}.$$

□

Using the definition of L^2-norm and Theorems 7.1 and 7.2, we have the following inequalities:

1. $\|f(qx)-f_N(qx)\|_{L^2}\leq\frac{\Upsilon_N}{\sqrt{q}}$,
2. $\|f_N(x)\|_{L^2}\leq\Big(\sum_{i=0}^{N}\frac{M_f^2\pi}{\mathfrak{h}_i}\Big(\sum_{l=0}^{i}\frac{\xi_{l,i}\Gamma(l+1/2)}{\Gamma(l+3)}(l^2+l+1)\Big)^2\Big)^{\frac{1}{2}} = Z_N$,
3. $\|f_N(qx)\|_{L^2}\leq\frac{Z_N}{\sqrt{q}}$.

Theorem 7.5 *Suppose that $y(x)$ is the exact solution and $y_N(x)$ is the approximate solution in terms of SFFKCPs of Eq. (1). If $\frac{|\lambda_1|\Gamma(1-\mu)}{\Gamma(\gamma-\mu+2)}M_{K_1}+\frac{|\lambda_2|\Gamma(1-\nu)}{\sqrt{q}\Gamma(\gamma-\nu+2)}M_{K_2}+\frac{q|\lambda_3|}{\Gamma(\gamma+2)}M_{K_3}+\frac{\beta_g+\sqrt{q}\beta_p}{\sqrt{q}\Gamma(\gamma+1)}<1$ where $M_{K_1}, M_{K_2}, M_{K_3}$ are maximum values of K_1, K_2, K_3, respectively. Then a bound for the method error can be gotten as follows:*

$$\|y(x)-y_N(x)\|_{L^2}\leq\frac{\mathbf{\Omega}_N+\Big(\frac{|\lambda_1|\Gamma(1-\mu)}{\Gamma(\gamma-\mu+2)}\mathbf{\Theta}_{1N}+\frac{|\lambda_2|\Gamma(1-\nu)}{\sqrt{q}\Gamma(\gamma-\nu+2)}\mathbf{\Theta}_{2N}+\frac{q|\lambda_3|}{\Gamma(\gamma+2)}\mathbf{\Theta}_{3N}\Big)Z_N}{1-\Big(\frac{|\lambda_1|\Gamma(1-\mu)}{\Gamma(\gamma-\mu+2)}M_{K_1}+\frac{|\lambda_2|\Gamma(1-\nu)}{\sqrt{q}\Gamma(\gamma-\nu+2)}M_{K_2}+\frac{q|\lambda_3|}{\Gamma(\gamma+2)}M_{K_3}+\frac{\beta_g+\sqrt{q}\beta_p}{\sqrt{q}\Gamma(\gamma+1)}\Big)}$$

Proof We use the Riemann–Liouville integral operator on Eq. (1) to get the following equation:

$$y(x) = \Psi(x) + \frac{\lambda_1 \Gamma(1-\mu)}{\Gamma(\gamma-\mu+1)} \int_0^x (x-z)^{\gamma-\mu} K_1(x,z) y(z)\, dz + \frac{\lambda_2 \Gamma(1-\nu)}{\Gamma(\gamma-\nu+1)} \int_0^x (x-z)^{\gamma-\nu} K_2(x,z) y(qz)\, dz + \lambda_3 \mathfrak{J}^\gamma \Big(\int_0^{qx} K_3(x,z) y(z)\, dz \Big) + \frac{1}{\Gamma(\gamma)} \int_0^x (x-z)^{\gamma-1} g(z) y(qz)\, dz + \frac{1}{\Gamma(\gamma)} \int_0^x (x-z)^{\gamma-1} p(z) y(z)\, dz, \tag{30}$$

where

$$\Psi(x) = \sum_{m=0}^{N-1} \frac{y^{(m)}(0)}{m!} x^m + \frac{1}{\Gamma(\gamma)} \int_0^x (x-z)^{\gamma-1} f(z)\, dz.$$

Also, the approximate equation corresponding to Eq. (30) can be written as follows:

$$y_N(x) = \Psi(x) + \frac{\lambda_1 \Gamma(1-\mu)}{\Gamma(\gamma-\mu+1)} \int_0^x (x-z)^{\gamma-\mu} K_{1N}(x,z) y_N(z)\, dz + \frac{\lambda_2 \Gamma(1-\nu)}{\Gamma(\gamma-\nu+1)} \int_0^x (x-z)^{\gamma-\nu} K_{2N}(x,z) y_N(qz)\, dz + \lambda_3 \mathfrak{J}^\gamma \Big(\int_0^{qx} K_{3N}(x,z) y_N(z)\, dz \Big) + \frac{1}{\Gamma(\gamma)} \int_0^x (x-z)^{\gamma-1} g(z) y_N(qz)\, dz + \frac{1}{\Gamma(\gamma)} \int_0^x (x-z)^{\gamma-1} p(z) y_N(z)\, dz + H_N(x). \tag{31}$$

By subtracting (30) from (31), we obtain the following equation:

$$y(x) - y_N(x) = -H_N(x) + \frac{\lambda_1 \Gamma(1-\mu)}{\Gamma(\gamma-\mu+1)} \int_0^x (x-z)^{\gamma-\mu} \big(K_1(x,z) y(z) - K_{1N}(x,z) y_N(z) \big)\, dz + \frac{\lambda_2 \Gamma(1-\nu)}{\Gamma(\gamma-\nu+1)} \int_0^x (x-z)^{\gamma-\nu} \big(K_2(x,z) y(qz) - K_{2N}(x,z) y_N(qz) \big)\, dz + \lambda_3 \mathfrak{J}^\gamma \Big(\int_0^{qx} \big(K_3(x,z) y(z) - K_{3N}(x,z) y_N(z) \big)\, dz \Big) + \frac{1}{\Gamma(\gamma)} \int_0^x (x-z)^{\gamma-1} g(z) \big(y(qz) - y_N(qz) \big)\, dz + \frac{1}{\Gamma(\gamma)} \int_0^x (x-z)^{\gamma-1} p(z) \big(y(z) - y_N(z) \big)\, dz, \tag{32}$$

where $H_N(x)$ is the residual function. First, we determine a bound for the perturbation term $H_N(x)$. For this aim, we apply the L^2-norm on Eq. (32):

$$\|H_N(x)\|_{L^2} \le \|y(x) - y_N(x)\|_{L^2} + \frac{|\lambda_1| \Gamma(1-\mu)}{\Gamma(\gamma-\mu+1)} \int_0^x |x-z|^{\gamma-\mu} \big(\|K_1(x,z) y(z) - K_{1N}(x,z) y_N(z)\|_{L^2} \big)\, dz + \frac{|\lambda_2| \Gamma(1-\nu)}{\Gamma(\gamma-\nu+1)} \int_0^x |x-z|^{\gamma-\nu} \big(\|K_2(x,z) y(qz) - K_{2N}(x,z) y_N(qz)\|_{L^2} \big)\, dz + |\lambda_3| \mathfrak{J}^\gamma \Big(\int_0^{qx} \big(\|K_3(x,z) y(z) - K_{3N}(x,z) y_N(z)\|_{L^2} \big)\, dz \Big) + \frac{1}{\Gamma(\gamma)} \int_0^x |x-z|^{\gamma-1} \|g(z)\|_{L^2} \big(\|y(qz) - y_N(qz)\|_{L^2} \big)\, dz + \frac{1}{\Gamma(\gamma)} \int_0^x |x-z|^{\gamma-1} \|p(z)\|_{L^2} \big(\|y(z) - y_N(z)\|_{L^2} \big)\, dz, \tag{33}$$

Using inequality 2 and Theorems 7.2 and 7.4, we get:

$$\frac{|\lambda_1| \Gamma(1-\mu)}{\Gamma(\gamma-\mu+1)} \int_0^x |x-z|^{\gamma-\mu} \|K_1(x,z) y(z) - K_{1N}(x,z) y_N(z)\|_{L^2}\, dz \le \frac{|\lambda_1| \Gamma(1-\mu)}{\Gamma(\gamma-\mu+2)} M_{K_1} \|y(x) - y_N(x)\|_{L^2} + \frac{|\lambda_1| \Gamma(1-\mu)}{\Gamma(\gamma-\mu+2)} \mathbf{\Theta}_{1N} \|y_N(x)\|_{L^2} \le \frac{|\lambda_1| \Gamma(1-\mu)}{\Gamma(\gamma-\mu+2)} (M_{K_1} \mathbf{\Upsilon}_N + Z_N \mathbf{\Theta}_{1N}). \tag{34}$$

From inequalities 1 and 3 and Theorem 7.4, we have:

$$\frac{|\lambda_2| \Gamma(1-\nu)}{\Gamma(\gamma-\nu+1)} \int_0^x |x-z|^{\gamma-\nu} \|K_2(x,z) y(qz) - K_{2N}(x,z) y_N(qz)\|_{L^2}\, dz \le \frac{|\lambda_2| \Gamma(1-\nu)}{\Gamma(\gamma-\nu+2)} M_{K_2} \|y(qx) - y_N(qx)\|_{L^2} + \frac{|\lambda_2| \Gamma(1-\nu)}{\Gamma(\gamma-\nu+2)} \mathbf{\Theta}_{2N} \|y_N(qx)\|_{L^2} \le \frac{|\lambda_2| \Gamma(1-\nu)}{\sqrt{q}\, \Gamma(\gamma-\nu+2)} (M_{K_2} \mathbf{\Upsilon}_N + Z_N \mathbf{\Theta}_{2N}), \tag{35}$$

And using Theorems 7.2, 7.4 and inequality 2, we obtain the following inequality:

$$|\lambda_3| \mathfrak{J}^\gamma \Big(\int_0^{qx} \|K_3(x,z) y(z) - K_{3N}(x,z) y_N(z)\|_{L^2}\, dz \Big) \le \frac{q|\lambda_3|}{\Gamma(\gamma+2)} (M_{K_3} \mathbf{\Upsilon}_N + Z_N \mathbf{\Theta}_{3N}). \tag{36}$$

Also, according to inequality 1 and $\|g(x)\|_{L^2} \le \beta_g$, we have:

$$\frac{1}{\Gamma(\gamma)} \int_0^x |x-z|^{\gamma-1} \|g(z)\|_{L^2} \|y(qz) - y_N(qz)\|_{L^2}\, dz \le \frac{\beta_g \mathbf{\Upsilon}_N}{\sqrt{q}\, \Gamma(\gamma+1)}, \tag{37}$$

and using Theorem 7.2 and $\|p(x)\|_{L^2} \le \beta_p$, we find:

$$\frac{1}{\Gamma(\gamma)}\int_0^x |x-z|^{\gamma-1}\|p(z)\|_{L^2}\|y(z)-y_N(z)\|_{L^2}\,dz \le \frac{\beta_p \Upsilon_N}{\Gamma(\gamma+1)}. \tag{38}$$

By substituting inequalities (34)–(38) into Eq. (33), we can get the following upper bound for $H_N(x)$:

$$\begin{aligned}\|H_N(x)\|_{L^2} \le &\Big(1+\frac{|\lambda_1|\Gamma(1-\mu)}{\Gamma(\gamma-\mu+2)}M_{K_1}+\frac{|\lambda_2|\Gamma(1-v)}{\sqrt{q}\Gamma(\gamma-v+2)}M_{K_2}\\ &+\frac{q|\lambda_3|}{\Gamma(\gamma+2)}M_{K_3}+\frac{\beta_g+\sqrt{q}\beta_p}{\sqrt{q}\Gamma(\gamma+1)}\Big)\Upsilon_N\\ &+\Big(\frac{|\lambda_1|\Gamma(1-\mu)}{\Gamma(\gamma-\mu+2)}\Theta_{1N}+\frac{|\lambda_2|\Gamma(1-v)}{\sqrt{q}\Gamma(\gamma-v+2)}\Theta_{2N}\\ &+\frac{q|\lambda_3|}{\Gamma(\gamma+2)}\Theta_{3N}\Big)Z_N=\Omega_N.\end{aligned}$$

Now, we again consider the residual function (32). Therefore, by substituting all the obtained boundaries into Eq. (32), the error bound of the method can be obtained as follows:

$$\begin{aligned}&\|y(x)-y_N(x)\|_{L^2}\\ &\le \|H_N(x)\|_{L^2}+\frac{|\lambda_1|\Gamma(1-\mu)}{\Gamma(\gamma-\mu+1)}\\ &\quad\int_0^x |x-z|^{\gamma-\mu}\big(\|K_1(x,z)y(z)-K_{1N}(x,z)y_N(z)\|_{L^2}\big)\,dz\\ &\quad+\frac{|\lambda_2|\Gamma(1-v)}{\Gamma(\gamma-v+1)}\int_0^x |x-z|^{\gamma-v}\big(\|K_2(x,z)y(qz)\\ &\quad-K_{2N}(x,z)y_N(qz)\|_{L^2}\big)\,dz\\ &\quad+|\lambda_3|\mathfrak{I}^{\gamma}\Big(\int_0^{qx}\big(\|K_3(x,z)y(z)-K_{3N}(x,z)y_N(z)\|_{L^2}\big)\,dz\Big)\\ &\quad+\frac{1}{\Gamma(\gamma)}\int_0^x |x-z|^{\gamma-1}\|g(z)\|_{L^2}\big(\|y(qz)-y_N(qz)\|_{L^2}\big)\,dz\\ &\quad+\frac{1}{\Gamma(\gamma)}\int_0^x |x-z|^{\gamma-1}\|p(z)\|_{L^2}\big(\|y(z)-y_N(z)\|_{L^2}\big)\,dz\\ &\le \Omega_N+\frac{|\lambda_1|\Gamma(1-\mu)}{\Gamma(\gamma-\mu+2)}M_{k_1}\|y(x)-y_N(x)\|_{L^2}\\ &\quad+\frac{|\lambda_1|\Gamma(1-\mu)}{\Gamma(\gamma-\mu+2)}Z_N\Theta_{1N}\\ &\quad+\frac{|\lambda_2|\Gamma(1-v)}{\sqrt{q}\Gamma(\gamma-v+2)}M_{K_2}\|y(x)-y_N(x)\|_{L^2}\\ &\quad+\frac{|\lambda_2|\Gamma(1-v)}{\sqrt{q}\Gamma(\gamma-v+2)}Z_N\Theta_{2N}\\ &\quad+\frac{q|\lambda_3|}{\Gamma(\gamma+2)}M_{K_3}\|y(x)-y_N(x)\|_{L^2}\\ &\quad+\frac{q|\lambda_3|}{\Gamma(\gamma+2)}Z_N\Theta_{3N}+\frac{\beta_g+\sqrt{q}\beta_p}{\sqrt{q}\Gamma(\gamma+1)}\|y(x)-y_N(x)\|_{L^2}.\end{aligned}$$

Finally, the desired result is achieved:

$$\begin{aligned}&\|y(x)-y_N(x)\|_{L^2}\\ &\le \frac{\Omega_N+\Big(\frac{|\lambda_1|\Gamma(1-\mu)}{\Gamma(\gamma-\mu+2)}\Theta_{1N}+\frac{|\lambda_2|\Gamma(1-v)}{\sqrt{q}\Gamma(\gamma-v+2)}\Theta_{2N}+\frac{q|\lambda_3|}{\Gamma(\gamma+2)}\Theta_{3N}\Big)Z_N}{1-\Big(\frac{|\lambda_1|\Gamma(1-\mu)}{\Gamma(\gamma-\mu+2)}M_{K_1}+\frac{|\lambda_2|\Gamma(1-v)}{\sqrt{q}\Gamma(\gamma-v+2)}M_{K_2}+\frac{q|\lambda_3|}{\Gamma(\gamma+2)}M_{K_3}+\frac{\beta_g+\sqrt{q}\beta_p}{\sqrt{q}\Gamma(\gamma+1)}\Big)}.\end{aligned}$$

□

8 Numerical Results

In this section, we consider some examples that demonstrate the applicability of the shifted fractional-order fifth-kind Chebyshev polynomials in the proposed operational method for solving fractional-order delay integro-differential equations. Also, we compute the absolute errors to indicate the accuracy of the presented method.

Example 8.1 Consider the following fractional-order integro-differential equation:

$$\begin{aligned}&\mathfrak{D}^{\frac{1}{2}}y(x)=\int_0^x y(z)dz+y(x)+y\Big(\frac{x}{2}\Big)+f(x),\\ &f(x)=\frac{8}{3\Gamma(\frac{1}{2})}x^{\frac{3}{2}}-x^2-\frac{1}{3}x^3-\frac{1}{4}x^2,\qquad x\in[0,1].\end{aligned}\tag{39}$$

The initial condition is $y(0)=0$ and the exact solution is $y(x)=x^2$. The Fig. 1 shows a comparison between the absolute error functions with $\sigma=0.5,1$ and $N=10$. Also, Table 1 presents values of absolute errors at equally spaced points for $N=5,10$ and $\sigma=0.5,1$. The results presented in Table 1 and Fig. 1 show that using the fractional-order fifth-kind Chebyshev polynomials increases the accuracy.

Example 8.2 Consider the following problem:

$$\begin{aligned}\mathfrak{D}y(x)=&\int_0^x e^{x+z}y(z)dz+\int_0^{\frac{x}{4}} zy(z)\,dz\\ &+\frac{1}{2}y(x)+y\Big(\frac{x}{4}\Big)+f(x),\\ f(x)=&\frac{1}{2}-\frac{x}{4}e^{\frac{x}{4}}+\frac{x^2}{32}-\frac{1}{2}e^{3x}+e^{2x},\qquad x\in[0,1],\end{aligned}\tag{40}$$

with initial condition $y(0)=0$ and exact solution $y(x)=e^x-1$. Figure 2 demonstrates maximum absolute errors for different values of N. Table 2 indicates the comparison between our operational method with other methods. Table 3 shows the maximum absolute errors of the approximate solutions for $\sigma=0.5,1$ and $N=4,5,6$. From the obtained results, it is concluded that in this case, the use of the shifted fifth-kind Chebyshev polynomials for $\sigma=1$ gives more accurate results than shifted fractional-order fifth-kind Chebyshev polynomials for $\sigma=0.5$. Also, the proposed method has provided acceptable results compared to other methods such that Chebyshev spectral and Euler operational methods. Furthermore, Table 2 confirms our method needs a lower number of basis functions in compared to the Sinc collocation method to converge to the exact solution.

Example 8.3 Consider the following pantograph-type Volterra integro-differential equation (PTVIDE):

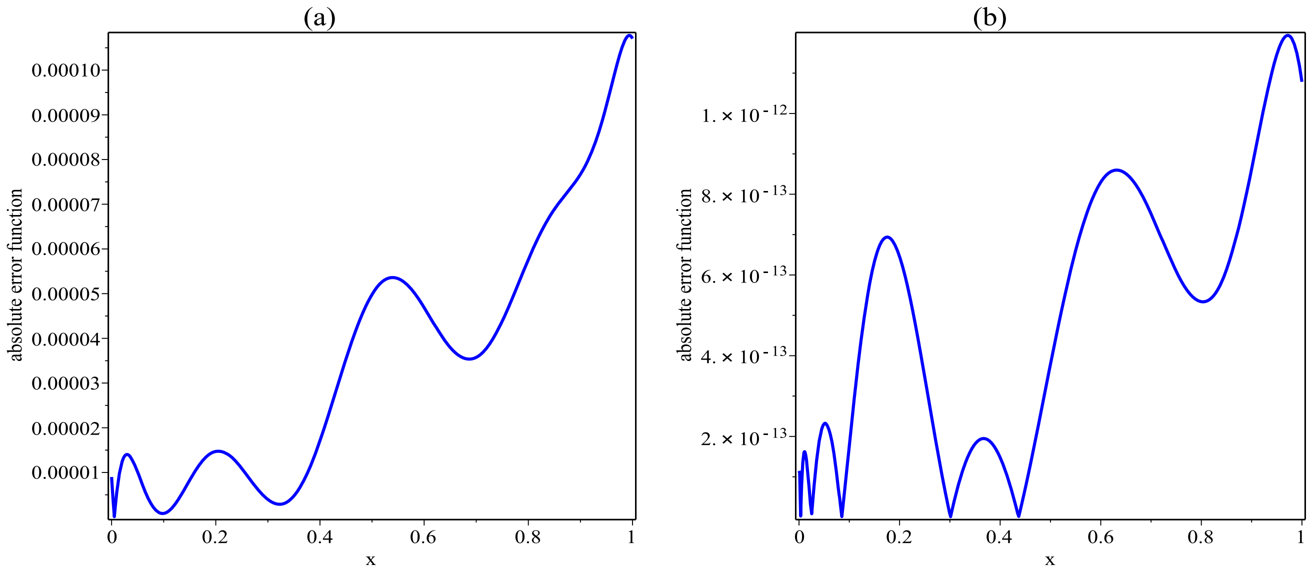


Fig. 1 **a** Absolute error function with $\sigma = 1$, **b** Absolute error function with $\sigma = 0.5$ and $N = 10$ in Example 8.1

Table 1 Absolute errors for $\sigma = 0.5, 1$ and $N = 5, 10$ in Example 8.1

x	$\sigma = 1, N = 5$	$\sigma = 1, N = 10$	$\sigma = 0.5, N = 10$
0.1	1.7302×10^{-4}	8.3780×10^{-7}	1.7754×10^{-13}
0.2	1.0729×10^{-4}	1.4695×10^{-5}	6.4545×10^{-13}
0.3	8.7989×10^{-5}	3.9424×10^{-6}	6.8549×10^{-15}
0.4	2.5405×10^{-4}	1.7088×10^{-5}	1.4756×10^{-13}
0.5	5.8837×10^{-4}	4.9806×10^{-5}	3.7867×10^{-13}
0.6	9.9164×10^{-4}	4.6884×10^{-5}	8.2943×10^{-13}
0.7	1.3562×10^{-3}	3.5759×10^{-5}	7.5163×10^{-13}
0.8	1.6398×10^{-3}	5.7626×10^{-5}	5.3343×10^{-13}
0.9	1.9395×10^{-3}	7.6691×10^{-5}	8.6295×10^{-13}
1	2.5657×10^{-3}	1.0705×10^{-4}	1.0786×10^{-12}

$$
\begin{aligned}
\mathfrak{D}y(x) &= y\left(\frac{x}{4}\right) + \frac{1}{2}y(x) + \int_0^x e^{x+z}y(z)\,dz \\
&\quad + \int_0^{\frac{x}{4}} zy(z)\,dz + f(x), \\
f(x) &= 2x - \frac{9}{16}x^2 - \frac{1}{4^5}x^4 - x^2e^{2x} \\
&\quad + 2xe^{2x} - 2e^{2x} + 2e^x, \qquad x \in [0,1],
\end{aligned} \tag{41}
$$

where initial condition is $y(0) = 0$ and the exact solution is $y(x) = x^2$. We solved the Eq. (41) using the presented method in Sect. 6 and we reported the numerical results in Table 4 and Fig. 3. Table 4 shows the maximum absolute errors for different values of N and Fig. 3 displays the absolute error functions for $\sigma = 0.5, 1$. Table 4 and Fig. 3 indicate the results obtained in $\sigma = 1$ are more accurate than $\sigma = 0.5$.

Example 8.4 Consider the following delay fractional integro-differential equation:

$$
\begin{aligned}
\mathfrak{D}^{\nu}y(x) &= \int_0^x \frac{zy(z)}{(x-z)^{\frac{1}{2}}}\,dz + \int_0^x \frac{zy\left(\frac{1}{2}\right)z}{(x-z)^{\frac{1}{3}}}\,dz \\
&\quad + y\left(\frac{1}{2}x\right) + f(x), \\
f(x) &= 2x - \frac{32}{35}x^{\frac{7}{2}} - \frac{243}{1760}x^{\frac{11}{3}} - \frac{1}{4}x^2, \qquad x \in [0,1].
\end{aligned} \tag{42}
$$

where the exact solution for $\nu = 1$ is $y(x) = x^2$. Figure 4 shows the approximate solutions for values of $\nu = 0.5, 0.6, 0.7, 0.8, 0.9, 1,\ \sigma = 1$, and $N = 3$. It can be seen the numerical solution converges to the exact solution when $\nu \longrightarrow 1$. We obtained the $MAE = 6.9722 \times 10^{-5}$ for $N = 3$.

Example 8.5 As the final example, consider the following delay fractional integro-differential equation with the solution $y(x) = x^{\frac{3}{2}}$:

$$
\begin{aligned}
\mathfrak{D}^{\frac{1}{2}}y(x) &= \int_0^x y(z)dz + y(x) + y\left(\frac{x}{2}\right) + f(x), \\
f(x) &= \frac{3}{4}\sqrt{\pi}x - \frac{2}{5}x^{\frac{5}{2}} - \left(1 + \frac{\sqrt{2}}{4}\right)x^{\frac{3}{2}} \qquad x \in [0,1].
\end{aligned} \tag{43}
$$

The initial condition is $y(0) = 0$. Figure 5 shows the absolute error functions for $\sigma = 0.5, 1$ with $N = 6$. Also, Table 5 presents absolute errors in equally spaced points for $\sigma = 0.25, 0.5, 0.75, 1$ with $N = 6$. As you can see, the

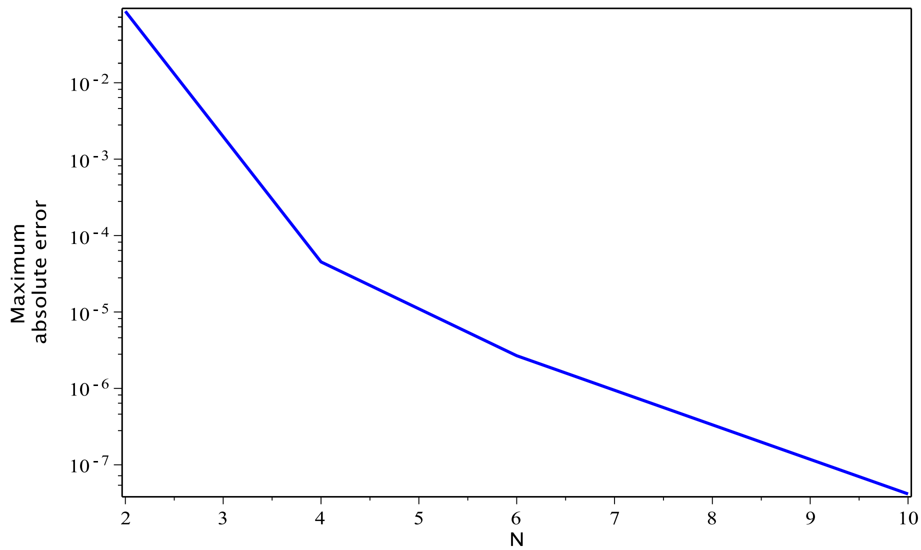


Fig. 2 Maximum absolute errors for different values of N and $\sigma = 1$ in Example 8.2

Table 2 Comparison of maximum absolute errors of SFFKCP operational method with other methods in Example 8.2

N	Sinc collocation method in Zhao et al. (2017)	N	Euler operational method in Rezabeyk et al. (2020)	N	Our method for $\sigma = 1$
10	2.2328×10^{-4}	4	5.1681×10^{-5}	4	4.5357×10^{-4}
20	4.7215×10^{-6}	5	2.6343×10^{-6}	6	2.6962×10^{-6}
60	2.2096×10^{-8}	6	1.9487×10^{-6}	8	4.5160×10^{-8}

Table 3 Maximum absolute errors for various values of σ and N in Example 8.2

N	$\sigma = 1$	$\sigma = 0.5$
4	4.5320×10^{-4}	7.1246×10^{-3}
5	1.5164×10^{-4}	2.0147×10^{-2}
6	2.6848×10^{-6}	8.7324×10^{-4}

Table 4 Maximum absolute errors in Example 8.3

N	$\sigma = 1$	$\sigma = 0.5$
2	5.7222×10^{-2}	9.7296×10^{-2}
4	2.1652×10^{-4}	3.5249×10^{-3}
6	5.6744×10^{-7}	7.6891×10^{-4}

obtained results demonstrate that the approximate solutions in $\sigma = 0.5$ are more accurate.

9 Conclusion

In this paper, the numerical method was presented for solving fractional-order delay integro-differential equations using operational matrices based on fractional-order fifth-kind Chebyshev polynomials. We substituted the obtained approximations into the main equation and got the algebraic system by applying the collocation method, then, we solved these equations by Newton's iterative method and obtained the approximation solutions. Moreover, we proved the existence and uniqueness of the solution to problem (1) and derived the error bound. The Holder exponent σ refers to the regularity of the function $y(x)$ in a Holder space $C^{m,\sigma}(J)$. The order of a fractional derivative $D^{\gamma}y(x)$ is related to σ as $\gamma = \sigma + r$ where r is the integer part of γ. The order of an Abel-type integral, μ, in a singular integral or integro-differential equation, has a relation as $\mu = \sigma - s$ where s is the integer part of σ. So, the relation between σ, γ, and μ is as $2\sigma - \gamma - \mu + r - s = 0$ or equivalently $\gamma - \mu - r - s = 0$. Based on some coefficients in terms of these factors in Theorem 7.5, one has $\gamma - \mu \neq -2$. We solved some examples to show the applicability of the proposed method. This method provides several advantages, such as simple calculation and easy implementation. Also, the numerical results reported

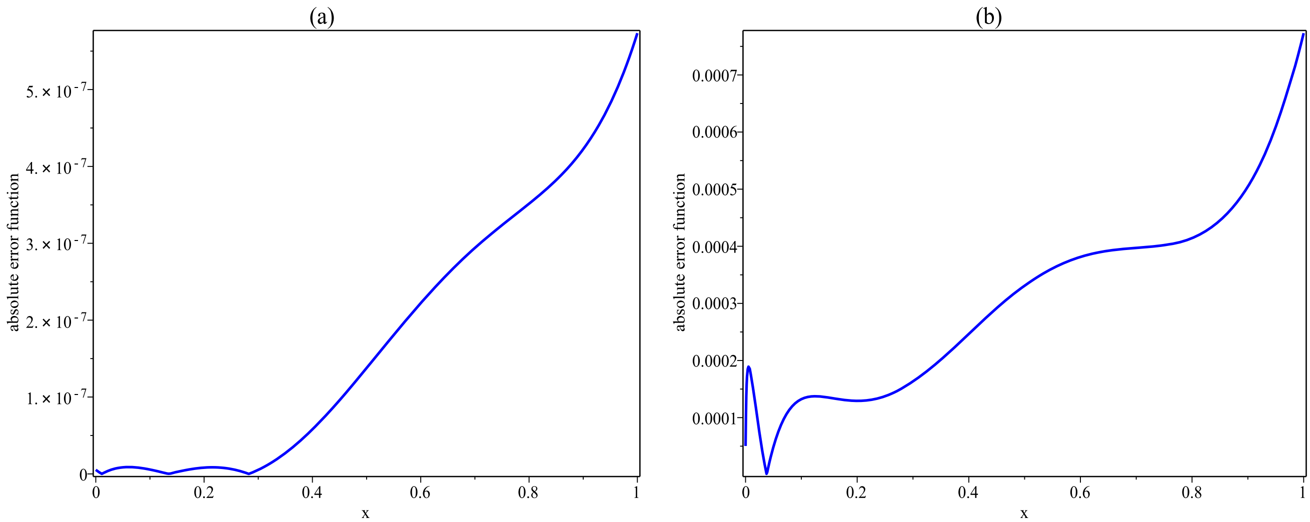
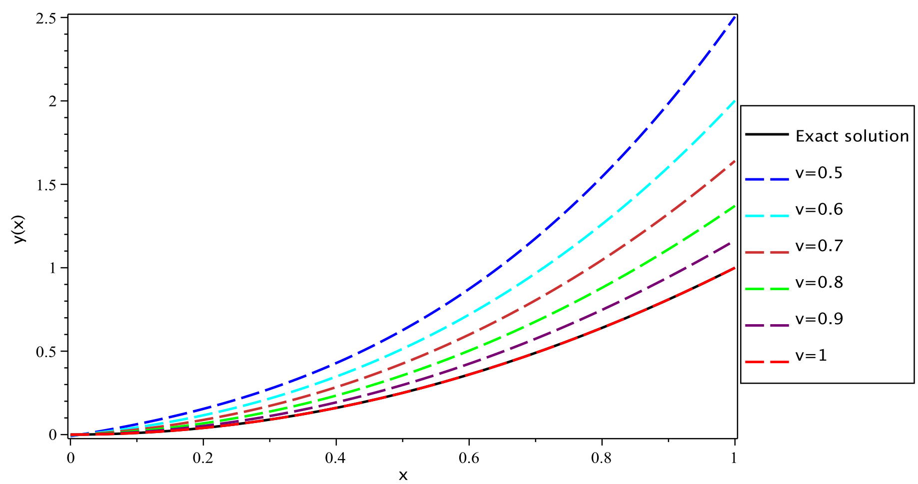


Fig. 3 **a** Absolute error function with $\sigma = 1$, **b** Absolute error function with $\sigma = 0.5$ and $N = 6$ in Example 8.3

Fig. 4 Exact and approximate solutions of the function $y(x)$ for $N = 3$ with $v = 0.5, 0.6, 0.7, 0.8, 0.9, 1$ and $\sigma = 1$ for Example 8.4



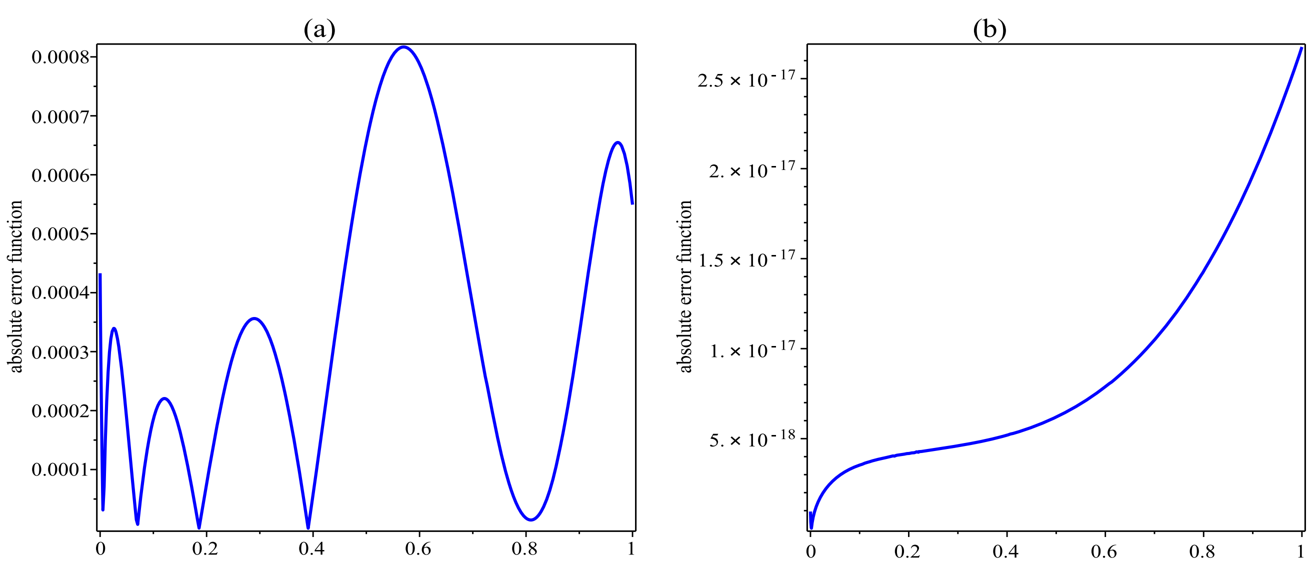


Fig. 5 **a** Absolute error function with $\sigma = 1$, **b** Absolute error function with $\sigma = 0.5$ and $N = 6$ in Example 8.5

Table 5 Absolute errors for $\sigma = 0.25, 0.5, 0.75, 1$ and $N = 6$ in Example 8.5

x	$\sigma = 0.25$	$\sigma = 0.5$	$\sigma = 0.75$	$\sigma = 1$
0.1	2.777×10^{-4}	3.538×10^{-18}	6.601×10^{-5}	1.853×10^{-4}
0.2	1.264×10^{-4}	4.183×10^{-18}	4.428×10^{-5}	7.545×10^{-5}
0.3	1.024×10^{-3}	4.600×10^{-18}	1.496×10^{-5}	3.518×10^{-4}
0.4	1.767×10^{-3}	5.200×10^{-18}	1.133×10^{-5}	5.788×10^{-5}
0.5	2.383×10^{-3}	6.170×10^{-18}	3.323×10^{-5}	6.561×10^{-4}
0.6	3.014×10^{-3}	7.740×10^{-18}	6.808×10^{-5}	7.877×10^{-4}
0.7	3.814×10^{-3}	1.058×10^{-17}	1.021×10^{-4}	3.767×10^{-4}
0.8	4.922×10^{-3}	1.439×10^{-17}	1.281×10^{-4}	1.742×10^{-5}
0.9	6.459×10^{-3}	1.954×10^{-17}	1.524×10^{-4}	3.296×10^{-4}
1	8.530×10^{-3}	2.681×10^{-17}	1.996×10^{-4}	5.490×10^{-4}

in the given examples confirmed that the SFFKCP operational method has an acceptable performance in comparison with other methods, such as Sinc collocation method Zhao et al. (2017), Chebyshev spectral method Ezz-Eldien and Doha (2019), and Euler operational method Rezabeyk et al. (2020). The results obtained from the tables and figures showed that there is a good agreement between the approximate and the exact solutions, even using a few terms of the proposed expansion. Furthermore, the errors of our presented method decreased with increasing N. By choosing diverse values of σ, values of absolute error can be controlled. However, our method can be helpful and efficient for solving these kinds of equations. Also, the other fractional operators such as the Caputo-Fabrizio and MSM operators have been introduced by researchers that the equations under study can be rewritten based on these fractional operators and obtained results can be compared with the results reported in papers Ata and Kıymaz (2023); Jafari et al. (2023); Bhatter et al. (2024); Singh et al. (2023). The authors of the current article will try to consider them in future works.

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Data Availability All results have been obtained by conducting the numerical procedure, and the ideas can be shared for the researchers.

Declarations

Conflict of interest The authors declare that there are no conflicts of interest regarding the publication of this paper.

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