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RESEARCH PAPER

Numerical Simulation of Kink Collisions, Analytical Solutions and Conservation Laws of the Potential Korteweg–de Vries Equation

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Abstract

In this study, we investigate the nonlinear potential Korteweg–de Vries equation (pKdVe) by making use of the Lie group analysis. We start by constructing Lie symmetries and thereafter utilize them to execute symmetry reductions of pKdVe. We then obtain solutions of the pKdVe by using the direct integration method. The obtained solutions are demonstrated in respect of Jacobi elliptic functions. Some of the obtained solutions are illustrated graphically. Moreover, we obtain four conserved vectors of the pKdVe by making use of the multiplier method and five conserved vectors by using the theorem owing to Ibragimov. Finally, we simulate collisions between kinks for the pKdVe.

Keywords Potential KdV equation · Lie symmetries · Analytic solutions · Conserved vectors · Multiplier method · Numerical simulation

1 Introduction

Most natural phenomena of the real world are modelled by nonlinear partial differential equations (NPDEs) (Márquez et al. 2023; Younas et al. 2023; Khalique and Lephoko 2023; Afrin 2023; Yin et al. 2023). To have a better understanding of these phenomena, one needs to determine the solutions of the NPDEs, in particular the exact solutions. Determining exact closed form solutions of NPDEs is a strenuous exercise as it is not easy to write down exact solutions of NPDEs. In spite of this fact, many researchers have come up with various techniques to determine exact particular solutions of NPDEs. A few of these techniques are homotopy perturbation technique (Shqair 2019), the sine–cosine method (Wazwaz 2005), Bäcklund transformation (Gu 1990), variation of parameters approach (Feng 2003), extended simplest equation technique (Kudryashov and Loguinova 2008), Hirota's technique (Hirota 2004), Lie group analysis (Lie 1891), Darboux transformation (Matveev and Salle 1991), F-expansion technique (Zhou et al. 2003), bifurcation technique (Zhang and Khalique 2018), Kudryashov's technique (Kudryashov 2005), sine-Gordon expanded equation approach (Chen and Yan 2005), multi-exponential function technique (Ma and Zhu 2012) homogeneous balance approach (Wang 1996), the $(G'/G)-$expansion method (Wang et al. 2005), as well as the inverse scattering transform (Ablowitz and Clarkson 1991), to mention but a few.

Among the above specified techniques, Lie group analysis is the most effective and power technique to determine exact solutions of the NPDEs. It was founded by the Norwegian mathematician, Marius Sophus Lie, who discovered the discipline of continuous groups of transformations and utilized it in the field of geometry and differential equations (DEs). Researchers have applied Lie's theory to problems in various scientific fields

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successfully. See for example (Ovsiannikov 1982; Olver 1993; Ibragimov 1999).

Conservation laws play a very salient role in engineering and physics from both a practical and theoretical standpoint. They describe quantities that are conserved over a certain period of time in a system. Conservation laws have many uses such as in the extensive study of uniqueness, existence and stability of solutions of DEs and in the evolution and manipulation of numerical methods (Noether 1918; Leveque 1992; Sarlet 2010; Yasar and Özer 2011; Gandarias and Bruzón 2017; Ibragimov 2007).

The Korteweg–de Vries (KdV) equation, that was first introduced by Boussinesq in 1877, and rediscovered by Diederik Korteweg and Gustav de Vries in 1895, is a NPDE of third-order which reads

$$u_{xxx} + u_t + 6uu_x = 0. \tag{1}$$

Here $u = u(t, x)$ represents the lengthening of the wave in the vicinity of x and at time t. The number 6 appearing in the equation is just traditional but of no great importance (Korteweg and de Vries 1895). The Kadomtsev–Petviashvili (KP) equation (Kadomtsev and Petviashvili 1970)

$$au_{yy} + (u_{xxx} + u_t + 6uu_x)_x = 0 \tag{2}$$

is a NPDE in one temporal and two spatial coordinates that narrates the development of non-linear, long waves of short amplitude accompanied by moderate relying on transverse coordinate. The 3D generalized KP equation

$$u_{xxxy} + u_{tx} + u_{ty} - u_{zz} + 3(u_x u_y)_x = 0 \tag{3}$$

was studied in Ma et al. (2011), using Wronskian and Grammian techniques. In Wazwaz (2012), Hirota's technique was invoked on Eq. (3) and multiple soliton along with multiple singular soliton solutions were constructed.

Of late, a new generalized 3D KP equation, which reads

$$u_{xxxy} + u_{tx} + u_{ty} + u_{tz} - u_{zz} + 3(u_x u_y)_x = 0 \tag{4}$$

was established in Wazwaz and El-Tantawy (2016), by adding u_{tz} in Eq. (3). It can be observed that this new equation (4) transforms to the classical KP Eq. (2) if we take $y = x$. However, when $x = y = z$, we note that (4) changes to the potential KdV (pKdV) equation that reads

$$3u_t - u_x + 3u_x^2 + u_{xxx} = 0, \tag{5}$$

which we shall study in this work.

2 Exact Solutions of the pKdV Eq. (5)

We work out Lie symmetries and perform symmetry reductions of the pKdV Eq. (5). On top of that, we construct travelling wave solution of (5) through integration.

2.1 Symmetries of (5)

We use Lie's theory to determine Lie symmetries of (5). The generator

$$Y = \tau^t(t, x, u)\frac{\partial}{\partial t} + \xi^x(t, x, u)\frac{\partial}{\partial x} + \eta^u(t, x, u)\frac{\partial}{\partial u} \tag{6}$$

is a symmetry of (5) provided

$$Y^{[3]}\Delta|_{\Delta=0} = 0, \tag{7}$$

where $\Delta \equiv 3u_t - u_x + 3u_x^2 + u_{xxx}$, $Y^{[3]}$ is the third extension of Y defined by

$$Y^{[3]} = Y + \zeta_1\frac{\partial}{\partial u_t} + \zeta_2\frac{\partial}{\partial u_x} + \zeta_{222}\frac{\partial}{\partial u_{xxx}} \tag{8}$$

and ζ_1, ζ_2 and ζ_{222} are given as

$$\begin{aligned}
\zeta_1 &= D_t(\eta^u) - u_x D_t(\xi^x) - u_t D_t(\tau^t),\\
\zeta_2 &= D_x(\eta^u) - u_x D_x(\xi^x) - u_t D_x(\tau^t),\\
\zeta_{12} &= D_x(\zeta_1) - u_{tt} D_x(\tau^t) - u_{tx} D_x(\xi^x),\\
\zeta_{222} &= D_x(\zeta_{22}) - u_{txx} D_x(\tau^t) - u_{xxx} D_x(\xi^x)
\end{aligned}$$

with D_t, D_x being the total derivatives formulated as

$$\begin{aligned}
D_t &= \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_{tx}\frac{\partial}{\partial u_x} + u_{tt}\frac{\partial}{\partial u_t} + \cdots,\\
D_x &= \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xx}\frac{\partial}{\partial u_x} + u_{tx}\frac{\partial}{\partial u_t} + \cdots,
\end{aligned}$$

respectively. Expanding Eq. (7) and splitting on different derivatives with respect to u, we acquire eight PDEs

$$\begin{aligned}
&\tau_u = 0,\ \tau_x = 0,\ xi_u = 0,\ \xi_{xx} = 0,\ \eta_u + \xi_x = 0,\\
&\tau_t - 3\xi_x = 0,\ 3\eta_t - \eta_x + \eta_{xxx} = 0,\ 6\eta_x - 3\xi_t - 2\xi_x = 0,
\end{aligned}$$

whose solution can be easily obtained and is given by

$$\begin{aligned}
\tau &= C_1 t + C_2,\\
\xi &= \frac{1}{3}C_1 x + C_3 t + C_4,\\
\eta &= -\frac{1}{3}C_1 u + \frac{1}{2}C_3 x + \frac{1}{9}C_1 x + \frac{1}{6}C_3 t + \frac{1}{27}C_1 t + C_5.
\end{aligned}$$

Thus, we conclude that the pKdV Eq. (5) admits a five-dimensional Lie algebra L_5 generated by

$$\begin{aligned}
&Y_1 = \frac{\partial}{\partial t},\quad Y_2 = \frac{\partial}{\partial x},\quad Y_3 = \frac{\partial}{\partial u},\quad Y_4 = 6t\frac{\partial}{\partial x} + (t + 3x)\frac{\partial}{\partial u},\\
&Y_5 = 27t\frac{\partial}{\partial t} + 9x\frac{\partial}{\partial x} + (t + 3x - 9u)\frac{\partial}{\partial u}.
\end{aligned}$$

2.2 Symmetry Reductions and Solutions

We now carry out symmetry reductions and find invariant solutions for the pKdV Eq. (5). We start off by constructing travelling wave solutions by invoking the translational symmetries $Y_1 = \partial/\partial t$ and $Y_2 = \partial/\partial x$. Let $Y = Y_1 + \theta Y_2$ with θ a constant. The symmetry Y has two invariants

$$\xi = x - \theta t \quad P = u, \tag{9}$$

and consequently we have the invariant solution $u = P(\xi)$. Substituting this invariant solution into (5) yields

$$P''' + 3(P')^2 - (3\theta + 1)P' = 0. \tag{10}$$

Letting $\mathcal{Z} = P'$, Eq. (10) becomes

$$\mathcal{Z}'' + 3\mathcal{Z}^2 - a\mathcal{Z} = 0,\ a = 3\theta + 1. \tag{11}$$

Multiplying the above equation by the integrating factor $\mathcal{Z}'$ and integrating yields

$$\mathcal{Z}'^2 + 2\mathcal{Z}^3 - a\mathcal{Z}^2 + 2C_1 = 0 \tag{12}$$

with C_1 a constant. Now if $\mathcal{Z}^3 - \frac{a}{2}\mathcal{Z}^2 + C_1 = 0$ has three roots $\rho_1 > \rho_2 > \rho_3$, then we can rewrite (12) as

$$\mathcal{Z}'^2 = -2(\mathcal{Z} - \rho_3)(\mathcal{Z} - \rho_2)(\mathcal{Z} - \rho_1), \tag{13}$$

whose solution is well familiar (Kudryashov 2004; Gradshteyn and Ryzhik 2007) and is given by

$$\mathcal{Z}(\xi) = \rho_2 + (\rho_1 - \rho_2)\mathrm{cn}^2\left(\sqrt{\frac{\rho_1 - \rho_3}{2}}\,\xi \,\middle|\, K^2\right), \quad K^2 = \frac{\rho_1 - \rho_2}{\rho_1 - \rho_3}, \tag{14}$$

with $\mathrm{cn}(\xi|K^2)$ being the Jacobi elliptic cosine function and ρ_1, ρ_2, ρ_3 are $\rho_1\rho_2 + \rho_2\rho_3 + \rho_3\rho_1 = 0$, $\rho_1\rho_2\rho_3 = -C_1$ and $\rho_1 + \rho_2 + \rho_3 = a/2$. Recall that if $K^2 \to 1$, $\mathrm{cn}(\xi|K^2) \to \mathrm{sech}(\xi)$ and if $K^2 \to 0$, $\mathrm{cn}(\xi|K^2) \to \cos(\xi)$ (Gradshteyn and Ryzhik 2007). Since $\mathcal{Z} = P'$, we integrate (14) to get $P(\xi)$. Now going back to variables t and x, the solution of pKdV Eq. (5) is

$$u(t,x) = \frac{\{\rho_2 - (1 - K^2)\rho_1\}\xi}{K^2} + \frac{(\rho_1 - \rho_2)\mathrm{dn}(A\,\xi|K^2)E(\mathrm{am}(A\,\xi|K^2)|K^2)}{AM^2\sqrt{\mathrm{dn}(A\,\xi|K^2)^2}}, \tag{15}$$

with $\xi = x - \theta t$, $\mathrm{dn}(\xi|K^2)$ being the delta amplitude function, $\mathrm{E}(\xi|K^2)$ being the elliptic integral of the second kind, $\mathrm{am}(\xi|K^2)$ being the amplitude function and $A = \sqrt{(\rho_1 - \rho_3)/2}$ (Gradshteyn and Ryzhik 2007). In Fig. 1 we exhibit the profile of solution (15) when $\theta = 1$, $\rho_1 = 3$, $\rho_2 = 2$, $\rho_3 = 1$ with $-1 \le t \le 1$ and $-1 \le x \le 1$.

Figure 1 illustrates the periodic nature of the obtained solution (15) achieved by making a suitable chioce of the parameters included in the solution. The wave profile further establishes the elliptic solution with solitonic structure peculiar to solutions with sinusodial wave form.

Solitary wave solutions of (12) arise when $\mathcal{Z}^3 - \frac{a}{2}\mathcal{Z}^2 + C_1 = 0$ has two equal roots. Assume $\rho_1 = \rho_2$, we have two cases of (12), namely

$$\mathcal{Z}'^2 = -2\mathcal{Z}^2\left(\mathcal{Z} - \frac{a}{2}\right) \quad \text{and} \quad \mathcal{Z}'^2 = -2\left(\mathcal{Z} + \frac{a}{6}\right)\left(\mathcal{Z} - \frac{a}{3}\right)^2, \tag{16}$$

which gives

$$\int \frac{d\mathcal{Z}}{\sqrt{2\mathcal{Z}^2\left(\mathcal{Z} - \frac{a}{2}\right)}} = -\xi, \quad \int \frac{d\mathcal{Z}}{\sqrt{2\left(\mathcal{Z} - \frac{a}{3}\right)^2\left(\mathcal{Z} + \frac{a}{6}\right)}} = -\xi.$$

Evaluating the above integrals, we obtain

$$\mathcal{Z}_1 = -\frac{2aC_5\exp(\xi\sqrt{a})}{\{1 - C_5\exp(\xi\sqrt{a})\}^2}$$

and

$$\mathcal{Z}_2 = a\left[\frac{1}{3} - \frac{1 + C_5^2}{2\cos^2(\xi\sqrt{a}/2)\{1 - C_5\tan(\xi\sqrt{a}/2)\}^2}\right].$$

Using $\mathcal{Z} = P'$ in the above equations and integrating we obtain two solitary wave solutions of Eq. (5):

$$u_1 = -\frac{2\sqrt{3\theta + 1}}{C_5\exp(\xi\sqrt{3\theta + 1}) - 1} \tag{17}$$

and

$$u_2 = (3\theta + 1)\left[\frac{1}{3}\xi + \frac{(C_5^2 + 1)\sin\left(\sqrt{3\theta + 1}\,\xi/2\right)}{\sqrt{3\theta + 1}\left[C_5\sin\left(\sqrt{3\theta + 1}\,\xi/2\right) - \cos\left(\sqrt{3\theta + 1}\,\xi/2\right)\right]}\right], \tag{18}$$

where $\xi = x - \theta t$, $3\theta + 1 > 0$ and C_5 is an arbitrary constant. In Fig. 2 we give the wave profile of (18) in form of 3D and 2D plots with dissimilar constant values $\theta = 1$, $C_5 = 20$ with $-10 \le t, x \le 10$.

2.2.1 Special Case

If we take the integration constant $C_1 = 0$ in the Eq. (12), we obtain

$$\frac{d\mathcal{Z}}{\mathcal{Z}\sqrt{a - 2\mathcal{Z}}} = \pm d\xi,$$

which gives

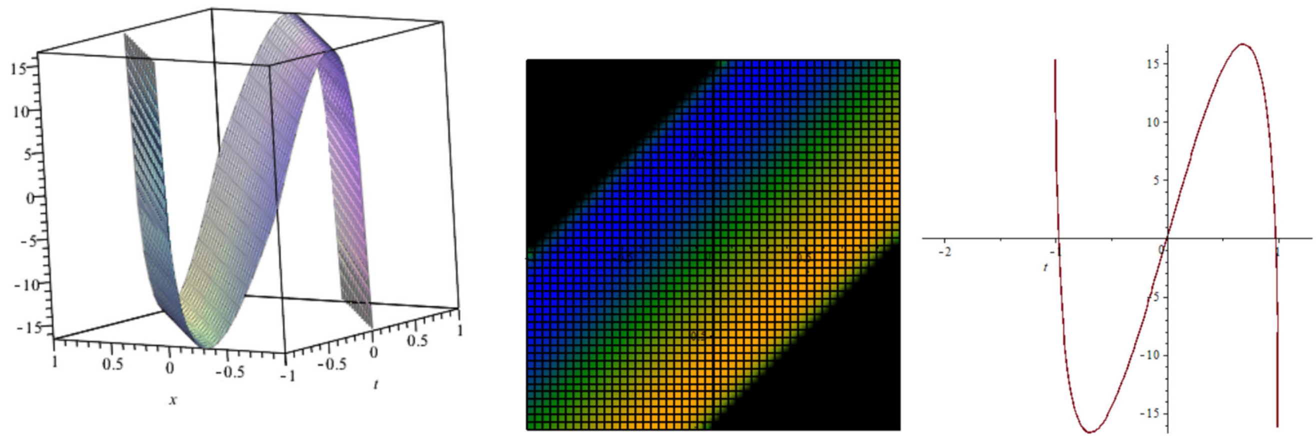

Fig. 1 Wave dynamics of periodic solution (15)

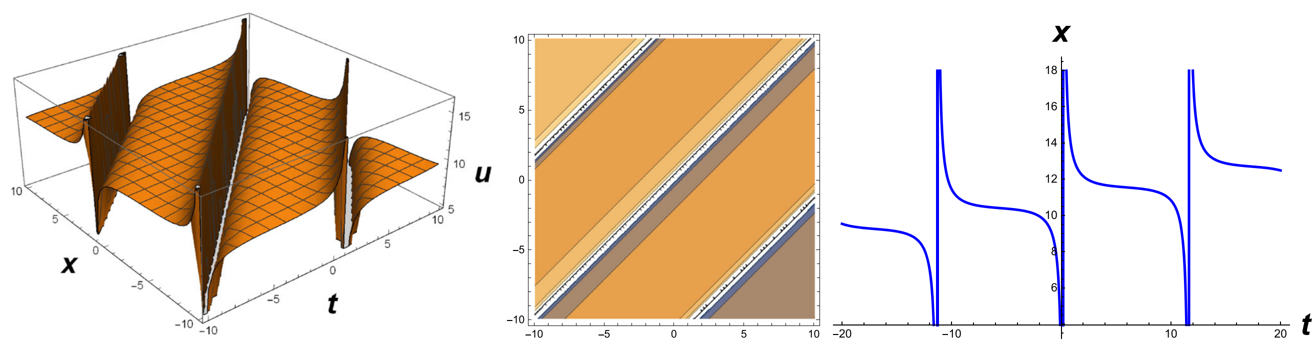

Fig. 2 Wave dynamics of trigonometric solution (18)

$$\frac{2}{\sqrt{-a}}\tan^{-1}\left(\sqrt{\frac{2\mathcal{Z}-a}{3\theta+1}}\right)=\pm\xi+K_1 \tag{19}$$

with K_1 a constant. Solving Eq. (19) for $\mathcal{Z}$ gives

$$\mathcal{Z}=\frac{1}{2}a\sec^2\left(\frac{1}{2}\sqrt{-a}(K_1\pm\xi)\right),$$

and consequently, we obtain

$$P(\xi)=\pm\sqrt{-a}\tan\left(\frac{1}{2}\sqrt{-a}(K_1\pm\xi)\right)+K_2,$$

where $a<0$ and K_2 is a constant of integration. Hence the travelling wave (invariant) solution under symmetry Y is

$$u(t,x)=\pm\sqrt{-(3\theta+1)}\tan\left(\frac{1}{2}\sqrt{-(3\theta+1)}(K_1\pm(x-\theta t))\right)+K_2,$$

provided $3\theta+1<0$. However, for the case $3\theta+1>0$, we can integrate Eq. (12) with $C_1=0$ to obtain

$$-\frac{2}{\sqrt{3\theta+1}}\operatorname{arctanh}\left\{\frac{\sqrt{3\theta+1-2\mathcal{Z}}}{\sqrt{3\theta+1}}\right\}=\pm\xi+K_1,$$

where K_1 is the constant of integration. This eventually leads to the travelling wave (invariant) solution under symmetry Y as

$$u(t,x)=\pm\sqrt{3\theta+1}\tanh\left\{K_1\pm\frac{\sqrt{3\theta+1}\,(x-\theta t)}{2}\right\}+K_2. \tag{20}$$

We give the wave profile of (20) as revealed in Fig. 3 with dissimilar constant values $\theta=0.1$, $K_1=10$, $K_2=20$ where $-10\leq t,x\leq 10$.

We now utilize the five Lie symmetries of (5) and perform symmetry reductions and compute group invariant solutions.

Symmetry 1. The time translational symmetry $Y_1=\partial/\partial t$ gives the invariants $I_1=x$, $I_2=u$. Thus, $u=f(x)$ is the invariant solution, where arbitrary function f depends on x. We note that this is a special case of (9) when $\theta=0$. Thus, the invariant solutions are obtained by taking $\theta=0$ in the above derived solutions.

Symmetry 2. Consider the space translational operator $Y_2=\partial/\partial x$. This operator yields the invariants $I_1=t$, $I_2=u$. Thus, $u=\phi(t)$ and substituting into (5) produces $\phi'(t)=0$, which yields $\phi(t)=C_1$, where C_1 is a constant. Hence, solution under the symmetry Y_2 is $u(t,x)=C_1$.

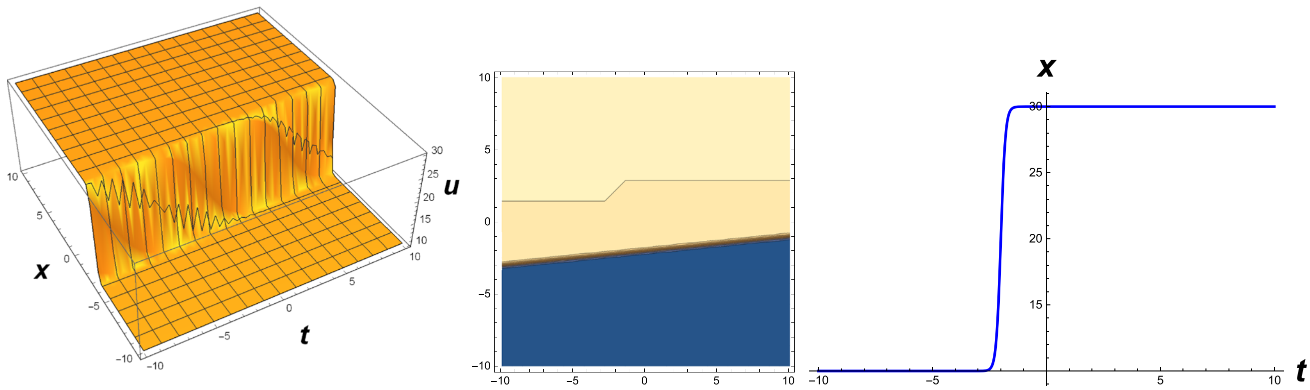

Fig. 3 Wave dynamics of hyperbolic solution (20)

Symmetry 3. The symmetry $Y_3 = \partial/\partial u$ does not yield any invariant solution.

Symmetry 4. Consider Y_4. Invariants associated with Y_4 are $I_1 = t$ and $I_2 = u - (1/6)x - x^2/(4t)$. Hence

$$u = \frac{1}{6}x + \frac{x^2}{4t} + h(t), \tag{21}$$

is the invariant solution, where arbitrary function h depends on t. Taking the above value of u and substituting it in (5) yields $h'(t) = 1/36$, whose solution is $h(t) = (1/36)t + C$, with C being a constant. Hence, the similarity solution under Y_4 is

$$u(t,x) = \frac{1}{36}(t + 6x) + \frac{x^2}{4t} + C.$$

Symmetry 5. Lastly, consider Y_5. This provides us with two invariants

$$J_1 = \frac{x}{\sqrt[3]{t}}, \quad J_2 = t^{1/3}\left(u - \frac{t}{36} - \frac{x}{6}\right).$$

Thus,

$$u(t,x) = \frac{1}{36}(t + 6x) + t^{-1/3}h(\xi), \quad \xi = \frac{x}{\sqrt[3]{t}},$$

where arbitrary function h depends on ξ. The substitution of u into Eq. (5) produces $h''' + 3h'^2 - \xi h' - h = 0$, which on integration gives $h''^2 - h'' + 2h'^3 - \xi h'^2 - 2h\, h' + \xi h + C_1 = 0$. Thus, the invariant solution under Y_5 is

$$u(t,x) = \frac{1}{36}(t + 6x) + t^{-1/3}h(\xi)$$

where $h(\xi)$ solves

$$h''^2 - h'' + 2h'^3 - \xi h'^2 - 2hh' + \xi h + C_1 = 0, \quad \xi = \frac{x}{\sqrt[3]{t}}.$$

3 Conservation Laws of (5)

Computation of conserved vectors for the pKdV Eq. (5) is done by using the method of multipliers and the theorem due to Ibragimov.

3.1 Conserved Vectors via the Multiplier Approach

We search for 2nd-order multipliers $\mathfrak{M}$, which means that $\mathfrak{M}$ depends on the variables t, x, u and their first and second derivatives. The equation that will determine $\mathfrak{M}$ is Olver (1993)

$$\frac{\delta}{\delta u}\left\{\mathfrak{M}\left(3u_t - u_x + 3u_x^2 + u_{xxx}\right)\right\} = 0, \tag{22}$$

where $\delta/\delta u$ denotes the Euler operator and

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_t\frac{\partial}{\partial u_t} - D_x\frac{\partial}{\partial u_x} - D_x^3\frac{\partial}{\partial u_{xxx}} + \cdots.$$

Expanding (22) and separating on appropriate derivatives of u produces eleven PDEs for the multiplier $\mathfrak{M}$:

$$\mathfrak{M}_x = 0,\ \mathfrak{M}_u = 0,\ \mathfrak{M}_{u_{xx}u_{xx}} = 0,\ \mathfrak{M}_{u_{tt}u_{tt}} = 0,$$
$$\mathfrak{M}_{u_{tx}u_{xx}} = 0,\ \mathfrak{M}_{u_{tx}u_{tx}} = 0,\ \mathfrak{M}_{u_{tt}u_{xx}} = 0,\ \mathfrak{M}_{u_{tt}u_{tx}} = 0,$$
$$3\mathfrak{M}_{u_x} - 2u_{tx}\mathfrak{M}_{u_{tt}} = 0,\ 3\mathfrak{M}_{u_t} + 2u_{xx}\mathfrak{M}_{u_{tt}} = 0,$$
$$\mathfrak{M}_t - 2u_{xx}^2\mathfrak{M}_{u_{xx}} - 2u_{xx}u_{tx}\mathfrak{M}_{u_{tx}} - 2u_{tt}u_{xx}\mathfrak{M}_{u_{tt}} + 2u_{xx}\mathfrak{M} = 0.$$

Solving the above equations for $\mathfrak{M}$, we get

$$\mathfrak{M} = (C_2 - 2C_1 t)u_{xx} - \frac{2}{3}C_4 u_t u_{xx} + C_4 u_{tt} + \frac{2}{3}C_4 u_x u_{tx} + C_3 u_{tx} + C_1. \tag{23}$$

This means that we get four multipliers

$$\mathfrak{M}_1 = u_{xx},\ \mathfrak{M}_2 = u_{tx},\ \mathfrak{M}_3 = tu_{xx} - \frac{1}{2},$$
$$\mathfrak{M}_4 = u_t u_{xx} - u_x u_{tx} - \frac{3}{2}u_{tt}.$$

A multiplier $\mathfrak{M}$ of (5) has the characteristic that

$$\mathfrak{M}(3u_t - u_x + 3u_x^2 + u_{xxx}) = D_t T^t + D_x T^x, \tag{24}$$

with T^t being density, and T^x flux. Hence, using (24) we acquire four conserved vectors given as

Case 1. Considering first $\mathfrak{M}_1 = u_{xx}$, the associated conserved vector is (T_1^t, T_1^x) whose components are

$$T_1^t = -\frac{3}{2}u_x^2,$$
$$T_1^x = u_x^3 + 3u_x u_t + \frac{1}{2}u_{xx}^2 - \frac{1}{2}u_x^2.$$

Case 2. The multiplier $\mathfrak{M}_2 = u_{tx}$ yields the conserved vector (T_2^t, T_2^x) with components

$$T_2^t = u_x^3 - \frac{1}{2}u_x^2 - \frac{1}{2}u_{xx}^2,$$
$$T_2^x = \frac{3}{2}u_t^2 + u_{tx}u_{xx}.$$

Case 3. For the multiplier $\mathfrak{M}_3 = tu_{xx} - 1/2$, the corresponding conserved vector is

$$T_3^t = \frac{3}{2}tuu_{xx} - \frac{3}{2}u,$$
$$T_3^x = \frac{1}{2}\left(2u_x^3 - 3uu_{tx} + 3u_x u_t - u_x^2 + u_{xx}^2\right)t - \frac{3}{2}uu_x + \frac{1}{2}u - \frac{1}{2}u_{xx}.$$

Case 4. For the fourth multiplier $\mathfrak{M}_4 = u_t u_{xx} - u_x u_{tx} - (3/2)u_{tt}$, we get

$$T_4^t = \frac{1}{2}u_x u_{xx}^2 - u_x^4 + \frac{1}{2}u_x^3 - \frac{9}{2}u_t u_x^2 + \frac{3}{2}u_t u_x - \frac{9}{4}u_t^2 + \frac{3}{2}u_{tx}u_{xx},$$
$$T_4^x = 3u_t^2 u_x - \frac{3}{4}u_t^2 + u_t u_x^3 - \frac{1}{2}u_t u_x^2 + \frac{1}{2}u_t u_{xx}^2 - u_x u_{tx} u_{xx} - \frac{3}{2}u_{tt}u_{xx} - \frac{3}{4}u_{tx}^2.$$

3.2 Conserved Vectors of (5) via Ibragimov's Method

We compute conserved vectors for pKdV Eq. (5) by invoking the conservation theorem owing to Ibragimov (2007).

The adjoint equation of (5) is

$$\mathfrak{E}^* \equiv \frac{\delta}{\delta u}\left\{v\left(u_{xxx} + 3u_t - u_x + 3u_x^2\right)\right\} = 0, \tag{25}$$

where v is a new variable. From (25) we get

$$\mathfrak{E}^* \equiv v_x - 3v_t - 6u_{xx}v - 6u_x v_x - v_{xxx} = 0. \tag{26}$$

Equations (5) and (26) have a Lagrangian

$$\mathcal{L} = 3vu_x^2 - vu_x + 3vu_t - v_x u_{xx}. \tag{27}$$

We now use symmetries admitted by pKdV Eq. (5) to determine conserved vectors associated with each symmetry. For this purpose the formulae (Ibragimov 2007)

$$C^i = \xi^i \mathcal{L} + W^\alpha\left(\frac{\partial \mathcal{L}}{\partial u_i^\alpha} - D_k \frac{\partial \mathcal{L}}{\partial u_{ik}^\alpha}\right) + D_k(W^\alpha)\frac{\partial \mathcal{L}}{\partial u_{ik}^\alpha} \tag{28}$$

is applied, where W^α is the Lie characteristic function formulated as $W^\alpha = \eta^\alpha - \xi^j u_j^\alpha$. Thus, invoking (28), we achieve the conserved vectors (C_i^t, C_i^x) for its five symmetries as

$$C_1^t = 3u_x^2 v - u_x v - u_{xx}v_x,$$
$$C_1^x = u_t v - 6u_t u_x v - u_t v_{xx} + v_t u_{xx} + v_x u_{tx};$$
$$C_2^t = -3u_x v,$$
$$C_2^x = -3u_x^2 v + 3u_t v - u_x v_{xx} + u_{xx}v_x;$$
$$C_3^t = 3v,$$
$$C_3^x = 6u_x v - v + v_{xx};$$
$$C_4^t = 3tv + 9xv - 18tu_x v,$$
$$C_4^x = 6tu_x v + 18xu_x v + 18tu_t v - tv - 3xv - 6tu_x v_{xx} + 6tu_{xx}v_x - 18tu_x^2 v + tv_{xx} - 3v_x + 3xv_{xx};$$
$$C_5^t = 81tu_x^2 v - 27tu_x v - 27xu_x v - 27uv + 3tv + 9xv - 27tu_{xx}v_x,$$
$$C_5^x = 6tu_x v - 27xu_x^2 v + 18xu_x v - 54u_x uv - 162tu_t u_x v - 9v_{xx}u + 27tu_t v + 27xu_t v + 9uv - tv - 3xv - 27tu_t v_{xx} + 27tv_t u_{xx} + 27tv_x u_{tx} + tv_{xx} + 18u_x v_x - 9xu_x v_{xx} + 9xu_{xx}v_x - 3v_x + 3xv_{xx}.$$

Remark. We observe that the two methods used here to compute conserved vectors provide us with different conserved vectors.

4 Numerical Simulation of Kink Collisions

Integrable systems are systems of differential equations (DEs) whose behaviour is determined by its initial conditions and which can be integrated from those initial conditions. The existence of many conserved vectors and the ability to construct many explicit solutions are recognized features of integrable systems. For PDEs, integrability has the characteristics of having infinitely many conservation

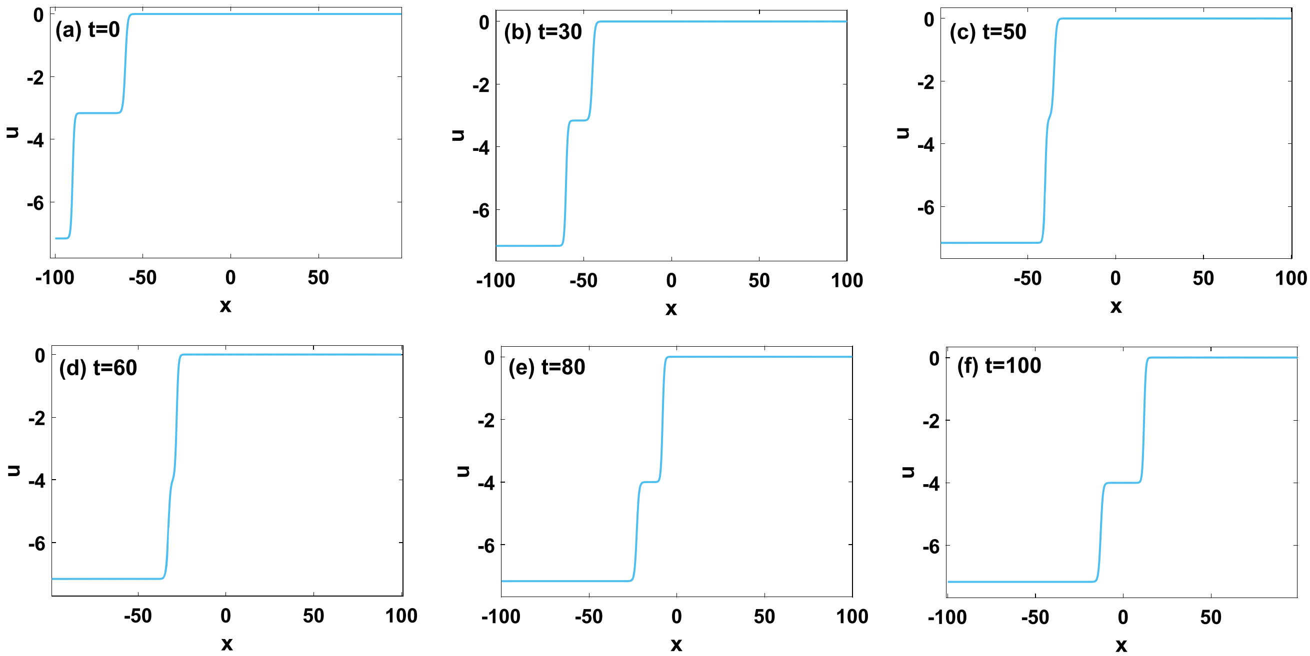

Fig. 4 Plots of two kinks moving at **a** $t = 0$, **b** $t = 30$, **c** $t = 50$, **d** $t = 60$, **e** $t = 80$ and **f** $t = 100$

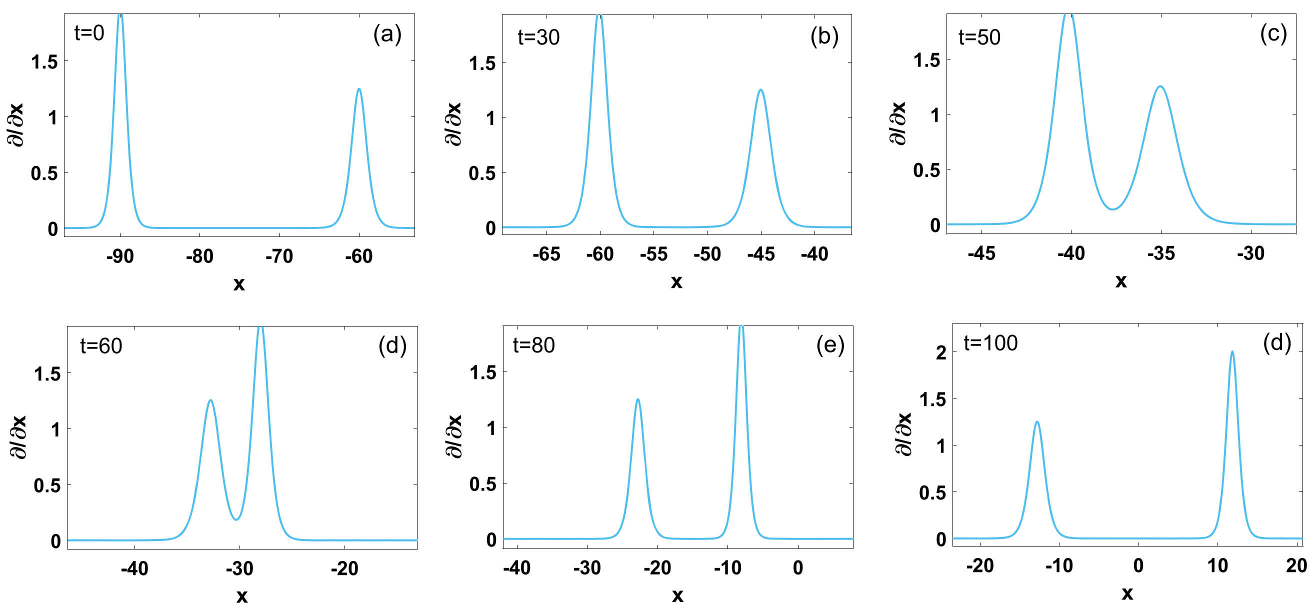

Fig. 5 Plots of derivatives of kinks moving at **a** $t = 0$, **b** $t = 30$, **c** $t = 50$, **d** $t = 60$, **e** $t = 80$ and **f** $t = 100$

laws, the existence of a Lax pair, and elastic collisions between solitons and/or kinks. Elastic collisions are collisions in which there is no net loss of kinetic energy in the system.

The aim of this section is to investigate integrability of the pKdV Eq. (5) numerically. To do this, we simulate collisions of kinks for the pKdV Eq. (5). For integrable equations, we expect these simulations to produce elastic collisions. While this does not conclusively establish integrability, it does provide a strong suggestion that the equation may be integrable.

For the numerical scheme we consider a large but finite spatial domain $x \in \left[-\frac{L}{2}, \frac{L}{2}\right]$ with interval length $L = 200$, and introduce a grid on this domain with $N = 2000$ points. In addition, we apply the Neumann boundary conditions

$$\frac{\partial u}{\partial x}\left(-\frac{L}{2}, t\right) = 0, \quad \frac{\partial u}{\partial x}\left(\frac{L}{2}, t\right) = 0,$$

and

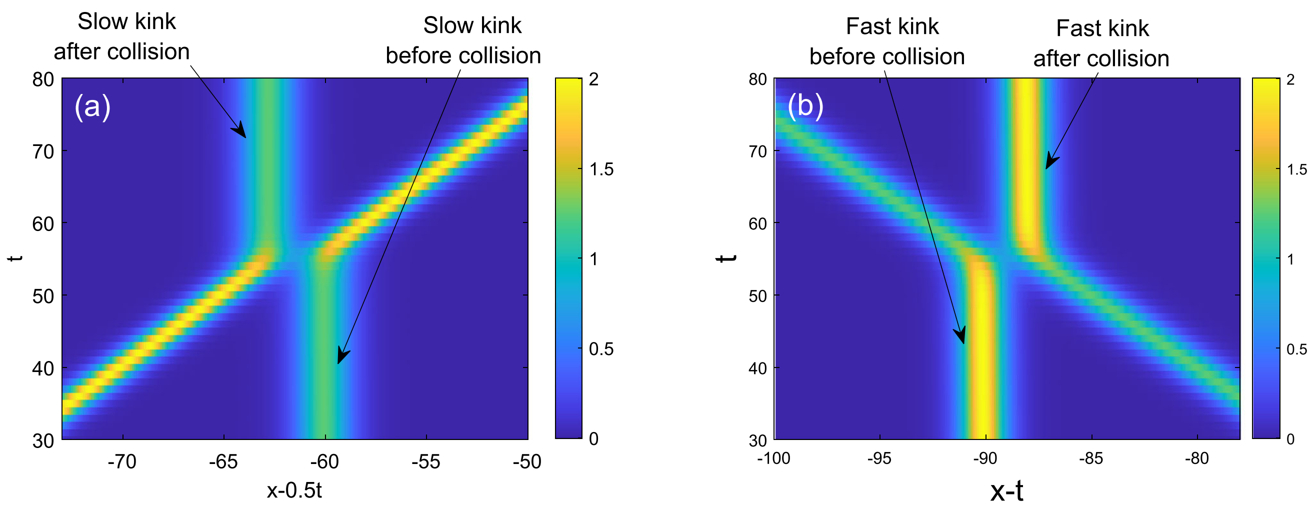

Fig. 6 **a** Kink moving at the speed of 0.5. **b** Kink moving at the speed of 1

$$\frac{\partial^2 u}{\partial x^2}\left(-\frac{L}{2},t\right)=0,\quad \frac{\partial^2 u}{\partial x^2}\left(\frac{L}{2},t\right)=0.$$

A finite difference approximation is then applied to all partial derivatives with respect to x. The resulting system of ODEs is then integrated numerically by making use of fourth-order Runge–Kutta method with a timestep of $\Delta t = 10^{-4}$ that is sufficiently small to secure the stability of the numerical scheme.

In order to simulate the collision of the kinks, we start off by taking the kink solution (20) into consideration

$$u(x,t)=\pm\sqrt{a}\tanh\left[K_1\pm\frac{\sqrt{a}(x-ct)}{2}\right], \tag{29}$$

where $a=3c+1$ and $K_2=0$. For our investigation, we consider solutions of the form (29) with $K_1=-1$ and using only the upper $+$ signs, resulting in initial conditions of the form

$$u(x,0)=\sqrt{a}\tanh\left[-1+\frac{\sqrt{a}(x)}{2}\right]. \tag{30}$$

We let the slow kink be $f(x)=u(x,0)$ when $c=0.5$ and the fast kink be $g(x)=u(x,0)$ when $c=1$. We add the kinks together and shift the fast kink 30 units behind the slow kink. The new initial condition becomes $h(x)=f(x)+g(x+30)$.

Figure 4 shows the results of two kinks before and after the collision. The initial condition of the solutions are presented in Fig. 4a. As time goes by, the kinks approach each other. This is clear from Fig. 4b where the solution is shown at $t=30$. Figure 4c and d show that the collision takes place where the two kinks seem to merge to form one large kink. After the collision, Fig. 4e and f show that the kinks re-emerge without any indication of a loss of energy.

As is clear, a kink is a sharp twist or curve in a function that is otherwise flat. The initial condition we constructed for the pKdV Eq. (5) produces two kinks that move at a constant velocity. If these kinks are differentiated with respect to x, the resulting structures have the appearance of solitons. This representation of the solution simplifies the analysis of the kink speeds before and after the collision.

Figure 5 shows the results of the derivatives of the kinks before and after the collision. In Fig. 5a, we show the initial condition of the solution. As time evolves, these structures approach each other. This is clear from Fig. 5b where the solution is shown at $t=30$. Here the tall kink starts to overtake the shorter one. Figure 5c and d show that the collision takes place where the two soliton-like structures are difficult to distinguish. However, comparing Fig. 5d with 5e it is clear that there is no dip during the collision. After the collision, Fig. 5e and f show that all the energy is retained by the solitons. Hence, this implies the collision is elastic.

To compare the speeds of the kinks before and after the collision, we plot their partial derivatives with respect to x in a moving frame. Figure 6a shows the results in a frame moving with the speed of the slow kink of 0.5. The green curve is stationary in this frame, we can see that on the interval $45<t<54$. After the collision, the green curve remains vertical, indicating that the kink does not lose or gain speed. Figure 6b shows the same result in the moving frame of the fast kink with a speed of 1. The curve is stationary in this frame, as seen on the interval $30<t<50$. After the collision, the yellow curve remains vertical, indicating that the kink does not lose or gain speed. One may therefore conclude that the collision is elastic.

5 Concluding Remarks

We investigated the nonlinear pKdV Eq. (5) by making use of the Lie group analysis in this paper. We started by computing Lie symmetries and used them to perform symmetry reductions of the pKdV equation. We then obtained solutions of (5) by using the direct integration method. The obtained solutions were expressed in terms of Jacobi elliptic functions. Thereafter, we obtained four conservation laws by utilizing the multiplier method and five conservation laws using the theorem due to Ibragimov. Finally, we investigated the integrability of the pKdV equation numerically. For this purpose, we simulated collisions between kinks for the pKdV equation. To establish the elastic or inelastic nature of the collisions, we used two criteria that were judged by the visual presentation of the results. The first was to see if there was any background perturbation arising during the collision. The second was to determine the speeds of the kinks after the collisions. The results showed that the kink collisions produced no background perturbation during the collision. Moreover, both kinks were shown to maintain their speeds after the collision. This suggested that the pKdV may be integrable.

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Declarations

Conflict of interest The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References

Ablowitz MJ, Clarkson PA (1991) Solitons, nonlinear evolution equations and inverse scattering. Cambridge University Press, Cambridge

Afrin FU (2023) Solitary wave solutions and investigation the effects of different wave velocities of the nonlinear modified Zakharov–Kuznetsov model for the wave propagation in nonlinear media. Partial Differ Equ Appl 8:100583

Chen Y, Yan Z (2005) New exact solutions of (2+1)-dimensional Gardner equation via the new sine–Gordon equation expansion method. Chaos Solit Fractals 26:399–406

Feng Z (2003) A note on "explicit exact solutions to the compound Burgers–Korteweg–de Vries equation''. Phys Lett 312:65–70

Gandarias ML, Bruzón MS (2017) Conservation laws for a Boussinesq equation. Appl Math Nonlinear Sci 2(2):465–472

Gradshteyn IS, Ryzhik IM (2007) Table of integrals, series, and products, 7th edn. Academic Press, New York

Gu CH (1990) Soliton theory and its application. Zhejiang Science and Technology Press, Zhejiang

Hirota R (2004) The direct method in soliton theory. Cambridge University Press, Cambridge

Ibragimov NH (1999) Elementary lie group analysis and ordinary differential equations. Wiley, Chichester, NY

Ibragimov NH (2007) A new conservation theorem. J Math Anal Appl 333:311–328

Kadomtsev BB, Petviashvili VI (1970) On the stability of solitary waves in weakly dispersive media. Sov Phys Dokl 15:539–541

Khalique CM, Lephoko MYT (2023) Conserved vectors and solutions of the two-dimensional potential KP equation. Open Phys 21(1):20230103

Korteweg DJ, de Vries F (1895) On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves. Philos Mag 39:422–443

Kudryashov NA (2004) Analytical theory of nonlinear differential equations. Institute of Computer Investigations, Moskow, Igevsk

Kudryashov NA (2005) Simplest equation method to look for exact solutions of nonlinear differential equations. Chaos Solit Fractals 24:1217–1231

Kudryashov NA, Loguinova NB (2008) Extended simplest equation method for nonlinear differential equations. Appl Math Comput 205:396–402

Leveque RJ (1992) Numerical methods for conservation laws, 2nd edn. Birkhäuser-Verlag, Basel

Lie S (1891) Vorlesungen über Differentialgleichungen mit bekannten infinitesimalen Transformation, G. Scheffers, ed., Teubner, Leipzig, (reprint, Chelsea, New York, 1967)

Ma WX, Zhu Z (2012) Solving the (3+1)-dimensional generalized KP and BKP equations by the multiple exp-function algorithm. Appl Math Comput 218:11871–11879

Ma WX, Abdeljabbar A, Asaad MG (2011) Wronskina and Grammian solutions to a (3+1)-dimensional generalized KP equation. Appl Math Comput 217:10016–1002

Márquez AP, de la Rosa R, Garrido TM, Gandarias ML (2023) Conservation laws and exact solutions for time-delayed Burgers–Fisher equations. Mathematics 11(17):640

Matveev VB, Salle MA (1991) Darboux transformations and solitons. Springer, New York

Noether E (1918) Invariante Variationsprobleme, Nacr. Konig. Gesell. Wissen., Gottingen, Math.-Phys. Kl. Heft 2:235–257 (English translation in Transp Theory Stat Phys 1(3):186–207 (1971))

Olver PJ (1993) Applications of lie groups to differential equations, 2nd edn. Springer, Berlin

Ovsiannikov LV (1982) Group analysis of differential equations. Academic Press, New York

Sarlet W (2010) Comment on 'Conservation laws of higher order nonlinear PDEs and the variational conservation laws in the class with mixed derivatives. J Phys A: Math Theor 43:458001

Shqair M (2019) Solution of different geometries reflected reactors neutron diffusion equation using the homotopy perturbation method. Results Phys 12:61–66

Wang M (1996) Exact solutions for a compound KdV–Burgers equation. Phys Lett 213:279–287

Wang M, Li X, Zhang J (2005) The (G'/G)-expansion method and travelling wave solutions for linear evolution equations in mathematical physics. Phys Lett A 24:1257–1268

Wazwaz AM (2005) The tanh and sine–cosine method for compact and noncompact solutions of nonlinear Klein–Gordon equation. Appl Math Comput 167:1179–1195

Wazwaz AM (2012) Multiple-soliton solutions for a (3+1)-dimensional generalized KP equation. Commun Nonlinear Sci Numer Simul 17:491–495

Wazwaz AM, El-Tantawy SA (2016) A new (3+1)-dimensional generalized Kadomtsev–Petviashvili equation. Nonlinear Dyn 84:1107–1112

Yasar E, Özer T (2011) On symmetries, conservations laws and similarity solutions of foam drainage equation. Int J Non-linear Mech 46:357–362

Yin T, Xing Z, Pang J (2023) Modified Hirota bilinear method to (3+1)-D variable coefficients generalized shallow water wave equation. Nonlinear Dyn 111(11):9741–9752

Younas U, Sulaiman TA, Ren J (2023) Dynamics of optical pulses in fiber optics with stimulated Raman scattering effect. Int J Mod Phys B 37(08):2350080

Zhang L, Khalique CM (2018) Classification and bifurcation of a class of second-order ODEs and its application to nonlinear PDEs. Discrete Contin Dyn Syst Ser S 11:777–790

Zhou Y, Wang M, Wang Y (2003) Periodic wave solutions to a coupled KdV equations with variable coefficients. Phys Lett A 308:31–36