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RESEARCH PAPER

q – Bézier Curves with Shifted Nodes

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Abstract

This article explores the applications of q-calculus in polynomial basis functions and curve modeling. We define the properties of q-Bernstein Cholodowsky basis polynomials. A novel approach to Bézier curves is introduced, utilizing basis polynomials to create generalized curves with shape-preserving properties. Additionally, the article presents degree elevation and De Casteljau algorithms tailored for these curves.

Keywords Bézier curves · Chlodowsky polynomials · De Casteljau algorithm · Degree elevation

Mathematics Subject Classification 39B82 · 41A35 · 41A44

1 Introduction

Bézier curves are elegant and fundamental mathematical constructions used in computer graphics, Computer-Aided Design (CAD), animation, and various other fields to describe and model smooth curves and shapes. Named after the French engineer Bézier (1972), who developed them in the 1960 while working at the automotive company Renault, Bézier curves have become an essential tool in the creation of digital imagery and design. He defined the curves with the help of Bernstein polynomial basis as follows:

$$C(\mathsf{v}) = \sum_{j=0}^{m} q_{m,j}(\mathsf{v}).P_j,$$

where $q_{m,j}(\mathsf{v}) = \binom{m}{j}\mathsf{v}^j(1-\mathsf{v})^{m-j}$ and $P_j, j = 0, 1, 2, \cdots, m$ are the control points.

At their core, Bézier curves are a way to represent mathematically and interpolate points within a curve's path. They are defined by a set of control points that influence the curve's shape. What makes Bézier curves especially powerful, is their ability to generate curves of varying complexity, from simple straight lines to intricate and smooth curves. This flexibility along with their intuitive control mechanisms has made Bézier curves a cornerstone in the world of computer graphics. Due to the importance of Bézier curves, they are widely studied as one can see (Hu et al. 2015; Ismail and Ali 1957; Kaur and Goyal 2023; Muraru 2010; Páleš and Rédl 2015).

For the purpose of altering the geometry of curves without altering the control points, parametric generalizations of Bézier curves emerged. There are several papers of generalization of the positive linear operators by introducing parameters as one can see (Ansari et al. 2022; Cai et al. 2022; Goyal 2022; Mursaleen et al. 2017, 2020, 2019). These parameters in basis polynomials work as a shape parameter. Han et al. (2008) used m shape parameters, such as $\lambda_i, i = 1, 2, \cdots, m$, to define $Q-$Bézier curves. The authors observed that the shape of the curves can be altered by modifying the values of shape parameters. Han et al. (2014) modified the Bézier curves by incorporating Lupaş $q-$analogue in order to regulate the curves' shape. When $q = 1$, the generalization transforms into traditional Bézier curves. Similarly, Khan et al. (2019) generalized Bézier curves based on shifted nodes using two real and non-negative parameters. The authors showed that the curves generated over any subinterval of [0, 1] with the shifted nodes are similar to classical Bézier curves.

Khatri and Mishra (2022) defined the generalized Bézier curves by choosing Bernstein-Stancu Chlodowsky basis polynomials defined by Aral et al. (2012) by using a

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positive increasing sequence $(b_n)_{n\geq 1}$ such that $\lim_{n\to\infty} b_n = \infty$, $\lim_{n\to\infty} \frac{b_n}{n} = 0$ in the following form:

$$H_{n,k}^{a,b}(t) = \binom{n}{k}\left(\frac{n+b}{n}\right)^n\left(\frac{t}{b_n} - \frac{a}{n+b}\right)^k \times \left(\frac{n+a}{n+b} - \frac{t}{b_n}\right)^{n-k}, \tag{1.1}$$

where $\frac{a}{n+b} b_n \leq t \leq \frac{n+a}{n+b} b_n$ and a, b are the real numbers such that $0 \leq a \leq b$. The authors studied the properties of basis polynomials and the curves induced by these polynomials. They also defined De Casteljau algorithm and degree elevation for these curves. Motivated by the advantages of parametric generalizations, we define the $q-$Bézier curves of basis polynomials (1.1).

The present article is structured as: In the next section, we present fundamental results of q-calculus, which serves as the foundation for the subsequent sections. Section 3 focuses on the properties and characteristics of q-Bernstein Cholodowsky basis polynomials, exploring their applications and associated results. In Sect. 4, we study a generalized approach to Bézier curves utilizing q-Bernstein Cholodowsky basis polynomials and their properties to preserve specific shapes. Section 5 discusses the degree elevation of these Bézier curves and in Sect. 6, we give De Casteljau algorithm tailored for these curves. These algorithms are essential tools for manipulating and optimizing these curves. In the last section, we present some examples to show the flexibility in the shape of the curve with the choice of the parameters.

2 Preliminaries

For a given real number $q > 0$ and any $m \in \mathbb{N}$, we have

$$[m]_q = \begin{cases} \frac{1-q^m}{1-q}, & q \neq 1 \\ m, & q = 1 \end{cases}. \tag{2.1}$$

Let $\mathbb{N}_q = \{[m]_q, m \in \mathbb{N}\}$, the set $\mathbb{N}_q$ is the generalization of the set of non-negative integers $\mathbb{N}$, which can be obtained by taking $q = 1$.

For given $m \in \mathbb{N}$, $[m]_q!$ is defined as

$$[m]_q! = \begin{cases} [m]_q.[m-1]_q \cdots [1]_q, & m \geq 1 \\ 1, & m = 0 \end{cases}. \tag{2.2}$$

The $q-$Binomial coefficient $\begin{bmatrix} m \\ j \end{bmatrix}_q$ is defined as:

$$\begin{bmatrix} m \\ j \end{bmatrix}_q = \frac{[m]_q!}{[j]_q!\,[m-j]_q!}, \tag{2.3}$$

for $m \geq j \geq 1$ and 0 otherwise.

The Pascal type relation between $q-$Binomial coefficients is defined as follows:

$$\begin{bmatrix} m \\ j \end{bmatrix}_q = \begin{bmatrix} m-1 \\ j-1 \end{bmatrix}_q + q^j \begin{bmatrix} m-1 \\ j \end{bmatrix}_q, \tag{2.4}$$

or

$$\begin{bmatrix} m \\ j \end{bmatrix}_q = q^{m-j} \begin{bmatrix} m-1 \\ j-1 \end{bmatrix}_q + \begin{bmatrix} m-1 \\ j \end{bmatrix}_q. \tag{2.5}$$

The product in q-analogue is defined in the following way:

$$(1+\mathsf{v})_q^m = \prod_{j=0}^{m-1}(1+q^j\mathsf{v}) = \sum_{j=0}^{m} q^{\frac{j(j-1)}{2}} \begin{bmatrix} m \\ j \end{bmatrix}_q \mathsf{v}^j. \tag{2.6}$$

3 Properties of q-Bernstein Cholodowsky Basis Polynomial

Firstly, we define the $q-$analogue of (1.1) as the basis polynomial for $n \in \mathbb{N}$ as follows:

$$p_{n,k}^q(\mathsf{v}) = \frac{\begin{bmatrix} n \\ k \end{bmatrix}_q \left(\frac{\mathsf{v}}{b_n} - \frac{\alpha}{[n]_q+\beta}\right)^k \left(\frac{[n]_q+\alpha}{[n]_q+\beta} - \frac{\mathsf{v}}{b_n}\right)^{n-k} q^{\frac{k(k-1)}{2}}}{\prod_{j=1}^{n}\left(\frac{[n]_q+\alpha}{[n]_q+\beta} - \frac{\mathsf{v}}{b_n} + q^{j-1}\left(\frac{\mathsf{v}}{b_n} - \frac{\alpha}{[n]_q+\beta}\right)\right)} := \frac{1}{\prod_{j=1}^{n}\left(\frac{[n]_q+\alpha}{[n]_q+\beta} - \frac{\mathsf{v}}{b_n} + q^{j-1}\left(\frac{\mathsf{v}}{b_n} - \frac{\alpha}{[n]_q+\beta}\right)\right)} \mu_{n,k}^q(\mathsf{v}), \tag{3.1}$$

where $\mathsf{v} \in \left[\frac{\alpha b_n}{[n]_q+\beta}, \frac{[n]_q+\alpha}{[n]_q+\beta} b_n\right]$.

Theorem 1 *The basis polynomial* (3.1) *has the following properties*:

(1) Non-negativity: $p_{n,k}^q(\mathsf{v}) \geq 0$ for $\mathsf{v} \in \left[\frac{\alpha b_n}{[n]_q+\beta}, \frac{[n]_q+\alpha}{[n]_q+\beta} b_n\right]$.

(2) Partition of unity: $\sum_{k=0}^{n} p_{n,k}^q(\mathsf{v}) = 1$.

(3) End point interpolation property $p_{n,k}^q\left(\frac{\alpha b_n}{[n]_q+\beta}\right) = \begin{cases} 1, & k = 0 \\ 0, & k \neq 0 \end{cases}$,
$p_{n,k}^q\left(\frac{[n]_q+\alpha}{[n]_q+\beta} b_n\right) = \begin{cases} 1, & k = n \\ 0, & k \neq n \end{cases}$.

(4) Reducibility: when $q = 1, b_n = 1$ and $\alpha = \beta = 0$, it reduces to classical Bernstein polynomials.

(5) Symmetry: $p^q_{n,n-k}(t) = p^{\frac{1}{q}}_{n,k}\left(\frac{[n]_q}{[n]_q+\beta} - t\right)$, where $t = \frac{v}{b_n} - \frac{\alpha}{[n]_q+\beta}$.

Proof The proofs for the properties (1), (3) and (4) are quite simple. So, we left it for the reader.

For property (2), consider

$$\begin{aligned}\sum_{k=0}^{n}\mu^q_{n,k}(v) &= \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q\left(\frac{v}{b_n} - \frac{\alpha}{[n]_q+\beta}\right)^k \\ &\quad\times\left(\frac{[n]_q+\alpha}{[n]_q+\beta} - \frac{v}{b_n}\right)^{n-k} q^{\frac{k(k-1)}{2}} \\ &= \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q\left(\frac{\frac{v}{b_n} - \frac{\alpha}{[n]_q+\beta}}{\frac{[n]_q+\alpha}{[n]_q+\beta} - \frac{v}{b_n}}\right)^k \\ &\quad\times\left(\frac{[n]_q+\alpha}{[n]_q+\beta} - \frac{v}{b_n}\right)^{n} q^{\frac{k(k-1)}{2}}.\end{aligned} \tag{3.2}$$

Using (2.6), it becomes:

$$\begin{aligned}\sum_{k=0}^{n}\mu^q_{n,k}(v) &= \left(\frac{[n]_q+\alpha}{[n]_q+\beta} - \frac{v}{b_n}\right)^n\left(1 + \frac{\frac{v}{b_n} - \frac{\alpha}{[n]_q+\beta}}{\frac{[n]_q+\alpha}{[n]_q+\beta} - \frac{v}{b_n}}\right) \\ &\quad\times\left(1 + q\frac{\frac{v}{b_n} - \frac{\alpha}{[n]_q+\beta}}{\frac{[n]_q+\alpha}{[n]_q+\beta} - \frac{v}{b_n}}\right)\cdots \\ &\quad\times\left(1 + q^{n-1}\frac{\frac{v}{b_n} - \frac{\alpha}{[n]_q+\beta}}{\frac{[n]_q+\alpha}{[n]_q+\beta} - \frac{v}{b_n}}\right) \\ &= \left(\frac{[n]_q+\alpha}{[n]_q+\beta} - \frac{v}{b_n} + \frac{v}{b_n} - \frac{\alpha}{[n]_q+\beta}\right) \\ &\quad\times\left(\frac{[n]_q+\alpha}{[n]_q+\beta} - \frac{v}{b_n} + q\left(\frac{v}{b_n} - \frac{\alpha}{[n]_q+\beta}\right)\right)\cdots \\ &\quad\times\left(\frac{[n]_q+\alpha}{[n]_q+\beta} - \frac{v}{b_n} + q^{n-1}\left(\frac{v}{b_n} - \frac{\alpha}{[n]_q+\beta}\right)\right) \\ &= \prod_{j=1}^{n}\left(\frac{[n]_q+\alpha}{[n]_q+\beta} - \frac{v}{b_n} + q^{j-1}\left(\frac{v}{b_n} - \frac{\alpha}{[n]_q+\beta}\right)\right).\end{aligned} \tag{3.3}$$

Thus, by using (3.1) and (3.3), we get

$$\begin{aligned}\sum_{k=0}^{n}p^q_{n,k}(v) &= \sum_{k=0}^{n}\frac{\mu^q_{n,k}(v)}{\prod_{j=1}^{n}\left(\frac{[n]_q+\alpha}{[n]_q+\beta} - \frac{v}{b_n} + q^{j-1}\left(\frac{v}{b_n} - \frac{\alpha}{[n]_q+\beta}\right)\right)} \\ &= 1.\end{aligned}$$

Now, for property (5), consider $t = \frac{v}{b_n} - \frac{\alpha}{[n]_q+\beta}$.

$$\begin{aligned}p^q_{n,n-k}(t) &= \frac{\mu^q_{n,n-k}(v)}{\prod_{j=1}^{n}\left(\frac{[n]_q+\alpha}{[n]_q+\beta} - \frac{v}{b_n} + q^{j-1}\left(\frac{v}{b_n} - \frac{\alpha}{[n]_q+\beta}\right)\right)} \\ &= \frac{\mu^q_{n,n-k}(v)}{\prod_{j=1}^{n}q^{j-1}\left(\frac{1}{q^{j-1}}\left(\frac{[n]_q+\alpha}{[n]_q+\beta} - \frac{v}{b_n}\right) + \frac{v}{b_n} - \frac{\alpha}{[n]_q+\beta}\right)} \\ &= \frac{\begin{bmatrix} n \\ n-k \end{bmatrix}_q\left(\frac{v}{b_n} - \frac{\alpha}{[n]_q+\beta}\right)^{n-k}\left(\frac{[n]_q+\alpha}{[n]_q+\beta} - \frac{v}{b_n}\right)^k q^{\frac{(n-k)(n-k-1)}{2}}}{q^{\frac{n(n-1)}{2}}\prod_{j=1}^{n}\left(\frac{1}{q^{j-1}}\left(\frac{[n]_q+\alpha}{[n]_q+\beta} - \frac{v}{b_n}\right) + \frac{v}{b_n} - \frac{\alpha}{[n]_q+\beta}\right)} \\ &= \frac{\begin{bmatrix} n \\ k \end{bmatrix}_{\frac{1}{q}}\left(\frac{v}{b_n} - \frac{\alpha}{[n]_q+\beta}\right)^{n-k}\left(\frac{[n]_q+\alpha}{[n]_q+\beta} - \frac{v}{b_n}\right)^k}{q^{\frac{k(k-1)}{2}}\prod_{j=1}^{n}\left(\frac{1}{q^{j-1}}\left(\frac{[n]_q+\alpha}{[n]_q+\beta} - \frac{v}{b_n}\right) + \frac{v}{b_n} - \frac{\alpha}{[n]_q+\beta}\right)} \\ &= \frac{\mu^{\frac{1}{q}}_{n,k}\left(\frac{[n]_q}{[n]_q+\beta} - t\right)}{\prod_{j=1}^{n}\left(\frac{1}{q^{j-1}}\left(\frac{[n]_q+\alpha}{[n]_q+\beta} - \frac{v}{b_n}\right) + \frac{v}{b_n} - \frac{\alpha}{[n]_q+\beta}\right)} \\ &= p^{\frac{1}{q}}_{n,k}\left(\frac{[n]_q}{[n]_q+\beta} - t\right).\end{aligned}$$

Hence, the proof is completed.

Now, we present the effect of different values of the parameter q in the basis functions and choosing $n = 4, \alpha = 0.2, \beta = 0.8, b_n = \sqrt{n}$. In Fig. 1 the values of $q = 1$, whereas $q = 2$ in Fig. 2. □

Theorem 2 *For $n \geq 1$, each n^{th} degree basis polynomial can be written as combination of two $(n-1)^{th}$ degree basis polynomials, that is*

$$p_{n,k}^q(\mathsf{v}) = \frac{q^{k-1}}{\left(\frac{[n]_q+\alpha}{[n]_q+\beta} - \frac{\mathsf{v}}{b_n} + q^{n-1}\left(\frac{\mathsf{v}}{b_n} - \frac{\alpha}{[n]_q+\beta}\right)\right)} \times \left(\frac{\mathsf{v}}{b_n} - \frac{\alpha}{[n]_q+\beta}\right) p_{n-1,k-1}^q(\mathsf{v}) + \frac{q^k}{\left(\frac{[n]_q+\alpha}{[n]_q+\beta} - \frac{\mathsf{v}}{b_n} + q^{n-1}\left(\frac{\mathsf{v}}{b_n} - \frac{\alpha}{[n]_q+\beta}\right)\right)} \times \left(\frac{[n]_q+\alpha}{[n]_q+\beta} - \frac{\mathsf{v}}{b_n}\right) p_{n-1,k}^q(\mathsf{v}).$$

Proof Consider

$$p_{n,k}^q(\mathsf{v}) = \frac{q^{\frac{k(k-1)}{2}}}{\prod_{j=1}^{n}\left(\frac{[n]_q+\alpha}{[n]_q+\beta} - \frac{\mathsf{v}}{b_n} + q^{j-1}\left(\frac{\mathsf{v}}{b_n} - \frac{\alpha}{[n]_q+\beta}\right)\right)} \times \begin{bmatrix} n \\ k \end{bmatrix}_q \left(\frac{\mathsf{v}}{b_n} - \frac{\alpha}{[n]_q+\beta}\right)^k \left(\frac{[n]_q+\alpha}{[n]_q+\beta} - \frac{\mathsf{v}}{b_n}\right)^{n-k}.$$

Using the recurrence Eq. (2.4) of $q-$binomial coefficients:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q + q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q.$$

Therefore,

$$\begin{aligned}
p_{n,k}^q(\mathsf{v}) &= \frac{q^{\frac{k(k-1)}{2}}}{\prod_{j=1}^{n}\left(\frac{[n]_q+\alpha}{[n]_q+\beta} - \frac{\mathsf{v}}{b_n} + q^{j-1}\left(\frac{\mathsf{v}}{b_n} - \frac{\alpha}{[n]_q+\beta}\right)\right)} \\
&\quad \times \left(\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q + q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q\right) \left(\frac{\mathsf{v}}{b_n} - \frac{\alpha}{[n]_q+\beta}\right)^k \\
&\quad \times \left(\frac{[n]_q+\alpha}{[n]_q+\beta} - \frac{\mathsf{v}}{b_n}\right)^{n-k} \\
&= \frac{q^{\frac{k(k-1)}{2}}}{\prod_{j=1}^{n}\left(\frac{[n]_q+\alpha}{[n]_q+\beta} - \frac{\mathsf{v}}{b_n} + q^{j-1}\left(\frac{\mathsf{v}}{b_n} - \frac{\alpha}{[n]_q+\beta}\right)\right)} \\
&\quad \times \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q \left(\frac{\mathsf{v}}{b_n} - \frac{\alpha}{[n]_q+\beta}\right)^k \left(\frac{[n]_q+\alpha}{[n]_q+\beta} - \frac{\mathsf{v}}{b_n}\right)^{n-k} \\
&\quad + \frac{q^{\frac{k(k-1)}{2}} q^k}{\prod_{j=1}^{n}\left(\frac{[n]_q+\alpha}{[n]_q+\beta} - \frac{\mathsf{v}}{b_n} + q^{j-1}\left(\frac{\mathsf{v}}{b_n} - \frac{\alpha}{[n]_q+\beta}\right)\right)} \\
&\quad \times \begin{bmatrix} n-1 \\ k \end{bmatrix}_q \left(\frac{\mathsf{v}}{b_n} - \frac{\alpha}{[n]_q+\beta}\right)^k \left(\frac{[n]_q+\alpha}{[n]_q+\beta} - \frac{\mathsf{v}}{b_n}\right)^{n-k} \\
&= \frac{q^{\frac{(k-1)(k-2)}{2}+k-1}}{\prod_{j=1}^{n}\left(\frac{[n]_q+\alpha}{[n]_q+\beta} - \frac{\mathsf{v}}{b_n} + q^{j-1}\left(\frac{\mathsf{v}}{b_n} - \frac{\alpha}{[n]_q+\beta}\right)\right)} \\
&\quad \times \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q \left(\frac{\mathsf{v}}{b_n} - \frac{\alpha}{[n]_q+\beta}\right)^k \left(\frac{[n]_q+\alpha}{[n]_q+\beta} - \frac{\mathsf{v}}{b_n}\right)^{n-k} \\
&\quad + \frac{q^{\frac{k(k-1)}{2}+k}}{\prod_{j=1}^{n}\left(\frac{[n]_q+\alpha}{[n]_q+\beta} - \frac{\mathsf{v}}{b_n} + q^{j-1}\left(\frac{\mathsf{v}}{b_n} - \frac{\alpha}{[n]_q+\beta}\right)\right)} \\
&\quad \times \begin{bmatrix} n-1 \\ k \end{bmatrix}_q \left(\frac{\mathsf{v}}{b_n} - \frac{\alpha}{[n]_q+\beta}\right)^k \left(\frac{[n]_q+\alpha}{[n]_q+\beta} - \frac{\mathsf{v}}{b_n}\right)^{n-k}.
\end{aligned}$$

Hence, we reach at the desired result. □

Theorem 3 *Each n^{th} degree basis polynomial can be expressed as linear combination of two $(n+1)^{th}$ degree polynomials in the following way*

$$p_{n,k}^q(\mathsf{v}) = \frac{[n-k+1]_q}{[n+1]_q} p_{n+1,k}^q(\mathsf{v}) + \left(1 - \frac{[n-k]_q}{[n+1]_q}\right) p_{n+1,k+1}^q(\mathsf{v}).$$

Proof

$$\begin{aligned}
p_{n,k}^q(\mathsf{v}) &= p_{n,k}^q(\mathsf{v})\left(1 - \frac{q^n\left(\frac{\mathsf{v}}{b_n} - \frac{\alpha}{[n]_q+\beta}\right)}{\frac{[n]_q+\alpha}{[n]_q+\beta} - \frac{\mathsf{v}}{b_n} + q^n\left(\frac{\mathsf{v}}{b_n} - \frac{\alpha}{[n]_q+\beta}\right)} + \frac{q^n\left(\frac{\mathsf{v}}{b_n} - \frac{\alpha}{[n]_q+\beta}\right)}{\frac{[n]_q+\alpha}{[n]_q+\beta} - \frac{\mathsf{v}}{b_n} + q^n\left(\frac{\mathsf{v}}{b_n} - \frac{\alpha}{[n]_q+\beta}\right)}\right) \\
&= \frac{\left(\frac{[n]_q+\alpha}{[n]_q+\beta} - \frac{\mathsf{v}}{b_n}\right) q^{\frac{k(k-1)}{2}}}{\prod_{j=1}^{n+1}\left(\frac{[n]_q+\alpha}{[n]_q+\beta} - \frac{\mathsf{v}}{b_n} + q^{j-1}\left(\frac{\mathsf{v}}{b_n} - \frac{\alpha}{[n]_q+\beta}\right)\right)} \\
&\quad \times \begin{bmatrix} n \\ k \end{bmatrix}_q \left(\frac{\mathsf{v}}{b_n} - \frac{\alpha}{[n]_q+\beta}\right)^k \left(\frac{[n]_q+\alpha}{[n]_q+\beta} - \frac{\mathsf{v}}{b_n}\right)^{n-k} \\
&\quad + \frac{q^n q^{\frac{k(k-1)}{2}}\left(\frac{\mathsf{v}}{b_n} - \frac{\alpha}{[n]_q+\beta}\right)}{\prod_{j=1}^{n+1}\left(\frac{[n]_q+\alpha}{[n]_q+\beta} - \frac{\mathsf{v}}{b_n} + q^{j-1}\left(\frac{\mathsf{v}}{b_n} - \frac{\alpha}{[n]_q+\beta}\right)\right)} \\
&\quad \times \begin{bmatrix} n \\ k \end{bmatrix}_q \left(\frac{\mathsf{v}}{b_n} - \frac{\alpha}{[n]_q+\beta}\right)^k \left(\frac{[n]_q+\alpha}{[n]_q+\beta} - \frac{\mathsf{v}}{b_n}\right)^{n-k}.
\end{aligned} \tag{3.4}$$

Since, we know, $\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n-k+1]_q}{[n+1]_q}\begin{bmatrix} n+1 \\ k \end{bmatrix}_q$, $\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[k+1]_q}{[n+1]_q}\begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_q$, and

$$q^{n-k}\frac{[k+1]_q}{[n+1]_q} = \left(1 - \frac{[n-k]_q}{[n+1]_q}\right)$$

Using these equalities and (3.4), we get

$$\begin{aligned} p^q_{n,k}(\mathsf{v}) =& \frac{[n-k+1]_q}{[n+1]_q}\begin{bmatrix} n+1 \\ k \end{bmatrix}_q \left(\frac{\mathsf{v}}{b_n} - \frac{\alpha}{[n]_q+\beta}\right)^k \\ &\times \frac{q^{\frac{k(k-1)}{2}}}{\prod_{j=1}^{n+1}\left(\frac{[n]_q+\alpha}{[n]_q+\beta} - \frac{\mathsf{v}}{b_n} + q^{j-1}\left(\frac{\mathsf{v}}{b_n} - \frac{\alpha}{[n]_q+\beta}\right)\right)} \\ &\times \left(\frac{[n]_q+\alpha}{[n]_q+\beta} - \frac{\mathsf{v}}{b_n}\right)^{n-k+1} \\ &+ \frac{[k+1]_q}{[n+1]_q}\frac{q^{\frac{k(k+1)}{2}+n-k}}{\prod_{j=1}^{n+1}\left(\frac{[n]_q+\alpha}{[n]_q+\beta} - \frac{\mathsf{v}}{b_n} + q^{j-1}\left(\frac{\mathsf{v}}{b_n} - \frac{\alpha}{[n]_q+\beta}\right)\right)} \\ &\times \begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_q \left(\frac{\mathsf{v}}{b_n} - \frac{\alpha}{[n]_q+\beta}\right)^{k+1} \\ &\times \left(\frac{[n]_q+\alpha}{[n]_q+\beta} - \frac{\mathsf{v}}{b_n}\right)^{n-k}. \end{aligned}$$

Hence, we get the required result. □

4 Construction of Bézier Curves

We define the generalization of Bézier curves with the help of the q-Bernstein Cholodowsky basis polynomial (3.1) in the following way:

$$C(\mathsf{v}, q) = \sum_{k=0}^{n} p^q_{n,k}(\mathsf{v}).C_k, \tag{4.1}$$

where C_k, $k = 0, 1, \cdots, n$ are the control points of the curves.

Theorem 4 *The Bézier curves defined by (4.1) have the following properties:*

(1) Bézier curves lie inside the control polygon determined by the control points.

(2) End point interpolation property: $C\left(\frac{\alpha b_n}{[n]_q+\beta}\right) = C_0$ and $C\left(\frac{[n]_q+\alpha}{[n]_q+\beta}b_n\right) = C_n$.

(3) Reducibility: for $q = 1$, $\alpha = \beta = 0$, $b_n = 1$, the curves (4.1) reduce to classical Bézier curves.

Proof The above properties are easy to prove by using the results of Theorem 1. Hence, we omit the details. □

Theorem 5 *The Bézier curves (4.1) have the end-point derivative property as:*

$$C'\left(\frac{\alpha b_n}{[n]_q+\beta}\right) = \frac{[n]_q(C_1 - C_0)}{b_n\left(\frac{[n]_q}{[n]_q+\beta}\right)},$$

$$C'\left(\frac{([n]_q+\alpha)b_n}{[n]_q+\beta}\right) = \frac{[n]_q(C_n - C_{n-1})}{b_n q^{n-1}\left(\frac{[n]_q}{[n]_q+\beta}\right)}.$$

Proof Let

$$\begin{aligned} C(\mathsf{v}) &= \sum_{k=0}^{n} C_k.p^q_{n,k}(\mathsf{v}) \\ &= \sum_{k=0}^{n} \frac{\mu^q_{n,k}(\mathsf{v})}{\prod_{j=1}^{n}\left(\frac{[n]_q+\alpha}{[n]_q+\beta} - \frac{\mathsf{v}}{b_n} + q^{j-1}\left(\frac{\mathsf{v}}{b_n} - \frac{\alpha}{[n]_q+\beta}\right)\right)}.C_k \\ &:= \frac{C_1(\mathsf{v})}{C_2(\mathsf{v})}. \end{aligned}$$

Thus,

$$C(\mathsf{v}).C_2(\mathsf{v}) = C_1(\mathsf{v}).$$

Now, differentiating on both sides w.r.t. v, we get:

$$C(\mathsf{v}).C_2'(\mathsf{v}) + C_2(\mathsf{v}).C'(\mathsf{v}) = C_1'(\mathsf{v}). \tag{4.2}$$

$C_1(\mathsf{v}) = \sum_{k=0}^{n} C_k.\mu^q_{n,k}(\mathsf{v})$, where

$$\mu^q_{n,k}(\mathsf{v}) = \begin{bmatrix} n \\ k \end{bmatrix}_q \left(\frac{\mathsf{v}}{b_n} - \frac{\alpha}{[n]_q+\beta}\right)^k \left(\frac{[n]_q+\alpha}{[n]_q+\beta} - \frac{\mathsf{v}}{b_n}\right)^{n-k} q^{\frac{k(k-1)}{2}},$$

and by using property (2) of Theorem 1,

$$C_2(\mathsf{v}) = \sum_{k=0}^{n} \mu^q_{n,k}(\mathsf{v}).$$

Now,

$$\begin{aligned}(\mu^q_{n,k})'(\mathsf{v}) &= \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{k}{b_n}\left(\frac{\mathsf{v}}{b_n} - \frac{\alpha}{[n]_q+\beta}\right)^{k-1} \\ &\quad\times \left(\frac{[n]_q+\alpha}{[n]_q+\beta} - \frac{\mathsf{v}}{b_n}\right)^{n-k} q^{\frac{k(k-1)}{2}} \\ &\quad- \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{n-k}{b_n}\left(\frac{\mathsf{v}}{b_n} - \frac{\alpha}{[n]_q+\beta}\right)^{k} \\ &\quad\times \left(\frac{[n]_q+\alpha}{[n]_q+\beta} - \frac{\mathsf{v}}{b_n}\right)^{n-k-1} q^{\frac{k(k-1)}{2}} \\ &= \frac{[n]_q}{[k]_q}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q \frac{k}{b_n}\left(\frac{\mathsf{v}}{b_n} - \frac{\alpha}{[n]_q+\beta}\right)^{k-1} \\ &\quad\times \left(\frac{[n]_q+\alpha}{[n]_q+\beta} - \frac{\mathsf{v}}{b_n}\right)^{n-k} q^{\frac{k(k-1)}{2}} \\ &\quad- \frac{[n]_q}{[n-k]_q}\begin{bmatrix} n-1 \\ k \end{bmatrix}_q \frac{n-k}{b_n}\left(\frac{\mathsf{v}}{b_n} - \frac{\alpha}{[n]_q+\beta}\right)^{k} \\ &\quad\times \left(\frac{[n]_q+\alpha}{[n]_q+\beta} - \frac{\mathsf{v}}{b_n}\right)^{n-k-1} q^{\frac{k(k-1)}{2}} \\ &= \frac{[n]_q}{[k]_q}\cdot\frac{k}{b_n}.q^{k-1}.\mu^q_{n-1,k-1}(\mathsf{v}) \\ &\quad- \frac{[n]_q}{[n-k]_q}\cdot\frac{n-k}{b_n}.\mu^q_{n-1,k}(\mathsf{v}). \end{aligned} \tag{4.3}$$

We can easily calculate:

$$C\left(\frac{\alpha b_n}{[n]_q+\beta}\right) = C_0,\ C_2\left(\frac{\alpha b_n}{[n]_q+\beta}\right) = \left(\frac{[n]_q}{[n]_q+\beta}\right)^n,$$

$$C_1'\left(\frac{\alpha b_n}{[n]_q+\beta}\right) = \frac{[n]_q C_1 - nC_0}{b_n}\left(\frac{[n]_q}{[n]_q+\beta}\right)^{n-1},$$

$$C_2'\left(\frac{\alpha b_n}{[n]_q+\beta}\right) = \frac{([n]_q - n)}{b_n}\left(\frac{[n]_q}{[n]_q+\beta}\right)^{n-1}.$$

Using these equalities and (4.2), we get:

$$C'\left(\frac{\alpha b_n}{[n]_q+\beta}\right) = \frac{[n]_q(C_1 - C_0)}{b_n\left(\frac{[n]_q}{[n]_q+\beta}\right)}.$$

Similarly, Using (4.3)

$$C\left(\frac{[n]_q+\alpha}{[n]_q+\beta}b_n\right) = C_n,\ C_2\left(\frac{[n]_q+\alpha}{[n]_q+\beta}b_n\right) = \left(\frac{[n]_q}{[n]_q+\beta}\right)^n q^{\frac{n(n-1)}{2}},$$

$$C_1'\left(\frac{[n]_q+\alpha}{[n]_q+\beta}b_n\right) = \frac{(nq^{n-1}C_n - [n]_q C_{n-1})q^{\frac{(n-1)(n-2)}{2}}}{b_n}\left(\frac{[n]_q}{[n]_q+\beta}\right)^{n-1},$$

$$C_2'\left(\frac{[n]_q+\alpha}{[n]_q+\beta}b_n\right) = \frac{(nq^{n-1} - [n]_q)q^{\frac{(n-1)(n-2)}{2}}}{b_n}\left(\frac{[n]_q}{[n]_q+\beta}\right)^{n-1}.$$

Again, with the help of identity (4.2), we find:

$$C'\left(\frac{([n]_q+\alpha)b_n}{[n]_q+\beta}\right) = \frac{[n]_q(C_n - C_{n-1})}{b_n q^{n-1}\left(\frac{[n]_q}{[n]_q+\beta}\right)}.$$

□

Remark 1 This property represents that tangent at the point $x = 0$ is the resultant of vector from P_0 to P_1. Similarly, the tangent at the end point $x = 1$ is also the resultant of vector from P_{n-1} to P_n.

5 Degree Elevation

Degree elevation has applications in computer graphics, CAD, and font design, where it improves curve smoothness and complexity control. It is also used for data fitting and interpolation, ensuring accurate curve representations.

Degree elevation of Bézier curves is a mathematical technique used to increase the degree of a Bézier curve and control points while preserving its shape. This process involves introducing new control points to create a higher-degree curve, allowing for comparison with compatible curves. It is a valuable tool in computer graphics and design for achieving smoother and more precise curves.

Let $C(\mathsf{v}) = \sum_{k=0}^{n} p^q_{n,k}(\mathsf{v}).C_k$ where $k = 0, 1, \ldots, n$.

This curve can be represented as $C(\mathsf{v}) = \sum_{k=0}^{n+1} p^q_{n+1,k}(\mathsf{v}).D_k$, by applying the technique of degree elevation. In this process, we get $(n+2)$ control points D_k to determine new curves in the following way:

$$D_0 = C_0, \quad D_{n+1} = C_n,$$

$$D_k = \left(1 - \frac{[n-k+1]_q}{[n+1]_q}\right)C_{k-1} + \frac{[n-k+1]_q}{[n+1]_q}C_k,$$

$$\text{where } k = 1, 2, \cdots, n.$$

Let $A = [C_0, C_1, \cdots, C_n]^T$ be the vector of (n+1) control points for the given Bézier curves of degree n and $B = [D_0, D_1, \cdots, D_{n+1}]^T$ be the vector of (n+2) control points of the new Bézier curves of degree $n+1$. We can find the relationship between vectors A and B denoted by $B = M_{n+1}A$, where M_{n+1} is the matrix of order $(n+2) \times (n+1)$ defined as follows:

$$M_{n+1} = \frac{1}{[n+1]_q}\begin{bmatrix} [n+1]_q & 0 & 0 & \cdots & 0 & 0 & 0 \\ [n+1]_q-[n]_q & [n]_q & 0 & \cdots & 0 & 0 & 0 \\ 0 & [n+1]_q-[n-1]_q & [n-1]_q & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & [n+1]_q-[2]_q & [2]_q & 0 \\ 0 & 0 & 0 & \cdots & 0 & [n+1]_q-[1]_q & [1]_q \\ 0 & 0 & 0 & \cdots & 0 & 0 & [n+1]_q \end{bmatrix}.$$

Similarly, we can also represent this curve to any higher degree curve without changing its shape.

For $m \in \mathbb{N}$, we can find Bézier curve of degree $n+m$. The vector of control points of degree elevated curve having degree $n+m$ is $R^T = [R_0, R_1, \cdots, R_{n+m}]$, where $R = M_{n+m}\cdots M_{n+2}M_{n+1}P$. For $m \to \infty$, the control polygon converges to Bézier curve.

6 De Casteljau Agorithm

The De Casteljau algorithm is a fundamental method for evaluating Bézier curves and surfaces. It works by recursively dividing control points to find a point on the curve. Starting with the original control points, it repeatedly computes intermediate points along the curve or surface until the desired level of precision is achieved.

Consider $P_i^0 = P_i$ where $i = 0, 1, \cdots, n$.

$$P_i^r(\mathsf{v};q) = \frac{\left(\frac{[n]_q+\alpha}{[n]_q+\beta} - \frac{\mathsf{v}}{b_n}\right)}{\left(\frac{[n]_q+\alpha}{[n]_q+\beta} - \frac{\mathsf{v}}{b_n} + q^{n-r}\left(\frac{\mathsf{v}}{b_n} - \frac{\alpha}{[n]_q+\beta}\right)\right)} P_i^{r-1}(\mathsf{v};q) + \frac{q^{n-r}\left(\frac{\mathsf{v}}{b_n} - \frac{\alpha}{[n]_q+\beta}\right)}{\left(\frac{[n]_q+\alpha}{[n]_q+\beta} - \frac{\mathsf{v}}{b_n} + q^{n-r}\left(\frac{\mathsf{v}}{b_n} - \frac{\alpha}{[n]_q+\beta}\right)\right)} P_{i+1}^{r-1}(\mathsf{v};q),$$
$$r = 1, 2, \cdots, n, \; k = 0, 1, \cdots, n-r. \tag{6.1}$$

The matrix to represent these points:

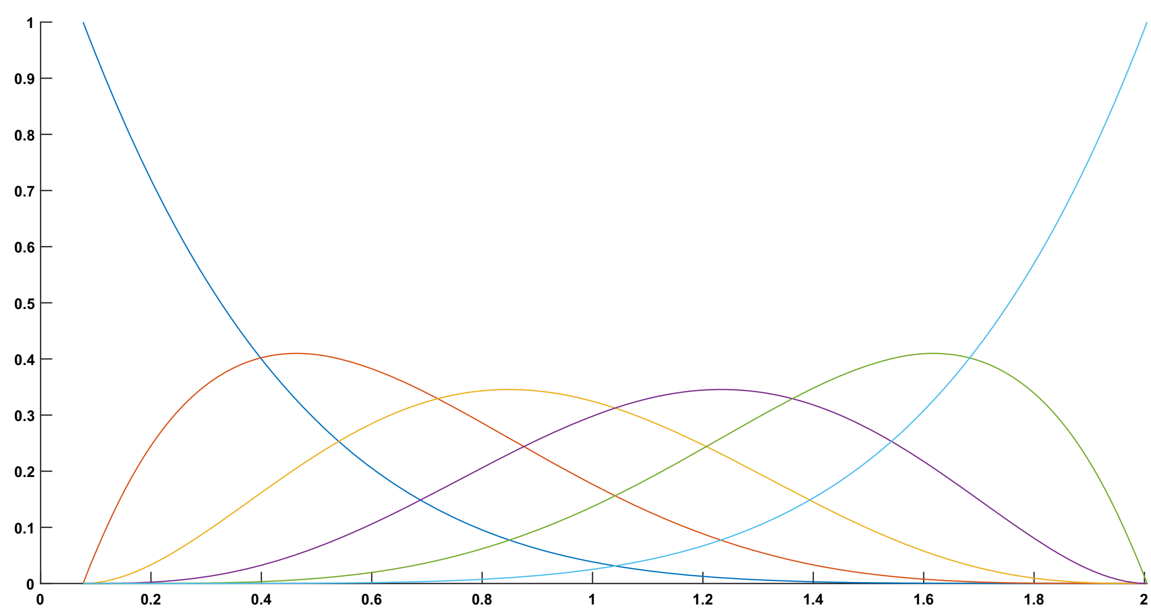

Fig. 1 Basis polynomials for $n = 4, \alpha = 0.2, \beta = 0.8, b_n = \sqrt{n}, q = 1$

$$Q_{n,r}=\left(\frac{[n]_q+\alpha}{[n]_q+\beta}-\frac{\mathsf{v}}{b_n}+q^{n-r}\left(\frac{\mathsf{v}}{b_n}-\frac{\alpha}{[n]_q+\beta}\right)\right)^{-1}$$
$$\times\begin{bmatrix}\frac{[n]_q+\alpha}{[n]_q+\beta}-\frac{\mathsf{v}}{b_n} & q^{n-r}\left(\frac{\mathsf{v}}{b_n}-\frac{\alpha}{[n]_q+\beta}\right) & 0 & \cdots & 0 & 0\\ 0 & \frac{[n]_q+\alpha}{[n]_q+\beta}-\frac{\mathsf{v}}{b_n} & q^{n-r}\left(\frac{\mathsf{v}}{b_n}-\frac{\alpha}{[n]_q+\beta}\right) & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & \frac{[n]_q+\alpha}{[n]_q+\beta}-\frac{\mathsf{v}}{b_n} & q^{n-r}\left(\frac{\mathsf{v}}{b_n}-\frac{\alpha}{[n]_q+\beta}\right)\end{bmatrix}.$$

7 Numerical Examples

In the present section, we provide some examples to represent the control of the new introduced parameters on the shape of Bézier curves.

Example 1 In Fig. 3, we choose the parameters $\alpha=0.2, \beta=0.8, b_n=\sqrt{n}$ and the control points (5, 0), (0, 10), (5, 20), (15, 20), (20, 10) and (15, 0). The degree of the $q-$Bernstein-Chlodowsky Bézier curves is 5. From the figure, it is clear that the parameter q is observed to alter the shape of the curves while keeping all other properties constant as well as maintaining the same control polygon. This ability to modify the curve's shape without changing the control polygon contributes to the flexibility of the curve.

Example 2 In Fig. 4, we choose the parameters $\alpha=0.5, \beta=0.9, b_n=\sqrt{n}$. and the control points $(2,5),(5,0),(15,10),\ (15,15),(-5,20),(-5,25),(5,35)$ and (8, 30). In this example, we observe the same nature of our new introduced parameter q as in Example 1.

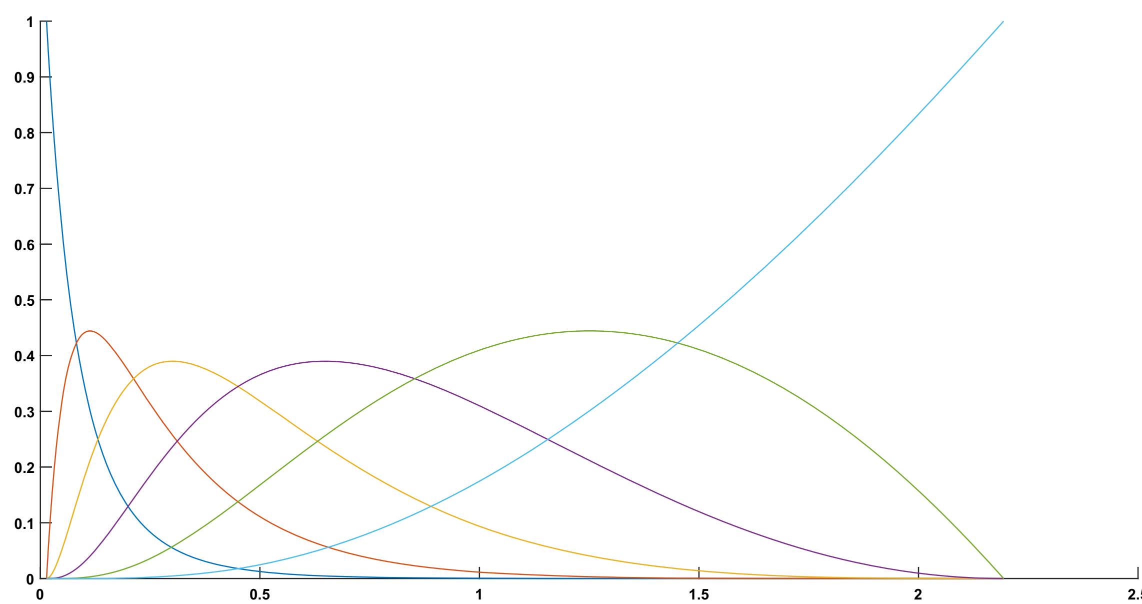

Fig. 2 Basis polynomials for $n=4,\alpha=0.2,\beta=0.8,b_n=\sqrt{n},q=2$

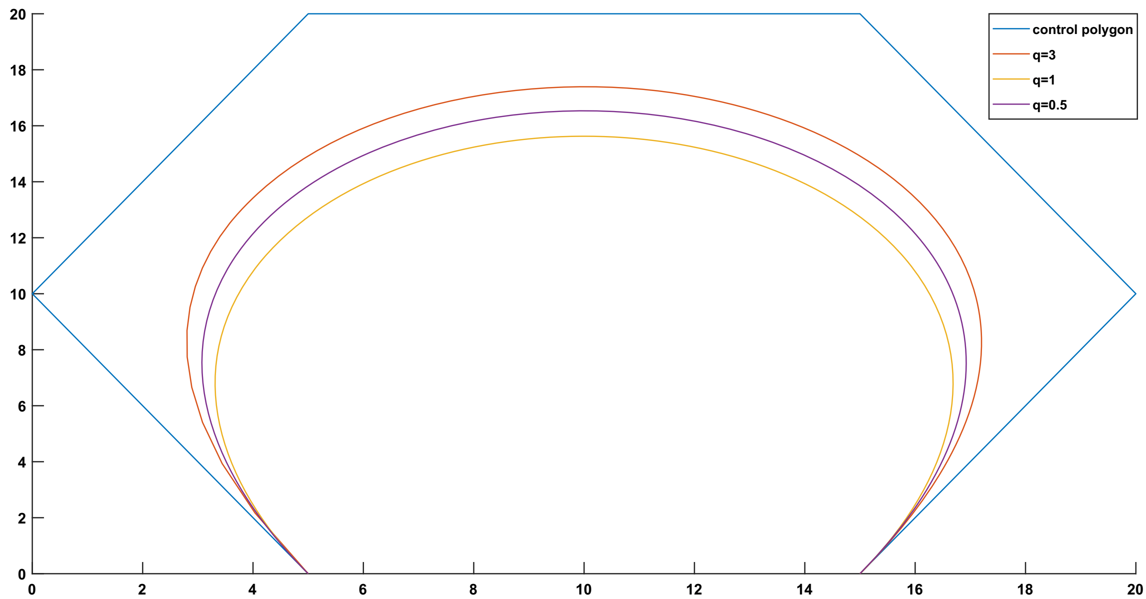

Fig. 3 Shape modification with different values of q

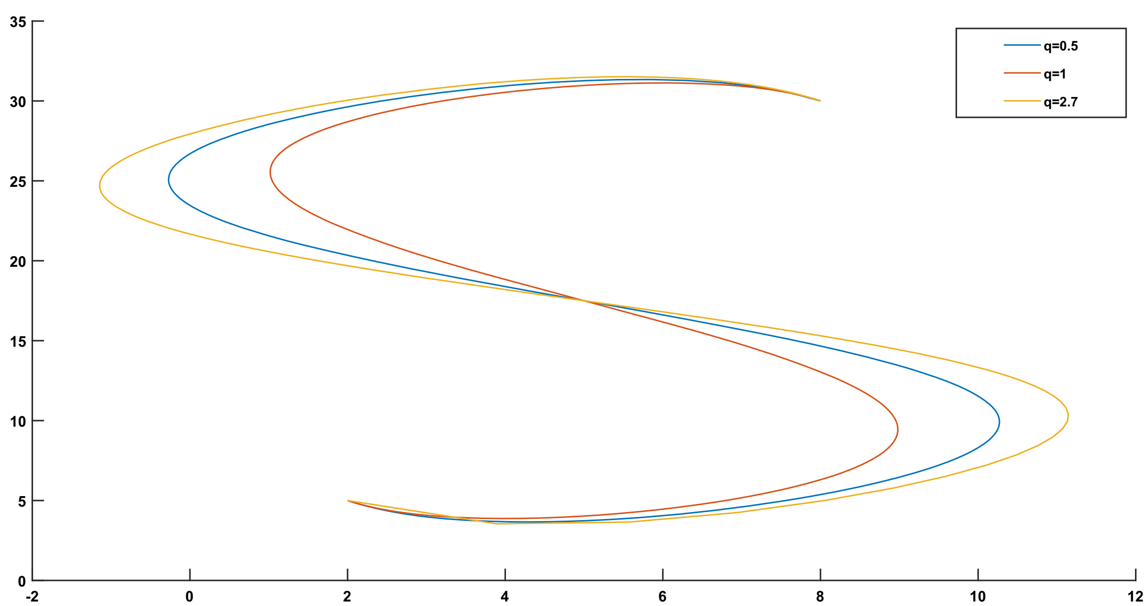

Fig. 4 Shape modification with different values of q

8 Conclusion

The present article deals with the Bézier curve's generalization, a technique that empowers us to fine-tune the shape of these curves through the manipulation of diverse parameters. Our exploration primarily focuses on $q-$analogue of Chlodowsky Bézier curves, a versatile extension of the classical Bézier curves. Additionally, we investigate the process of degree elevation and De Casteljau's algorithm, an essential tool for working with Bézier curves. This novel approach will give the flexibility to control the shape of the curves by choosing the parameters.

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Declarations

Conflict of interest The corresponding author, on behalf of all the authors, declares the absence of any Conflict of interest.

Human Participant and/or Animals Not Applicable.

Ethical Approval Not Applicable.

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