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RESEARCH PAPER

On the Functorial Properties of the *p*-Analog of the Fourier–Stieltjes Algebras

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Abstract
In this paper, some known results about the functorial properties of the Fourier–Stieltjes algebra, $B(G)$, will be generalized. First of all, the idempotent theorem on the Fourier–Stieltjes algebra will be promoted and linked to the p-analog one. Next, the p-analog of the π-Fourier space introduced by Arsac will be given, and by taking advantage of the theory of ultrafilters, the connection between the dual space of the algebra of p-pseudofunctions and the p-analog of the π-Fourier space will be fully investigated. As the main result, one of the significant and applicable functorial properties of the p-analog of the Fourier–Stieltjes algebras will be achieved.

Keywords p-analog of the Fourier–Stieltjes algebras · π-Fourier space · Ultrafilters · p-pseudofunctions · QSL_p-spaces

Mathematics Subject Classification Primary 43A30 · Secondary 46M07, 46A22

1 Introduction

For a locally compact group G, the Fourier algebra, $A(G)$, and the Fourier–Stieltjes algebra, $B(G)$, have been found by Eymard in 1964 Eymard (1964). He investigated almost all functorial properties of such algebras. On the other hand, idempotent elements of $B(G)$ are introduced by Host Host (1986) and have gotten accurate by Ilie and Spronk in Ilie and Spronk (2005). Even Runde (2007) went beyond and add some specific conditions to it, by benefiting from the theory of uniformly convex Banach spaces. In the next attempt on studying Fourier-type algebras, for a representation $(\pi, \mathcal{H})$ of G, on a Hilbert space $\mathcal{H}$, Arsac Arsac (1976) introduced π-Fourier and π-Fourier–Stieltjes spaces A_π and B_π and tremendously studied their functorial properties. Meanwhile, Figà-Talamanca-Herz algebra was initially defined for abelian locally compact groups Figà-Talamanca (1965) and then for general locally compact groups in Herz (1971). Afterwards, in Runde (2005), Runde took main step and determined the true p-analog of Eymard's $B(G)$, as it is indicated via $B_p(G)$. He has indicated that the space $B_p(G)$ is a communicative unital Banach algebra, and in the case that the underlying group G is amenable, it can be identified with the multiplier algebra of the Figà-Talamanca-Herz algebra, i.e. the Banach space $\mathcal{M}(A_p(G))$. Accordingly, the vast majority of functorial properties of the p-analog of the Fourier–Stieltjes algebras $B_p(G)$ have been remained unknown, and they would be the cause for a huge amount of studies. For instance, in Neufang and Runde (2009), one of the possible p-operator space structure on $B_p(G)$ is studied, while another one is introduced by authors Ahmadpoor and Shams Yousefi (2021). Besides, considerable amount of primary questions about the element of such algebras is still open. In this paper, we try to come up with detailed explanation, to touch on some topics, which can be considered as initial steps in the studying on the p-analog of the Fourier–Stieltjes algebras. The current paper is organized as follows: In Sect. 2, we give some preliminaries about representations of a locally compact group on QSL_p-spaces which are building blocks of Runde's $B_p(G)$. It is indicated that the communicative Banach space $B_p(G)$ is the dual space of the algebra of universal p-pseudofunctions $\mathrm{UPF}_p(G)$. Next, in the main section, Sect. 3, we divide our results into three main subsections. In one

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approach, Sect. 3.1 is devoted to the generalization of the idempotent theorem (Theorem 2) in terms of p-analog of the Fourier–Stieltjes algebra. Indeed, we have benefited from the aforementioned theory of Banach spaces and added one more equivalent statement to the latest version of it, that is done by Runde (2007, Theorem 1.5). In the next attempt, we have touched briefly on the generalization of π-Fourier spaces and simply introduced it, then through restating the result in Runde (2005), in respect to the notion of ultrafilter, we have reached to the appropriate description of the dual space of the algebra of p-pseudofunctions $\mathrm{PF}_{p,\pi}(G)$, for an arbitrary representation (π, E) of the locally compact group G on a QSL_p-space E, which is denoted by $B_{p,\pi}$ (Proposition 2). At this aim, we have generated somewhat crucial properties on the representations of a locally compact group, via the notion of ultrafilter and ultrapower space.

Finally, as a conclusion of previous results, in Proposition 5 we have shown that a function in $u \in B_p(G_0)$ can be extended to a function $u^\circ \in B_p(G)$, where G_0 is an open amenable subgroup of the locally compact group G, and the general form of this proposition channels us to Theorem 3.

2 Preliminaries

In this paper, G and H are locally compact groups, and for $p \in (1, \infty)$, the number p' is its complex conjugate, i.e. $1/p + 1/p' = 1$. In the first step, we give essential notions and definitions on QSL_p-spaces, and representations of groups on such spaces. For more information, one can see Runde (2005).

Definition 1 A representation of a locally compact group G is a pair (π, E), where E is a Banach space and π is a group homomorphism from G into the invertible isometries on E, that is continuous with respect to the given topology on G and the strong operator topology on $\mathcal{B}(E)$.

Remark 1 Every representation (π, E) of a locally compact group G induces a representation of the group algebra $L_1(G)$ on E, i.e. a contractive algebra homomorphism from $L_1(G)$ into $\mathcal{B}(E)$, which we shall denote likewise by π, through

$$\begin{aligned} &\pi(f) = \int f(x)\pi(x)\mathrm{d}x,\ f \in L_1(G), \\ &\langle \pi(f)\xi, \eta\rangle = \int f(x)\langle \pi(x)\xi, \eta\rangle \mathrm{d}x, \quad \xi \in E,\ \eta \in E^*, \end{aligned} \tag{1}$$

where the integral (1) converges with respect to the strong operator topology.

Definition 2 Let (π, E) and (ρ, F) be representations of the locally compact group G. Then,

1. (π, E) and (ρ, F) are called equivalent, if there exists an invertible isometry $\varphi : E \to F$ such that
$$\varphi\pi(x)\varphi^{-1} = \rho(x), \qquad x \in G.$$
2. (ρ, F) is said to be a subrepresentation of (π, E), if F is a closed subspace of E, and for every $x \in G$ we have $\pi(x)|_F = \rho(x)$.
3. (ρ, F) is said to be contained in (π, E), if it is equivalent to a subrepresentation of (π, E), and will be denoted by $(\rho, F) \subset (\pi, E)$.

Definition 3

1. A Banach space is called an L_p-space if it is of the form $L_p(X)$ for some measure space X.
2. A Banach space is called a QSL_p-space if it is isometrically isomorphic to a quotient of a subspace of an L_p-space.

We denote by $\mathrm{Rep}_p(G)$ the collection of all (equivalence classes) of representations of G on a QSL_p-space.

Definition 4 A representation of a Banach algebra $\mathcal{A}$ is a pair (π, E), where E is a Banach space, and π is a contractive algebra homomorphism from $\mathcal{A}$ to $\mathcal{B}(E)$. We call (π, E) isometric if π is an isometry and essential if the linear span of $\{\pi(a)\xi \,:\, a \in \mathcal{A},\ \xi \in E\}$ is dense in E.

Remark 2 If G is a locally compact group and (π, E) is a representation of G in the sense of Definition 1, then (1) induces an essential representation of $L_1(G)$. Conversely, every essential representation of $L_1(G)$ arises in this fashion.

Definition 5

1. A representation $(\pi, E) \in \mathrm{Rep}_p(G)$ is called cyclic, if there exists $\xi_0 \in E$ such that $\pi(L_1(G))\xi_0$ is dense in E. The set of cyclic representations of G on QSL_p-spaces is denoted by $\mathrm{Cyc}_p(G)$.
2. A representation $(\pi, E) \in \mathrm{Rep}_p(G)$ is called p-universal, if it contains every cyclic representation.

Remark 3 By Gardella and Thiel (2015, Remark 2.9-(3), and 2015, Proposition 2.4), it is easy to see that every p-universal representation of G contains every cyclic representation of G on a QSL_p-space, in the sense of equivalency. In addition, every representation in $\mathrm{Rep}_p(G)$ is contained in a p-universal representation. Actually, one could make a new p-universal representation by constructing l_p-direct sum of an arbitrary representation with a p-universal representation.

Now we are ready to describe the Figà-Talamanca-Herz and the p-analog of the Fourier–Stieltjes algebras.

Definition 6 Figà-Talamanca-Herz algebra on the locally compact group G, which is denoted by $A_p(G)$, is the collection of functions $u : G \to \mathbb{C}$ of the form

$$u(x) = \sum_{n=1}^{\infty} \langle \lambda_p(x)\xi_n, \eta_n \rangle, \quad x \in G, \tag{2}$$

with

$$(\xi_n)_{n\in\mathbb{N}} \subset L_p(G), \quad (\eta_n)_{n\in\mathbb{N}} \subset L_{p'}(G), \quad \text{and} \quad \sum_{n=1}^{\infty} \|\xi_n\|\|\eta_n\| < \infty, \tag{3}$$

where λ_p is the left regular representation of G on $L^p(G)$, defined as

$$\lambda_p : G \to \mathcal{B}(L^p(G)), \quad \lambda_p(x)\xi(y) = \xi(x^{-1}y), \quad \xi \in L^p(G),\ x, y \in G.$$

The norm of $A_p(G)$ is defined as:

$$\|u\| = \inf\left\{ \sum_{n=1}^{\infty} \|\xi_n\|\|\eta_n\| \ :\ u(\cdot) = \sum_{n=1}^{\infty} \langle \lambda_p(\cdot)\xi_n, \eta_n \rangle \right\},$$

where the infimum is taken over all expressions of u in (2) with (3). With this norm and pointwise operations, $A_p(G)$ turns into a commutative regular Banach algebra.

Remark 4 The p-analog of the Fourier–Stieltjes algebra has been studied, for example in Cowling (1979), Forrest (1994), Miao (1996) and Pier (1984), as the multiplier algebra of the Figà-Talamanca-Herz algebra. In this paper, we follow the construction of Runde in definition and notation (see Runde 2005), which we swap indexes p and p'.

Definition 7 The set of all functions of the form

$$u(x) = \sum_{n=1}^{\infty} \langle \pi_n(x)\xi_n, \eta_n \rangle, \quad \xi_n \in E_n,\ \eta_n \in E_n^*,\ x \in G, \tag{4}$$

where

$$(\pi_n, E_n)_{n\in\mathbb{N}} \subseteq \mathrm{Cyc}_p(G), \quad \text{and} \quad \sum_{n=1}^{\infty} \|\xi_n\|\|\eta_n\| < \infty,$$

equipped with the norm

$$\|u\| = \inf\left\{ \sum_{n=1}^{\infty} \|\xi_n\|\|\eta_n\| \ :\ u(x) = \sum_{n=1}^{\infty} \langle \pi_n(x)\xi_n, \eta_n \rangle,\ x \in G \right\},$$

is denoted by $B_p(G)$ and is called the p-analog of the Fourier–Stieltjes algebra of the locally compact group G.

Remark 5

1. By Runde (2005, Lemma 4.6), the space $B_p(G)$ can be defined to be the set of all coefficient functions of a p-universal representation (π, E), and the norm of an element $u \in B_p(G)$ is the infimum of all values $\sum_{n=1}^{\infty} \|\xi_n\|\|\eta_n\| < \infty$, which such vectors exist in the representation of u as a coefficient function of (π, E), i.e. $u(\cdot) = \sum_{n=1}^{\infty} \langle \pi(\cdot)\xi_n, \eta_n \rangle$.
2. In Runde (2007, Lemma 2.4), the following identification is shown for an open subgroup G_0 of a locally compact group G

   $$A_p(G_0) \cong \{ f \in A_p(G) \ :\ \mathrm{supp}(f) \subset G_0 \},$$

   and through this fact, one can assume that functions in $A_p(G_0)$ are restriction of functions in $A_p(G)$ to the open subgroup G_0.

Definition 8 Let $(\pi, E) \in \mathrm{Rep}_p(G)$.

1. For each $f \in L_1(G)$, let $\|f\|_\pi := \|\pi(f)\|_{\mathcal{B}(E)}$, then $\|\cdot\|_\pi$ defines an algebra seminorm on $L_1(G)$.
2. By $\mathrm{PF}_{p,\pi}(G)$, we mean the p-pseudofunctions associated with (π, E), which is the closure of $\pi(L_1(G))$ in $\mathcal{B}(E)$.
3. If $(\pi, E) = (\lambda_p, L_p(G))$, we denote $\mathrm{PF}_{p,\lambda_p}(G)$ by $\mathrm{PF}_p(G)$.
4. If (π, E) is p-universal, we denote $\mathrm{PF}_{p,\pi}(G)$ by $\mathrm{UPF}_p(G)$ and call it the algebra of universal p-pseudofunctions.

Remark 6

1. For $p = 2$, the algebra $\mathrm{PF}_p(G)$ is the reduced group C^*-algebra, and $\mathrm{UPF}_p(G)$ is the full group C^*-algebra of G.
2. If $(\rho, F) \in \mathrm{Rep}_p(G)$ is such that (π, E) contains every cyclic subrepresentation of (ρ, F), then $\|\cdot\|_\rho \le \|\cdot\|_\pi$ holds. In particular, the definition of $\mathrm{UPF}_p(G)$ is independent of a particular p-universal representation.
3. With $\langle \cdot, \cdot \rangle$ denoting $L_1(G) - L_\infty(G)$ duality, and with (π, E) a p-universal representation of G, we have

   $$\|f\|_\pi = \sup\{ |\langle f, g \rangle| \ :\ g \in B_p(G),\ \|g\|_{B_p(G)} \le 1 \}, \quad f \in L_1(G).$$

Next lemma states that $B_p(G)$ is a dual space.

Lemma 1 (Runde 2005, Lemma 6.5) *Let $(\pi, E) \in \mathrm{Rep}_p(G)$. Then, for each $\phi \in \mathrm{PF}_{p,\pi}(G)^*$, there is a unique $g \in B_p(G)$, with $\|g\|_{B_p(G)} \le \|\phi\|$ such that*

$$\langle \pi(f), \phi \rangle = \int_G f(x)g(x)\mathrm{d}x, \qquad f \in L_1(G). \tag{5}$$

Moreover, if (π, E) *is* p*-universal, we have* $\|g\|_{B_p(G)} = \|\phi\|$.

3 Generalized Functorial Properties

As the aim of the present paper, here we generalize some properties of the Fourier–Stieltjes algebra, $B(G)$, to the p-analog of the Fourier–Stieltjes algebra, $B_p(G)$, for a locally compact group G. In the first step, we generalize the idempotent theorem, which was stated in the most possible version as Theorem 1.5 in Runde (2007) in Theorem 2. Next, we introduce the p-analog of the π-Fourier and π-Fourier–Stieltjes spaces given by Arsac in Arsac (1976) in Definition 9 and state the expected properties in lemma and proposition afterwards. By such generalization, we then approach to extending result, Theorem 3, which would be crucial on solving similar problems around $B(G)$ for the p-analog case. For instance, in studies on homomorphisms on the p-analog of the Fourier–Stieltjes algebra, $B_p(G)$, which can be considered as a p-analog of what has been done in Ilie and Spronk (2005), Theorem 3 would be essential.

3.1 Idempotent Theorem

In order to being prepared for Theorem 2, which is a generalization of Runde (2007, Theorem 1.5), we need some elementary definitions and facts, and we give them in the following.

Definition 9

1. A Banach space $(E, \|\cdot\|)$ is said to be uniformly convex if for every $0<\epsilon\leq 2$ there is $\delta > 0$ so that for any two vectors x and y in E with $\|x\| = \|y\| = 1$, the condition $\|x-y\| \geq \epsilon$ implies that $\|\frac{x+y}{2}\| \leq 1-\delta$. Intuitively, the centre of a line segment inside the unit ball must lie deep inside the unit ball unless the segment is short.
2. A Banach space E is said to be smooth if for each $\xi \in E\backslash\{0\}$ there exists a unique $\eta \in E^*$ such that $\|\eta\| = 1$ and $\langle \xi, \eta\rangle = \|\xi\|$.

Remark 7 It is worthwhile to note that by Definition 9, every closed subspace of a uniformly convex Banach space is again a uniformly convex Banach space.

Now we state an immensely important theorem about a quotient space which can be found in Istratescu (1983).

Theorem 1 (Istratescu 1983, Theorem 2.4.18) *Let E be a uniformly convex Banach space and F be a closed linear subspace of E. Then, the quotient space E/F is uniformly convex Banach space.*

Now we can conclude the following statement.

Corollary 1 *Every* QSL_p*-space E is uniformly convex and smooth.*

Proof Uniformly convexity of QSL_p-space E can be derived from Remark 7 and Theorem 1. Since E is uniformly convex, by Fabian et al. (2001, Lemma 8.4(i) and Theorem 9.10), it is concluded that E^* is smooth, but E^* is a $\mathrm{QSL}_{p'}$-space so is uniformly convex, and then, E^{**} is smooth, but $E = E^{**}$ so E is smooth. □

Theorem 2 *For a subset $C \subset G$, the following statements are equivalent.*

1. *C is a left open coset,*
2. *$\chi_C \in B(G)$ with $\|\chi_C\|_{B(G)} = 1$,*
3. *$\chi_C \neq 0$ is a normalized coefficient function of a representation (π, E) where E or E^* is smooth,*
4. *$\chi_C \in B_p(G)$ with $\|\chi_C\|_{B_p(G)} = 1$.*

Proof Equivalency of the first three statements has been proved in Runde (2007, Theorem 1.5). We demonstrate (2)⇒ (4) ⇒(3). Let (2) hold. Then from the fact that $B(G) \subset B_p(G)$ and this embedding is a contraction, we have $\chi_C \in B_p(G)$ with $\|\chi_C\|_{B_p(G)} \leq 1$, which by inequality $\|\cdot\|_{C_b(G)} \leq \|\cdot\|_{B_p(G)}$, we have $\|\chi_C\|_{B_p(G)} = 1$ which shows (2) implies (4).

Now let $\chi_C \in B_p(G)$ with $\|\chi_C\|_{B_p(G)} = 1$. So, by Definition 7, the function χ_C is a normalized coefficient function of an isometric group representation on a QSL_p-space, which is smooth by Corollary 1 that is (3). □

Corollary 2 *Let G be a locally compact group and $Y \in \Omega_0(G)$, and then, we have $\chi_Y \in B_p(G)$. Moreover, we have*

$$1 \leq \|\chi_Y\|_{B_p(G)} \leq 2^{m_Y}, \quad \text{with} \\ m_Y = \inf\{m \in \mathbb{N} \ : \ Y = Y_0 \backslash \cup_{i=1}^m Y_i\}, \tag{6}$$

where for $i = 0, 1, \ldots, m$ sets Y_i, are as (15).

Proof Since $Y \in \Omega_0(G)$, then by (15), there exist open coset Y_0 and open subcosets $Y_i \subset Y_0$, for $i = 1, \ldots, m$ and $m \in \mathbb{N}$ such that $Y = Y_0 \backslash \cup_{i=1}^m Y_i$. By Theorem 2-(4), we have $\chi_{Y_i} \in B_p(G)$, with $\|\chi_{Y_i}\|_{B_p(G)} = 1$, for $i = 0, 1, \ldots, m$. On the other hand, since

$$\chi_Y = \chi_{Y_0} - \left(\sum_{i=1}^{m} \chi_{Y_i} - \sum_{i,j=1} \chi_{Y_i \cap Y_j} + \sum_{i,j,k=1} \chi_{Y_i \cap Y_j \cap Y_k} + \cdots + (-1)^{m+1} \chi_{Y_1 \cap Y_2 \cap \ldots \cap Y_m} \right), \tag{7}$$

then we have $\|\chi_Y\|_{B_p(G)} \le 2^{m_Y}$, and by taking infimum on all possible decomposition of Y as (15) relation (6) holds. □

3.2 *p*-Analog of the π-Fourier Spaces

In the sequel, we will give some extensions of results in Arsac (1976). For a unitary representation $(\pi, \mathcal{H}_\pi)$ with Hilbert space $\mathcal{H}_\pi$, the π-Fourier space has been defined to be closed linear span of the set of the coefficient functions of the representation $(\pi, \mathcal{H}_\pi)$, and is denoted by A_π, with the norm in usual way. Moreover, π-Fourier–Stieltjes algebra, B_π, for such representation is defined to be w^*-closure of A_π. Additionally, if we let $C^*_\pi(G)$ be the C^*-algebra associated with π, we have $B_\pi = C^*_\pi(G)^*$. Here we introduce p-generalization of these results; however, these generalizations are not new (see Cowling and Fendler (1984)), but in terms of Fourier-type algebras is somewhat new.

Definition 10 For a representation $(\pi, E) \in \mathrm{Rep}_p(G)$, we define the *p-analog of the π-Fourier space*, $A_{p,\pi}$, to be closed linear span of the collection of the coefficient functions of representation (π, E), i.e. functions of the form

$$u(x) = \sum_n \langle \pi(x)\xi_n, \eta_n \rangle, \quad x \in G, (\xi_n)_{n\in\mathbb{N}} \subseteq E, \ (\eta_n)_{n\in\mathbb{N}} \subseteq E^*,$$

equipped with the norm

$$\|u\|_{A_{p,\pi}} = \inf \left\{ \sum_{n=1}^{\infty} \|t_n\| \|s_n\| \ : \ u(x) = \sum_n \langle \pi(x)t_n, s_n \rangle, \ x \in G \right\},$$

and evidently, infimum is taken over all possible equivalent representative of u so that the value is convergent.

Remark 8

1. For $(\pi, E) \in \mathrm{Rep}_p(G)$ consider the map $\Psi_{p,\pi} : E^* \widehat{\otimes} E \to C_b(G)$, defined via

$$\Psi_{p,\pi}\left(\sum_n \eta_n \otimes \xi_n \right) = \sum_n \langle \pi(x)\xi_n, \eta_n \rangle, \ x \in G.$$

   This map is onto to its range, which is $A_{p,\pi}$, and so we can identify it with the Banach space $E^* \widehat{\otimes} E / \ker \Psi_{p,\pi}$, and the norm on $A_{p,\pi}$ is the quotient norm, i.e.

$$\begin{aligned} &\left\| \sum_n \xi_n \otimes \eta_n + \ker \Psi_{p,\pi} \right\| \\ &= \inf \left\{ \sum_n \|t_n\| \|s_n\| \ : \sum_n \langle \pi(\cdot)t_n, s_n \rangle = \sum_n \langle \pi(\cdot)\xi_n, \eta_n \rangle \right\} \\ &= \left\| \sum_n \langle \pi(\cdot)\xi_n, \eta_n \rangle \right\|_{A_{p,\pi}}. \end{aligned}$$

   So, one can identify $A_{p,\pi}$ with the quotient space $E^* \widehat{\otimes} E / \ker \Psi_{p,\pi}$.
2. Since we have $A_{p,\pi} \cong E^* \widehat{\otimes} E / \ker \Psi_{p,\pi}$, then the space $A_{p,\pi}$ is a Banach space.

In the next proposition, we give an equivalent formula of computing norm on the space $A_{p,\pi}$, for a representation $(\pi, E) \in \mathrm{Rep}_p(G)$. For this aim, we denote the set of cyclic subrepresentations of a representation (π, E) by $\mathrm{Cyc}_{p,\pi}(G)$.

Proposition 1 *Let $(\pi, E) \in \mathrm{Rep}_p(G)$ and $u \in A_{p,\pi}$. Then, we have*

$$\|u\|_{A_{p,\pi}} = \inf \left\{ \sum_n \|t_n\| \|s_n\| \ : \ u(x) = \sum_n \langle \rho_n(x)t_n, s_n \rangle, \ x \in G \right\}, \tag{8}$$

where the infimum is taken on all representations of u, in which $\left((\rho_n, F_n)\right)_{n\in\mathbb{N}} \subseteq \mathrm{Cyc}_{p,\pi}(G)$ with $(t_n)_{n\in\mathbb{N}} \subseteq F_n$ and $(s_n)_{n\in\mathbb{N}} \subseteq F_n^*$.

Proof Let us denote the infimum in (8) by C. Assume that for $x \in G$, we have $u(x) = \sum_n \langle \pi(x)\xi_n, \eta_n \rangle$ with $\sum_n \|\xi_n\| \|\eta_n\| < \infty$. For each $n \in \mathbb{N}$, if we put

$$F_n = \overline{\pi(L_1(G))\xi_n}^{\|\cdot\|_E}, \quad \rho_n : G \to \mathcal{B}(F_n), \quad \rho_n(x) = \pi(x)|_{F_n}, \ x \in G,$$

and $t_n = \xi_n, \ s_n = \eta_n|_{F_n}$, then we have

$$\left((\rho_n, F_n)\right)_{n\in\mathbb{N}} \subseteq \mathrm{Cyc}_{p,\pi}(G), \quad u(x) = \sum_n \langle \rho_n(x)t_n, s_n \rangle,$$

with $C \le \sum_{n=1}^{\infty} \|t_n\| \|s_n\| \le \sum_n \|\xi_n\| \|\eta_n\|$. Since $(\xi)_{n\in\mathbb{N}} \subset E$ and $(\eta)_{n\in\mathbb{N}} \subset E^*$ are arbitrary in the representing of u, we have $C \le \|u\|_{A_{p,\pi}}$.

For the inverse inequality, let $\epsilon > 0$ is given. Then, there exist $((\rho_n, F_n))_{n\in\mathbb{N}} \subseteq \mathrm{Cyc}_{p,\pi}(G)$, $(t_n)_{n\in\mathbb{N}} \subseteq F_n$, $(s_n)_{n\in\mathbb{N}} \subseteq F_n^*$, such that for each $n \in \mathbb{N}$, we have $(\rho_n, F_n) \subset (\pi, E)$ and

$$\sum_n \|t_n\| \|s_n\| < C + \epsilon, \qquad u(x) = \sum_n \langle \rho_n(x)t_n, s_n \rangle, \quad x \in G.$$

Now for each $n \in \mathbb{N}$, by applying Hahn–Banach theorem, we can extend each $s_n \in F_n^*$ to the $\eta_n \in E^*$ such that $\|\eta_n\| = \|s_n\|$. Therefore,

$$\|u\|_{A_{p,\pi}} \le \sum_n \|t_n\| \|\eta_n\| = \sum_n \|t_n\| \|s_n\| < C + \epsilon,$$

and it means $\|u\|_{A_{p,\pi}} \le C$. □

In the next attempt, we will generalize some known results, and for this aim, we need to be precise about notions there. To clarify everything, we bring some definitions and their requirement in the sequel. To do this, we briefly state some facts about ultrafilters on Banach spaces. The main reference here is Heinrich (1980), and the more applicable one is Daws (2004).

Let $(E_i)_{i\in\mathbb{I}}$ be a collection of Banach spaces for an indexing set $\mathbb{I}$. Consider the Banach space $L_\infty(\mathbb{I}, E_i)$ of elements $(\xi_i)_{i\in\mathbb{I}}$ equipped with the pointwise operations and the supremum norm $\|(\xi_i)_i\| = \sup_{i\in\mathbb{I}}\|\xi_i\|_{E_i}$. Now, let $\mathcal{U}$ be an ultrafilter on $\mathbb{I}$, and let $N_\mathcal{U}$ be as following

$$N_\mathcal{U} = \left\{(\xi_i)_{i\in\mathbb{I}} \ : \ \lim_\mathcal{U}\|\xi_i\| = 0\right\},$$

where by $\lim_\mathcal{U}\|\xi_i\|$ we mean the limit of $(\xi_i)_{i\in\mathbb{I}}$ along the ultrafilter $\mathcal{U}$, that exists due to the fact that the values $\|\xi_i\|$ belong to the compact interval $[0, M]$, where $\sup_{i\in\mathbb{I}}\|\xi_i\| = M < \infty$. Note that $N_\mathcal{U}$ is a closed subspace; therefore, the completion of the quotient space $L_\infty(\mathbb{I}, E_i)/N_\mathcal{U}$ is a Banach space. This Banach space is denoted by $(E_i)_\mathcal{U}$, and it is called the ultraproduct of $(E_i)_{i\in\mathbb{I}}$ with respect to the ultrafilter $\mathcal{U}$. Besides, the quotient norm of an element $(\xi_i)_\mathcal{U} \in (E_i)_\mathcal{U}$ coincides with the $\lim_\mathcal{U}\|\xi_i\|$.

In general, we have $(E_i^*)_\mathcal{U} \subseteq (E_i)^*_\mathcal{U}$, isometrically. In addition, when for each $i \in I$ we have $E_i = E$, then the ultraproduct space is called the ultrapower of the Banach space E and is denoted by $(E)_\mathcal{U}$. Moreover, if E is a super-reflexive space, then we have $(E_i)^*_\mathcal{U} = (E_i^*)_\mathcal{U}$, and this is a well-known fact that an ultrapower of a QSL_p-space E is again a QSL_p-space.

For a Banach space E, the natural embedding $J : E \to (E)_\mathcal{U}$, defined via $J(\xi) = (\xi_i)_\mathcal{U}$ where $\xi_i \equiv \xi$, is an isometric one. Furthermore, for the case that E is super-reflexive, if $(\eta_i)_\mathcal{U} \in (E^*)_\mathcal{U} = (E)^*_\mathcal{U}$ then we have

$$\langle J(\xi), (\eta_i)_\mathcal{U}\rangle = \lim_\mathcal{U}\langle \xi, \eta_i\rangle.$$

Additionally, there is a well-known canonical isometric map $\kappa_E : E \to E^{**}$ that is defined by $\langle \eta, \kappa_E(\xi)\rangle = \langle \xi, \eta\rangle$, and it is a surjection, if and only if E is reflexive. For a Banach space E, and an ultrafilter $\mathcal{U}$, since the unit ball of E^{**} is compact, then the following map is well-defined and contractive

$$\mathcal{J} : (E)_\mathcal{U} \to E, \quad \mathcal{J}((\xi_i)_\mathcal{U}) = w^* - \lim_\mathcal{U}\kappa_E(\xi_i),$$

and for every $\eta \in E$ we have $\langle \eta, \mathcal{J}((\xi_i)_\mathcal{U})\rangle = \lim_\mathcal{U}\langle \xi_i, \eta\rangle$.

According to Daws (2004, Proposition 2), for a Banach space E there exist an ultrafilter $\mathcal{U}$ and an isometric map, $\overline{J} : E^{**} \to (E)_\mathcal{U}$ such that $\overline{J}|_E = J$, $\mathcal{J} \circ \overline{J} = \mathrm{id}_{E^{**}}$, and $\overline{J} \circ \mathcal{J}$ is a norm one projection from $(E)_\mathcal{U}$ onto $\overline{J}(E^{**})$.

For a Banach space E, and a complex number $p \in (1, \infty)$, the Banach space $L_p(E) = L_p(\mathbb{N}, E)$, is as following

$$L_p(E) = \left\{(\xi_n)_n \ : \ \|(\xi_n)_n\| = \left(\sum_n \|\xi_n\|^p\right)^{\frac{1}{p}} < \infty\right\},$$

which by Daws (2004, Proposition 4) it is super-reflexive whenever E is so. Additionally, in the case that E is a QSL_p-space, then $L_p(E)$ is again a QSL_p-space.

Now, for a representation $(\pi, E) \in \mathrm{Rep}_p(G)$, we introduce two representations $(\pi^\infty, L_p(E))$ and $(\pi_\mathcal{U}, (E)_\mathcal{U})$, as follows:

$$\pi^\infty : G \to \mathcal{B}(L_p(E)),$$
$$\pi^\infty(x)((\xi_n)_n) = (\pi(x)\xi_n)_n, \quad x \in G,\ (\xi_n)_n \in L_p(E),$$
$$\pi_\mathcal{U} : G \to \mathcal{B}((E)_\mathcal{U}),$$
$$\pi_\mathcal{U}(x)((\xi_i)_\mathcal{U}) = (\pi(x)\xi_i)_\mathcal{U}, \quad x \in G,\ (\xi_i)_\mathcal{U} \in (E)_\mathcal{U}.$$

Lemma 2 *For a representation $(\pi, E) \in \mathrm{Rep}_p(G)$, consider the aforementioned representations $(\pi^\infty, L_p(E))$ and $(\pi_\mathcal{U}, (E)_\mathcal{U})$. Then, the following statements hold.*

1. *The ranges of both representations are the subspaces of invertible and isometric operators on their associated QSL_p-spaces, and therefore, they belong to $\mathrm{Rep}_p(G)$.*
2. *These representations are related to (π, E) as representations of the group algebra $L_1(G)$ as the same as they are related as representations of the group G. Precisely, for an element $f \in L_1(G)$, we have*
$$\pi^\infty(f) = (\pi(f))^\infty, \qquad \pi_\mathcal{U}(f) = (\pi(f))_\mathcal{U}.$$
3. *$(\pi^\infty, L_p(E))$ and $(\pi_\mathcal{U}, (E)_\mathcal{U})$ are essential representations of $L_1(G)$.*

Proof The first two parts hold naturally. For the third part, we briefly assure the reader about our claim. *Case one:* $(\pi^\infty, L_p(E))$. Let $(\xi_n)_n \in L_p(E)$. Since (π, E) is an essential representation, then for every given $\epsilon > 0$, for each $n \in \mathbb{N}$, there exist $f_n \in L_1(G)$ and $t_n \in E$ such that

$$\|\pi(f_n)t_n - \xi_n\| < \frac{\epsilon}{nM}, \quad M = \left(\sum_n \frac{1}{n^p}\right)^{\frac{1}{p}} < \infty \ (\text{since } p > 1),$$

then we have

$$\|(\pi(f_n)t_n)_n - (\xi_n)_n\| < \epsilon.$$

So, the arbitrary element $(\xi_n)_n$ is approximated by the element $(\pi(f_n)t_n)_n$, that lives in the range of π^∞, as a representation of $L_1(G)$. *Case two:* $(\pi_\mathcal{U}, (E)_\mathcal{U})$. Let $(\xi_i)_\mathcal{U} \in (E)_\mathcal{U}$ and consider a representative $(\xi_i)_{i\in\mathbb{I}} \in L_\infty(\mathbb{I}, E)$ of it. By the same argument as above, for every given $\epsilon > 0$, for each $i \in \mathbb{I}$, there exist $f_i \in L_1(G)$, and $t_i \in E$ such that

$$\|\pi(f_i)t_i - \xi_i\| < \frac{\epsilon}{2},$$

and consequently,

$$\|(\pi(f_i)t_i)_i - (\xi_i)_i\| \leq \frac{\epsilon}{2} < \epsilon.$$

On the other hand, since for an element $(r_i)_i \in L_\infty(\mathbb{I}, E)$ we have $\|(r_i)_{\mathcal{U}}\| \leq \|(r_i)_i\|$ it is obtained that

$$\|(\pi(f_i)t_i)_{\mathcal{U}} - (\xi_i)_{\mathcal{U}}\| < \epsilon.$$

Throughout the fact that the element $(\pi(f_i)t_i)_{\mathcal{U}}$ belongs to the range of $\pi_{\mathcal{U}}$ as representation of $L_1(G)$, we are done. □

Corollary 3 *For a representation $(\pi, E) \in \mathrm{Rep}_p(G)$, the representation $(\pi_{\mathcal{U}}^\infty, (L_p(E))_{\mathcal{U}}$, defined in the obvious way, is an essential representation and belongs to $\mathrm{Rep}_p(G)$.*

Next proposition is somewhat a restatement of Lemma 1, which is Lemma 6.5 in Runde (2005). This restatement is beneficial due to a detailed proof.

Proposition 2 *Let $(\pi, E) \in \mathrm{Rep}_p(G)$. Then*

1. *there exists a free ultrafilter $\mathcal{U}$, such that the canonical representation of $\mathrm{PF}_{p,\pi}(G)$ on $F = (L_p(E))_{\mathcal{U}}$ is weak-weak * continuous, essential and isometric,*
2. *the identification $\mathrm{PF}_{p,\pi}(G)^* = \overline{A_{p,\pi_{\mathcal{U}}^\infty}}^{w^*} = A_{p,\pi_{\mathcal{U}}^\infty}$ holds.*

Proof Here, we sometimes use F instead of $(L_p(E))_{\mathcal{U}}$, for ease of notation, and sometimes do not use to highlight the associated space and actions. To prove part one, by the proof of Daws (2004, Proposition 5), there exists an ultrafilter $\mathcal{U}$ on an indexing set $\mathbb{I}$, such that by considering the above-mentioned map $\bar{J}$ for $E^*\widehat{\otimes}E$ and using the fact that $(E^*\widehat{\otimes}E)^* = \mathcal{B}(E)$, we have the following isometry:

$$J : \mathcal{B}(E)^* \to (E^*\widehat{\otimes}E)_{\mathcal{U}}.$$

Through daws (2004, Theorem 1 or Proposition 5), for the obtained ultrafilter above, the map $P : F^*\widehat{\otimes}F \to \mathcal{B}(E)^*$ defined for $t = ((t_{i,n})_n)_{\mathcal{U}} \in F$ and $s = ((s_{i,n})_n)_{\mathcal{U}} \in F^*$ via

$$\langle T, P(s \otimes t)\rangle = \lim_{\mathcal{U}} \sum_n \langle T(t_{i,n}), s_{i,n}\rangle, \quad T \in \mathcal{B}(E).$$

is a linear isometric surjection. Therefore, the embedding $P^* : \mathcal{B}(E)^{**} \to \mathcal{B}(F)$ is an isometric homomorphism. So, the canonical representation of $\mathrm{PF}_{p,\pi}(G) \subseteq \mathcal{B}(E)$ on $F = (L_p(E))_{\mathcal{U}}$ that is $P^*|_{\mathrm{PF}_{p,\pi}(G)} = P_r^*$, is weak-weak* continuous and isometric. Precisely, the following map satisfies mentioned properties:

$$P_r^* : \mathrm{PF}_{p,\pi}(G) \to \mathcal{B}(F), \quad P_r^*(\pi(f)) = \pi_{\mathcal{U}}^\infty(f),\ f \in L_1(G).$$

In fact, as an application of Lemma 2-(3), through the following diagram,

the map P_r^* is an essential representation of $\mathrm{PF}_{p,\pi}(G)$. In detail, the map π is contractive with dense range, while $\pi_{\mathcal{U}}^\infty$ is an essential representation as it is described at the end of previous lemma. Since we have $P_r^* \circ \pi = \pi_{\mathcal{U}}^\infty$, then our claim is true and P_r^* is an isometric, weak-weak* continuous, and essential representation of $\mathrm{PF}_{p,\pi}(G)$. Subsequently, we have

$$\mathrm{PF}_{p,\pi}(G) = \mathrm{PF}_{p,\pi_{\mathcal{U}}^\infty}(G).$$

For the second part, since $(P_r^*)_r^*$, the restriction of the conjugate map $(P_r^*)^*$ to the subspace $F^*\widehat{\otimes}F$ is a quotient map onto $\mathrm{PF}_{p,\pi}(G)^*$, then we have the map

$$(P_r^*)_r^* : F^*\widehat{\otimes}F \to \mathrm{PF}_{p,\pi}(G)^*.$$

It is obtained that

$$\mathrm{PF}_{p,\pi}(G)^* = (L_{p'}(E^*))_{\mathcal{U}}\widehat{\otimes}(L_p(E))_{\mathcal{U}}/\ker(P_r^*)_r^*.$$

For an element $\phi \in \mathrm{PF}_{p,\pi}(G)^*$, there exists a unique $\tau \in F^*\widehat{\otimes}F/\ker(P_r^*)_r^*$ such that for a given $\epsilon > 0$ there exist $(\xi_k)_k \subseteq F$ and $(\eta_k)_k \subseteq F^*$ with $\tau = \sum_k \xi_k \otimes \eta_k$, and

$$\|\phi\| \leq \sum_k \|\xi_k\|\|\eta_k\| < \|\phi\| + \epsilon.$$

Additionally, for every $f \in L_1(G)$, we have

$$\langle \pi(f), \phi\rangle = \sum_k \langle P_r^* \circ \pi(f)\xi_k, \eta_k\rangle = \sum_k \langle \pi_{\mathcal{U}}^\infty(f)\xi_k, \eta_k\rangle = \langle \pi_{\mathcal{U}}^\infty(f), u\rangle, \tag{9}$$

where

$$u(x) = \sum_k \langle \pi_{\mathcal{U}}^\infty(x)\xi_k, \eta_k\rangle \in A_{p,\pi_{\mathcal{U}}^\infty}, \qquad x \in G,$$

and $\langle \pi_{\mathcal{U}}^\infty, u\rangle$ means the $L_1 - L_\infty$ duality between f and u, as it is described in Lemma 1. Now, consider the map $\Psi_{p,\pi_{\mathcal{U}}^\infty}$, as Remark 8-(1), associated with the representation $(\pi_{\mathcal{U}}^\infty, F)$. We have

$$A_{p,\pi_{\mathcal{U}}^\infty} = (L_{p'}(E^*))_{\mathcal{U}}\widehat{\otimes}(L_p(E))_{\mathcal{U}}/\ker \Psi_{\pi,\pi_{\mathcal{U}}^\infty}.$$

Since $\mathrm{PF}_{p,\pi}(G) = \mathrm{PF}_{p,\pi_{\mathcal{U}}^\infty}(G)$, then relation (9) reveals that kernels of the maps $(P_r^*)_r^*$ and $\Psi_{p,\pi_{\mathcal{U}}^\infty}$ coincide and we are done. □

Proposition 3 *Let $(\pi, E) \in \mathrm{Rep}_p(G)$. Then, we have the following identification*

$$A_{p,\pi} = A_{p,\pi^\infty}.$$

Proof Let $u \in A_{p,\pi^\infty}$, and $\epsilon > 0$ be given. There exist the sequence of vectors $((\xi_{n,m})_n)_m \subseteq L_p(E)$ and $((\eta_{n,m})_n)_m \subseteq L_p(E)^* = L_{p'}(E^*)$ such that

$$u(x) = \sum_m \langle \pi^\infty(x)(\xi_{n,m})_n, (\eta_{n,m})_n \rangle = \sum_{m,n} \langle \pi(x)\xi_{n,m}, \eta_{n,m} \rangle, \tag{10}$$

and

$$\|u\|_{\pi^\infty} + \epsilon > \sum_m \|(\xi_{n,m})_n\| \|(\eta_{n,m})_n\| \geq \sum_{n,m} \|\xi_{n,m}\| \|\eta_{n,m}\|. \tag{11}$$

In the last inequality, we utilized the Hölder inequality of positive numbers. From (10), it is evident that $u \in A_{p,\pi}$ and (11) shows that $\|u\|_{p,\pi} \leq \|u\|_{p,\pi^\infty}$, which means that $A_{p,\pi^\infty} \subseteq A_{p,\pi}$, contractively. We shall show the inverse inclusion holds contractively. To do so, let $u \in A_{p,\pi}$, and for a given $\epsilon > 0$ let vectors $(t_n)_n \subseteq E$ and $(s_n)_n \subseteq E^*$ be such that

$$u(x) = \sum_n \langle \pi(x)t_n, s_n \rangle, \qquad x \in G,$$

and

$$\|u\| + \epsilon > \sum_n \|t_n\| \|s_n\|.$$

Now, if we put

$$\xi_n = \|t_n\|^{-1+\frac{1}{p}} \|s_n\|^{\frac{1}{p}} t_n, \qquad \eta_n = \|s_n\|^{-1+\frac{1}{p'}} \|t_n\|^{\frac{1}{p'}} s_n,$$

then we have $(\xi_n)_n \in L_p(E)$ and $(\eta_n)_n \in L_{p'}(E^*)$. Moreover,

$$\Big(\sum_n \|\xi_n\|^p\Big)^{\frac{1}{p}} = \Big(\sum_n \|t_n\| \|s_n\|\Big)^{\frac{1}{p}} < \Big(\|u\|_\pi + \epsilon\Big)^{\frac{1}{p}},$$

$$\Big(\sum_n \|\eta_n\|^{p'}\Big)^{\frac{1}{p'}} = \Big(\sum_n \|t_n\| \|s_n\|\Big)^{\frac{1}{p'}} < \Big(\|u\|_\pi + \epsilon\Big)^{\frac{1}{p'}},$$

and

$$u(x) = \langle \pi^\infty(x)(\xi_n)_n, (\eta_n)_n \rangle = \sum_n \langle \pi(x)\xi_n, \eta_n \rangle = \sum_n \langle \pi(x)t_n, s_n \rangle, \quad x \in G.$$

Therefore,

$$u \in A_{p,\pi^\infty}, \quad \text{and} \quad \|u\|_{\pi^\infty} \leq \|(\xi_n)_n\| \|(\eta_n)_n\| < \|u\|_\pi + \epsilon.$$

□

Corollary 4 *For a representation* $(\pi, E) \in \mathrm{rep}_p(G)$, *we have the following identification*:

$$\mathrm{PF}_{p,\pi}(G)^* = A_{p,\pi_\mathcal{U}}.$$

Proof It is a straightforward. □

Remark 9

1. In the light of previous proposition, due to the fact that $(\pi_\mathcal{U}^\infty, F)$ is weak-weak* continuous, essential an isometric representation of $\mathrm{PF}_{p,\pi}(G)$, then we have
$$\mathrm{PF}_{p,\pi}(G) = \mathrm{PF}_{p,\pi_\mathcal{U}^\infty}(G).$$
2. In the case that the representation (π, E) is a p-universal representation, then since $(\pi_\mathcal{U}^\infty, (l_p(E))_\mathcal{U})$ is also a p-universal representation, our notation coincides with Runde's one in Runde (2005).
3. We follow Arsac (1976) in notation and denote $A_{p,\pi_\mathcal{U}}$ by $B_{p,\pi}$, and we call it *p-analog of the π-Fourier–Stieltjes algebra*, which by Proposition 2 is the dual space of the space of p-pseudofunctions associated with $(\pi, E) \in \mathrm{Rep}_p(G)$, i.e. the dual space of $\mathrm{PF}_{p,\pi}(G)$ through the following duality
$$\langle \pi(f), u \rangle = \int_G u(x)f(x)\mathrm{d}x, \quad f \in L_1(G),\ u \in B_{p,\pi},$$
and as we expect that, we have
$$\|u\| = \sup_{\|f\|_\pi \leq 1} |\langle \pi(f), u \rangle| = \sup_{\|f\|_\pi \leq 1} \Big| \int_G u(x)f(x)\mathrm{d}x \Big|, \quad u \in B_{p,\pi},$$
$$\|f\|_\pi = \sup_{\|u\| \leq 1} |\langle \pi(f), u \rangle| = \sup_{\|u\| \leq 1} \Big| \int_G u(x)f(x)\mathrm{d}x \Big|, \quad f \in L_1(G).$$
So, we have set $\mathrm{PF}_{p,\pi}(G)^* = A_{p,\pi_\mathcal{U}} = B_{p,\pi}$, and in the case that $(\pi, E) = (\lambda_{p,G}, L_p(G))$ we usually use the symbol $\mathrm{PF}_p(G)^*$.
4. It is obvious that $B_{p,\pi} \subseteq B_p(G)$ is a contractive inclusion for every $(\pi, E) \in \mathrm{Rep}_p(G)$, and if (π, E) is a p-universal representation, it will become an isometric isomorphism.
5. It is valuable to note that the ultrafilter $\mathcal{U}$ is the one for which the embedding $\mathcal{B}(E)^* \subseteq (E^* \widehat{\otimes} E)_\mathcal{U}$ is isometric, so the space $A_{p,\pi_\mathcal{U}^\infty}$ is determined. Furthermore, if $\mathcal{V}$ is another free ultrafilter that makes the similar embedding $\mathcal{B}(E)^* \subseteq (E^* \widehat{\otimes} E)_\mathcal{V}$ into an isometry, then we have
$$A_{p,\pi_\mathcal{U}} = \mathrm{PF}_{p,\pi}(G)^* = A_{p,\pi_\mathcal{V}}.$$
So, our definition is independent of choosing suitable free ultrafilter; therefore, it is well defined.
6. For a locally compact group G, we have the following contractive inclusions:
$$\mathrm{PF}_p(G)^* = B_{p,\lambda_p} \subset B_p(G) \subset \mathcal{M}(A_p(G)).$$

All inclusions will become equalities in the case that G is amenable [see Runde (2005, Theorem 6.6 and Theorem 6.7)].

3.3 Extension Theorem

In the following, we study some functorial properties of the p-analog of the Fourier–Stieltjes algebras. One of the earliest questions about such algebras is when an extension of a function defined on a subgroup belongs to the p-analog of the Fourier–Stieltjes algebra on the larger group . For this aim, we deal with the following notation. Let $G_0 \subset G$ be any subset, and $u : G_0 \to \mathbb{C}$ be a function. By u°, we mean

$$u^\circ = \begin{cases} u & \text{on } G_0 \\ 0 & \text{o.w.} \end{cases}.$$

The next lemma is going to express the relation between representation of an open subgroup G_0 with the one of the initial group G.

Lemma 3 *Let $(\pi, E) \in \mathrm{Rep}_p(G)$. Then, the restriction of π to the open subgroup G_0, which is denoted by (π_{G_0}, E) belongs to $\mathrm{Rep}_p(G_0)$. Moreover, for each $f \in L_1(G_0)$ and each $g \in L_1(G)$, we have the following relations*

$$\pi_{G_0}(f) = \pi(f^\circ), \quad \text{and} \quad \pi_{G_0}(g|_{G_0}) = \pi(g\chi_{G_0}). \tag{12}$$

Proof It is evident that $(\pi_{G_0}, E) \in \mathrm{Rep}_p(G_0)$. For the second part, simple calculations below reveal that our claim is true. For $\xi \in E$ and $\eta \in E^*$, if $f \in L_1(G_0)$ and $g \in L_1(G)$, then we have

$$\begin{aligned}\langle \pi_{G_0}(f)\xi, \eta\rangle &= \int_{G_0} f(x)\langle \pi_{G_0}(x)\xi, \eta\rangle \mathrm{d}x \\ &= \int_{G_0} f(x)\langle \pi(x)\xi, \eta\rangle \mathrm{d}x \\ &= \int_{G} f^\circ(x)\langle \pi(x)\xi, \eta\rangle \mathrm{d}x \\ &= \langle \pi(f^\circ)\xi, \eta\rangle,\end{aligned}$$

and

$$\begin{aligned}\langle \pi_{G_0}(g|_{G_0})\xi, \eta\rangle &= \int_{G_0} g|_{G_0}(x)\langle \pi_{G_0}(x)\xi, \eta\rangle \mathrm{d}x \\ &= \int_{G_0} g|_{G_0}(x)\langle \pi(x)\xi, \eta\rangle \mathrm{d}x \\ &= \int_{G} g(x)\chi_{G_0}(x)\langle \pi(x)\xi, \eta\rangle \mathrm{d}x \\ &= \langle \pi(g\chi_{G_0})\xi, \eta\rangle.\end{aligned}$$

So, we have

$$\langle \pi_{G_0}(f)\xi, \eta\rangle = \langle \pi(f^\circ)\xi, \eta\rangle, \tag{13}$$

$$\langle \pi_{G_0}(g|_{G_0})\xi, \eta\rangle = \langle \pi(g\chi_{G_0})\xi, \eta\rangle. \tag{14}$$

and since (13) and (14) hold for every $\xi \in E$ and $\eta \in E^*$, then the relations in (12) are obtained. □

Proposition 4 *Let G be a locally compact group and G_0 be its open subgroup, and let $(\pi, E) \in \mathrm{Rep}_p(G)$. Then, the following statements hold.*

1. *The map $S_{\pi_{G_0}} : \mathrm{PF}_{p,\pi_{G_0}}(G_0) \to \mathrm{PF}_{p,\pi}(G)$ defined via $S_{\pi_{G_0}}(\pi_{G_0}(f)) = \pi(f^\circ)$, for $f \in L_1(G_0)$, is an isometric homomorphism. In fact, we have the following isometric identification*

$$\mathrm{PF}_{p,\pi_{G_0}}(G_0) = \overline{\{\pi(f) \,:\, f \in L_1(G),\ \mathrm{supp}(f) \subseteq G_0\}}^{\|\cdot\|_{\mathcal{B}(E)}} \subseteq \mathrm{UPF}_{p,\pi}(G).$$

2. *The linear restriction mapping $R_\pi : B_{p,\pi} \to B_{p,\pi_{G_0}}$ which is defined for $u \in B_{p,\pi}$, as $R_\pi(u) = u|_{G_0}$ is the dual map of $S_{\pi_{G_0}}$ and is a quotient map.*
3. *The extension map $E_\pi : B_{p,\pi_{G_0}} \to B_{p,\pi}$, defined via $E_\pi(u) = u^\circ$ is an isometric map.*
4. *The restriction mapping $R : B_p(G) \to B_p(G_0)$ is a contraction.*
5. *When (π, E) is also a p-universal representation, we have the following contractive inclusions:*

$$\mathrm{PF}_p(G_0)^* \subseteq B_{p,\pi_{G_0}} \subseteq B_p(G_0) \subseteq \mathcal{M}(A_p(G_0)).$$

Under the assumption that G_0 is amenable, we have isometric identification below

$$\mathrm{PF}_p(G_0)^* = B_{p,\pi_{G_0}} = B_p(G_0) = \mathcal{M}(A_p(G_0)).$$

Proof

1. Through Lemma 3, the map $S_{\pi_{G_0}}$ is an isometric homomorphism with the range containing the dense space $\{\pi(f) \,:\, f \in L_1(G),\ \mathrm{supp}(f) \subseteq G_0\}$. So, the algebra $\mathrm{PF}_{p,\pi_{G_0}}(G_0)$ and the subalgebra $\overline{\{\pi(f) \,:\, f \in L_1(G),\ \mathrm{supp}(f) \subseteq G_0\}}^{\|\cdot\|_{\mathcal{B}(E)}}$ of $\mathrm{UPF}_p(G)$ are identified.
2. Evidently, we have $R_\pi = S^*_{\pi_{G_0}}$; therefore, $S_{\pi_{G_0}}$ is a quotient map.
3. Before showing E_π is an isometric map, it is needed to take notice of the fact that since R_π is onto, then we have

$$B_{p,\pi_{G_0}} = \mathrm{PF}_{p,\pi_{G_0}}(G_0)^* = \mathrm{PF}_{p,\pi}(G)^*/\mathrm{PF}^{\perp}_{p,\pi_{G_0}} = B_{p,\pi}/\mathrm{PF}^{\perp}_{p,\pi_{G_0}}.$$

Obviously, E_π is contraction. Furthermore, we have

$\|u\| = \|R_\pi(E_\pi(u))\| \le \|E_\pi(u)\| \le \|u\|.$

Moreover, one may be inclined to gain this conclusion through the following argument. Define the map $\mathcal{E}_\pi : \mathrm{PF}_{p,\pi}(G) \to \mathrm{PF}_{p,\pi_{G_0}}(G_0)$ via $\mathcal{E}_\pi(\pi(f)) = \pi_{G_0}(f|_{G_0})$, for $f \in L_1(G)$. Then, $\mathcal{E}_\pi$ is contraction and $(\mathcal{E}_\pi)^* = E_\pi$. So, E_π is well defined and contraction, as well.

4. From the part (2), the restriction map from $B_p(G)$ onto $B_{p,\pi_{G_0}}$ is a contraction, and due to Remark 9-(4), for every $(\rho, F) \in \mathrm{Rep}_p(G_0)$ the identity map from $B_{p,\rho}$ into $B_p(G_0)$ is a contraction; then, we have the result of this part.
5. Let $u \in \mathrm{PF}_p(G_0)^*$. Then by the part (3), we have $u^\circ \in \mathrm{PF}_p(G)^*$, and $\mathrm{PF}_p(G)^* \subseteq B_p(G)$, contractively, via Remark 9-(6). Since $R(u^\circ) = u$, it follows that $u \in B_{p,\pi_{G_0}}$, where (π, E) is a p-universal representation of G. For the case that G_0 is amenable, since through aforementioned remark, we have $\mathrm{PF}_p(G_0)^* = B_p(G_0)$, so $\mathrm{PF}_p(G_0)^* = B_{p,\pi_{G_0}} = B_p(G_0)$.

□

The next proposition is the consequence of the previous one and is one of the applicable result in dealing with problems about p-analog of the Fourier–Stieltjes algebras.

Proposition 5 *Let G be a locally compact group and G_0 be its open subgroup. Then*

1. *the extension mapping $E_{MM} : \mathcal{M}(A_p(G_0)) \to \mathcal{M}(A_p(G))$, defined for $u \in \mathcal{M}(A_p(G_0))$ via $E_{MM}(u) = u^\circ$, is an isometric map.*
2. *for every $u \in B_p(G_0)$, we have $u^\circ \in \mathcal{M}(A_p(G))$, and the map $E_{BM} : B_p(G_0) \to \mathcal{M}(A_p(G))$, with $u \mapsto u^\circ$, is a contraction.*
3. *if G_0 is also an amenable subgroup, then for every $u \in B_p(G_0)$, we have $u^\circ \in B_p(G)$, and the associated extending map $E_{BB} : B_p(G_0) \to B_p(G)$ is an isometric one.*

Proof

1. By the following relation for $u \in \mathcal{M}(A_p(G_0))$ and $v \in A_p(G)$ $u^\circ \cdot v = (u \cdot v|_{G_0})^\circ$, it can be concluded that $u^\circ \in \mathcal{M}(A_p(G))$, and obviously we have $\|u^\circ\|_{\mathcal{M}(A_p(G))} = \|u\|_{\mathcal{M}(A_p(G_0))}$
2. This part can be concluded by the inclusions in Proposition 4-(5) and the part (1).
3. Since G_0 is amenable, then by Proposition 4-(5) (or directly from Remark 9-(6)) we have the result.

□

One of the interesting problems on the Fourier–Stieltjes-type algebras is to study weighted homomorphism associated with a piecewise affine map as it has been considered in Ilie and Spronk (2005) and Ilie (2014). At this aim, it is crucial to be sure that such a homomorphism is well defined. Precisely, answering to the question that the homomorphism $\Phi_\alpha : B_p(G) \to B_p(H)$, defined via $\Phi_\alpha(u) = (u \circ \alpha)^\circ$, for $u \in B_p(G)$ is well-defined or not, would be precious. Here, $\alpha : Y \subseteq H \to G$ is a continuous piecewise affine map. So, we give some preliminaries here. For a locally compact topological group H, let $\Omega_0(H)$ denote the ring of subsets which generated by open cosets of H. By Ilie (2014), we have

$$\Omega_0(H) = \left\{ Y \setminus \cup_{i=1}^n Y_i \ : \ \begin{array}{c} Y \text{ is an open coset of } H, \\ Y_1, \ldots, Y_n \text{ open subcosets of infinite index in } Y \end{array} \right\}. \tag{15}$$

Moreover, for a set $Y \subseteq H$, by $\mathrm{Aff}(Y)$ we mean the smallest coset containing Y, and if $Y = Y_0 \setminus \cup_{i=1}^n Y_i \in \Omega_0(H)$, then $\mathrm{Aff}(Y) = Y_0$. Similarly, let us denote by $\Omega_{\mathrm{am}-0}(H)$ the ring of open cosets of open amenable subgroups of H. Now, we give the definition of a piecewise affine map.

Definition 11 Let $\alpha : Y \subseteq H \to G$ be a map.

1. The map α is called an affine map on an open coset Y of an open subgroup H_0, if

   $\alpha(xy^{-1}z) = \alpha(x)\alpha(y)^{-1}\alpha(z), \qquad x, y, z \in Y,$

2. The map α is called a piecewise affine map if

   3. there are pairwise disjoint $Y_i \in \Omega_0(H)$, for $i = 1, \ldots, n$, such that $Y = \cup_{i=1}^n Y_i$,
   4. there are affine maps $\alpha_i : \mathrm{Aff}(Y_i) \subseteq H \to G$, for $i = 1, \ldots, n$, such that $\alpha|_{Y_i} = \alpha_i|_{Y_i}$.

Remark 10 Ilie (2004, Remark 2.2) If $Y = h_0 H_0$ is an open coset of an open subgroup $H_0 \subset H$, and $\alpha : Y \subseteq H \to G$ is an affine map, then there exists a group homomorphism β associated with α such that

$$\beta : H_0 \subseteq H \to G, \quad \beta(h) = \alpha(h_0)^{-1}\alpha(h_0 h), \quad h \in H_0. \tag{16}$$

The next lemma is straightforward, and we leave it without proof, and it will be utilized in Theorem 3.

Lemma 4 *Let G and H are locally compact groups and $(\pi, E) \in \mathrm{Rep}_p(G)$.*

1. *For an element $x \in G$, let $L_x : B_p(G) \to B_p(G)$ be the left translation mapping defined through $L_x(u)(y) = u(xy)$ for $y \in G$ and $u \in B_p(G)$. Then, L_x is an invertible isometric map.*
2. *For a continuous homomorphism $\beta : H \to G$, the pair $(\pi \circ \beta, E)$ belongs to $\mathrm{Rep}_p(H)$ and the homomorphism $\Phi_\beta : B_p(G) \to B_p(H)$ is well-defined contractive homomorphism.*

The following theorem is one of our important results in this paper. Here, for a continuous piecewise affine map $\alpha : Y \subseteq H \to G$, we prove that the homomorphism $\Phi_\alpha : B_p(G) \to B_p(H)$, defined via

$$\Phi_\alpha(u) = \begin{cases} u \circ \alpha & \text{on } Y \\ 0 & \text{o.w. } Y \end{cases}$$

is well defined, and we determine its bound.

Theorem 3 *Let G and H be locally compact groups, and $\alpha : Y = \cup_{k=1}^n Y_k \subseteq H \to G$ be a continuous piecewise affine map with disjoint $Y_k \in \Omega_{\mathrm{am}-0}(H)$, for $k = 1, \ldots, n$. Then, $u \in B_p(G)$ implies that $(u \circ \alpha)^\circ \in B_p(H)$,* and consequently, *the weighted homomorphism $\Phi_\alpha : B_p(G) \to B_p(H)$ is well-defined bounded homomorphism.*

Proof We divide our proof into two steps. *Step 1*: First, we let $\alpha : Y = y_0 H_0 \subseteq H \to G$ be a continuous affine map, and $\beta : H_0 \to G$ be the homomorphism associated with α, as it is explained in Remark 10, for an open amenable subgroup H_0 of H. As we initially explained in Lemma 4-(2), the map $u \mapsto u \circ \beta$ is an algebra homomorphism from $\Phi_\beta : B_p(G) \to B_p(H_0)$. For the element y_0, consider the translation map $L_{\alpha(y_0)} : B_p(G) \to B_p(G)$, then by the following relation, and applying Proposition 5-(3), we have the result

$$(u \circ \alpha)^\circ = E_{BB} \circ \Phi_\beta \circ L_{\alpha(y_0)}, \quad u \in B_p(G).$$

where $E_{BB} : B_p(H_0) \to B_p(H)$, is the extension mapping. By the last relation, it is obtained that the extension of the function $u \circ \alpha$ belongs to $B_p(H)$, and evidently Φ_α is contractive as it is the combination of isometric and contractive maps. *Step 2*: Now, let $\alpha : Y \subseteq H \to G$ be a continuous piecewise affine map, so by our assumption of amenability, and similar to Definition 11, there exist pairwise disjoint sets $Y_k \in \Omega_{\mathrm{am}-0}(H)$, for $k = 1, \ldots, n$ with $n \in \mathbb{N}$, and affine maps $\alpha_k : \mathrm{Aff}(Y_k) \subseteq H \to G$ such that $Y = \cup_{k=1}^n Y_k$, and $\alpha_k|_{Y_k} = \alpha|_{Y_k}$. By previous step, we know that $(u \circ \alpha_k)^\circ \in B_p(H)$, and since

$$(u \circ \alpha_k)^\circ = \sum_{k=1}^n (u \circ \alpha_k)^\circ \cdot \chi_{Y_k},$$

we have the result via Corollary 2, and the fact that $B_p(H)$ is a Banach algebra. Moreover, we have

$$\|(u \circ \alpha)^\circ\| \le \|u\| \sum_{k=1}^n 2^{m_{Y_k}},$$

where the number m_{Y_k} is as it is described in Corollary 2. So, we have $\|\Phi_\alpha\| \le \sum_{k=1}^n 2^{m_{Y_k}}$. □

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