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RESEARCH PAPER

Some Pre-C*-algebras Generated by a C*-algebra $\mathcal{A}$ with Completion C($[-1, 1], \mathcal{A}$)

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Abstract

For any C*-algebra $\mathcal{A}$, we give a Banach $*$-algebra with approximate identity which $C([-1,1],\mathcal{A})$, the C*-algebra of all $\mathcal{A}$-valued continuous functions on [0, 1], is its C*-envelope. We show that $C([-1,1],\mathcal{A})$ is $*$-isomorphic to a C*-subalgebra of bounded continuous functions from self-adjoint elements of the closed unital ball of $\mathcal{E}$ to $\mathcal{A}\otimes\mathcal{E}$ for any unital C*-algebra $\mathcal{E}$. Furthermore, for any C*-algebra $\mathcal{A}$ and numerical semigroup S we give a pre-C*-algebra with completion $C([0,1],\mathcal{A})$ via Cauchy extensions of C*-algebras. It is also shown that the Dirichlet extension of $\mathcal{A}$ is $*$-isomorphic to $C([0,1],\mathcal{A})$. Finally, we introduce the notion of M-Cauchy envelope of C*-algebras, where M is an at most countable commutative monoid.

Keywords Pre-C*-algebra · Extension of C*-algebras · Cauchy extension · Dirichlet extension · Crossed product

Mathematics Subject Classfication 46L05 · 46M15

1 Introduction and Preliminaries

For any C*-algebra $\mathcal{A}$ finding some pre-C*-algebras with completion $C([0,1],\mathcal{A})$, the C*-algebra of all $\mathcal{A}$-valued continuous functions on [0, 1] is interesting. In fact, the idea may be interpreted as recovering $C([0,1],\mathcal{A})$ by some pre-C*-algebras. One approach to this is given in Nourouzi and Reza (2019) via a type of extension of C*-algebras which are called Cauchy extensions. For the extension of C*-algebras, we refer the reader to, e.g., Arveson (1977) and Busby (1968). In this paper, starting with a C*-algebra $\mathcal{A}$ we give some pre-C*-algebras with completion $C([0,1],\mathcal{A})$ through Cauchy and Dirichlet extensions of C*-algebras. In the last section, we introduce M-Cauchy envelope of C*-algebras, where M is an at most countable commutative monoid M.

We recall some definitions and results from Nourouzi and Reza (2019) which will be needed.

Let $\mathcal{A}$ be a C*-algebra. We denote by $\mathcal{A}[Z]$ the set of all formal power series $F(Z)=\sum_{n=0}^{\infty}a_nZ^n$, where each $a_n\in\mathcal{A}$ and $\sum_{n=0}^{\infty}\|a_n\|<\infty$. Then, $\mathcal{A}[Z]$ becomes a complex involutive algebra with the pointwise addition, scalar multiplication and involution and with the Cauchy product as multiplication. That is, for any formal power series $F(Z)=\sum_{n=0}^{\infty}a_nZ^n$ and $G(Z)=\sum_{n=0}^{\infty}b_nZ^n$ in the complex involutive algebra $\mathcal{A}[Z]$ and scalar $\lambda\in\mathbb{C}$ we have

$$\sum_{n=0}^{\infty}a_nZ^n+\sum_{n=0}^{\infty}b_nZ^n=\sum_{n=0}^{\infty}(a_n+b_n)Z^n,$$
$$\lambda\sum_{n=0}^{\infty}a_nZ^n=\sum_{n=0}^{\infty}\lambda a_nZ^n,$$
$$F^*(Z)=\sum_{n=0}^{\infty}a_n^*Z^n,$$
$$F(Z)G(Z)=\sum_{n=0}^{\infty}\left(\sum_{n=p+q}a_pb_q\right)Z^n.$$

Suppose that K is a subset of $[-1,1]$ such that 0 is a limit point of K. The norm $\|\cdot\|_K$ defined by

$$\|F\|_K=\sup_{t\in K}\left\|\sum_{n=0}^{\infty}a_nt^n\right\|,$$

for all $F(Z)=\sum_{n=0}^{\infty}a_nZ^n\in\mathcal{A}[Z]$ satisfies $\|FF^*\|_K=\|F\|_K^2$. Note that $(\mathcal{A}[Z],*,\|\cdot\|_K)$ is a pre-C*-algebra which

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is not a C*-algebra. The completion $[\mathcal{A}]_K$ of $(\mathcal{A}, *, \|\cdot\|_K)$ is called the Cauchy extension of $\mathcal{A}$ which is $*$-isomorphic to the set of all uniformly continuous functions from K to $\mathcal{A}$ [Nourouzi and Reza 2019, Theorem 5 (iii)]. If $K = [0, 1]$, then $[\mathcal{A}]_K \cong C([0, 1], \mathcal{A})$ [Nourouzi and Reza 2019, Theorem 5 (ii)]. Also, $(\mathcal{A}[Z], *, \|\cdot\|_1)$ is a Banach $*$-algebra where

$$\|F\|_1 = \sum_{n=0}^{\infty} \|a_n\|$$

for all $F(Z) = \sum_{n=0}^{\infty} a_n Z^n \in \mathcal{A}[Z]$ [see (Nourouzi and Reza 2019, Proposition 1)].

2 $C([-1, 1], \mathcal{A})$ as Enveloping C*-algebra

It is worth mentioning that if $(U_\lambda)_{\lambda\in\Lambda}$ is an approximate identity for C*-algebra $\mathcal{A}$, then $(U_\lambda)_{\lambda\in\Lambda}$ is also an approximate identity for $(\mathcal{A}[Z], \|\cdot\|_1)$. Indeed, let $F(Z) = \sum_{n=0}^{\infty} a_n Z^n \in \mathcal{A}[Z]$ and $\varepsilon > 0$ be given. There is a positive integer N such that $\sum_{n=N+1}^{\infty} 2\|a_n\| < \varepsilon$. For any $\lambda \in \Lambda$, we have

$$\begin{aligned} \|F(Z) - U_\lambda F(Z)\| &= \sum_{n=0}^{\infty} \|a_n - U_\lambda a_n\| \\ &= \sum_{n=0}^{N} \|a_n - U_\lambda a_n\| \\ &\quad + \sum_{n=N+1}^{\infty} \|a_n - U_\lambda a_n\| \\ &\le \sum_{n=0}^{N} \|a_n - U_\lambda a_n\| + \varepsilon. \end{aligned}$$

Therefore,

$$\limsup_{\lambda} \|F(Z) - U_\lambda F(Z)\| \le \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we have

$$\lim_{\lambda} \|F(Z) - U_\lambda F(Z)\| = 0.$$

Similarly, we get

$$\lim_{\lambda} \|F(Z) - F(Z)U_\lambda\| = 0.$$

That is, $(U_\lambda)_{\lambda\in\Lambda}$ is also an approximate identity for $(\mathcal{A}[Z], \|\cdot\|_1)$.

Theorem 1 *For any* C**-algebra* $\mathcal{A}$, $C([-1, 1], \mathcal{A})$ *is the enveloping* C**-algebra of* $(\mathcal{A}[Z], *, \|\cdot\|_1)$.

Proof We first show that (i): the C*-envelope C$^*(\mathcal{A}[Z])$ of $(\mathcal{A}[Z], \|\cdot\|_1)$ is not trivial and (ii): $C^*(\mathbb{C}[Z]) \cong C[-1, 1]$. To prove (i), let Φ be the universal representation of C*-algebra $C(J, \mathcal{A})$, where $J = [-1, 1]$. Then, $\Phi \circ i$ is a faithful representation of Banach $*$-algebra $(\mathcal{A}[Z], \|\cdot\|_1)$, where $i : \mathcal{A}[Z] \hookrightarrow C(J, \mathcal{A})$ is the inclusion map. Consider the (nontrivial) universal representation π of $(\mathcal{A}[Z], \|\cdot\|_1)$ [see (Dixmier 1977, 2.7.6)]. Let *Rep* denote the set of all representations of $\mathcal{A}[Z]$. We have

$$\|F\|_J \le \sup_{\omega\in Rep} \|\omega(F)\| = \|\pi(F)\|.$$

This implies that π is faithful. Therefore, $\mathcal{A}[Z]$ equipped with the norm $\|\cdot\|'$ defined by $\|F\|' = \|\pi(F)\|$ for any $F(Z) = \sum_{n=0}^{\infty} a_n Z^n \in \mathcal{A}[Z]$ is a pre-C*-algebra with completion C$^*(\mathcal{A}[Z])$.

To see (ii), since $\|\cdot\|_J \le \|\cdot\|'$, one can consider $B = C^*(\mathbb{C}[Z])$ as a $*$-subalgebra of $E = C[-1, 1]$. Let $f \in B$. We have $\sigma_E(f) = f(J) = \sigma_B(f)$ (σ stands for the spectrum), and therefore, $r_B(f) = r_E(f)$, where r_B and r_E are the spectral radii with respect to B and E, respectively. We have

$$\|f\|_J^2 = \|f\bar{f}\|_J = r_E(f\bar{f}) = r_B(f\bar{f}) = \|f\bar{f}\|' = \|f\|'^2.$$

That is $\|f\|' = \|f\|_J$. This implies that $C^*(\mathbb{C}[Z]) \cong C[-1, 1]$.

Now, from (Grothendieck 1955, Theorem 2) we have

$$\mathcal{A}[Z] \cong \mathbb{C}[Z]\hat{\otimes}_\gamma\mathcal{A},$$

where $\mathbb{C}[Z]\hat{\otimes}_\gamma\mathcal{A}$ is the projective tensor product of $\mathbb{C}[Z]$ and $\mathcal{A}$, i.e., the completion of the algebraic tensor product $\mathbb{C}[Z] \otimes \mathcal{A}$ with respect to the norm

$$\|u\|_\gamma = \inf\left\{\sum_{i=1}^{n} \|x_i\|\|y_i\| : u = \sum_{i=1}^{n} x_i \otimes y_i \in \mathbb{C}[Z] \otimes \mathcal{A}\right\}.$$

On the other hand, by (Okayasu 1966, Theorem 3) we have the isometric $*$-isomorphism

$$C^*(\mathbb{C}[Z]\hat{\otimes}_\gamma\mathcal{A}) \cong C^*(\mathbb{C}[Z])\hat{\otimes}_{\max}\mathcal{A}.$$

Finally, we have

$$\begin{aligned} C^*(\mathcal{A}[Z]) &\cong C^*(\mathbb{C}[Z]\hat{\otimes}_\gamma\mathcal{A}) \\ &\cong C^*(\mathbb{C}[Z])\hat{\otimes}_{\max}\mathcal{A} \\ &\cong C[-1, 1] \otimes \mathcal{A} \cong C([-1, 1], \mathcal{A}). \end{aligned}$$

□

Suppose that $\mathcal{A}$ and $\mathcal{E}$ are two unital C*-algebras. Let $S(\mathcal{E})$ be the set of all self-adjoint elements in the closed unit ball of $\mathcal{E}$. Define the $*$-homomorphism $\theta : \mathcal{A}[Z] \to C_b(S(\mathcal{E}), \mathcal{A} \otimes \mathcal{E})$ by $\theta(F)(b) = \sum_{n=0}^{\infty} a_n \otimes b^n$, where $b \in S(\mathcal{E})$, and $C_b(S(\mathcal{E}), \mathcal{A} \otimes \mathcal{E})$ is the set of all bounded continuous functions from $S(\mathcal{E})$ to $\mathcal{A} \otimes \mathcal{E}$ and $\otimes$ is the minimal (maximal) tensor product.

Corollary 1 *The* C**-algebra* $C([-1, 1], \mathcal{A})$ *is* $*$*-isomorphic to* $\operatorname{Im}\tilde{\theta}$, *where* $\tilde{\theta}$ *is the extension of* θ.

Proof By Theorem 1 we have

$$\begin{aligned}\|\theta(F)\| &= \sup_{b\in S(\mathcal{E})}\left\|\sum_{n=0}^{\infty} a_n \otimes b^n\right\| \\ &\leq \sup_{t\in J}\left\|\sum_{n=0}^{\infty} a_n t^n\right\| \\ &= \|F\|_J \\ &= \|F\|'\end{aligned}$$

On the other hand, since $\lambda 1 \in S(\mathcal{E})$ for $-1 \leq \lambda \leq 1$, we have $\|F\|_J \leq \|\theta(F)\|$. Therefore, $\|\theta(F)\| = \|F\|_J$ and the extension

$$\tilde{\theta} : C([-1,1],\mathcal{A}) \rightarrow C_b(S(\mathcal{E}), \mathcal{A}\otimes\mathcal{E})$$

is an isometry. □

3 Numerical Semigroups and Cauchy Extensions

A numerical semigroup is a submonoid of the additive monoid $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ such that the greatest common divisor of its nonzero elements is equal to one. A submonoid $S \subseteq \mathbb{N}$ is a numerical semigroup if and only if $\mathbb{N} \setminus S$ is a finite set. Let S be a nontrivial submonoid of $\mathbb{N}$ and $0 \neq \delta$ be the greatest common divisor of the elements of S. The function $\sigma : S \rightarrow \mathbb{N}$ defined by $\sigma(s) = s/\delta$ is bijective homomorphism between S and $\operatorname{Im} \sigma$. This means that every nontrivial submonoid of $\mathbb{N}$ is isomorphic to a numerical semigroup. If S is a numerical semigroup, then $\delta S = \{\delta s : s \in S\}$ is a nontrivial submonoid of $\mathbb{N}$ for any integer $\delta > 0$ [see Rosales et al. (2006)].

Let $\mathcal{A}$ be a C*-algebra and S a submonoid of $\mathbb{N}$. Then, the set $(\mathcal{A}, S)[Z]$ consisting of all power series of the form $F(Z) = \sum_{s\in S} a_s Z^s$ belonging to $\mathcal{A}[Z]$ equipped with the norm

$$\|F(Z)\|_K = \sup_{t\in K}\|F(t)\|.$$

is a sub-pre-C*-algebra of $\mathcal{A}[Z]$, where K is a subset of [0, 1] with 0 as limit point. The completion $[\mathcal{A}, S]_K$ of $(\mathcal{A}, S)[Z]$ is a C*-algebra which is in fact, (S, K)-Cauchy extension of $\mathcal{A}$. In particular, $[\mathcal{A}]_K = [\mathcal{A}, \mathbb{N}]_K$.

Theorem 2 *If S is a numerical semigroup and $I = [0, 1]$,then $[\mathcal{A}, S]_I \cong [\mathcal{A}]_I$.*

Proof Put

$$\mathcal{A}_m = \left\{F(Z) = \sum_{n=0}^{\infty} a_n Z^n \in \mathcal{A}[Z] : a_0 = a_1 = \cdots = a_{m-1} = 0\right\},$$

where $m \geq 1$ is an integer. Since S is a numerical semigroup, there exists an integer $m \geq 1$ such that $\mathcal{A}_m \subset (\mathcal{A}, S)[Z]$. The completion $\hat{\mathcal{A}}_m$ of $\mathcal{A}_m$ is equal to $\hat{\mathcal{A}}_1$. In fact, we have

$$at^{\alpha} = \lim_{N\rightarrow\infty}\sum_{n=0}^{N} at^{2\alpha}(1-t^{\alpha})^n,$$

for $\alpha = 1, 2, 3, \ldots$ and any $a \in \mathcal{A}$ and $t \in I$. This shows that

$$a_1 t + a_2 t^2 + \cdots + a_{m-1}t^{m-1} \in \hat{\mathcal{A}}_m,$$

for any element $a_1, a_2, \ldots, a_{m-1}$ of $\mathcal{A}$. Therefore, $\hat{\mathcal{A}}_1 = \hat{\mathcal{A}}_m$ from which we have $\hat{\mathcal{A}}_1 \subset [\mathcal{A}, S]_I$. Since $0 \in S$, we have $\mathcal{A} \subset [\mathcal{A}, S]_I$, and therefore, $\mathcal{A}[Z] \subseteq [\mathcal{A}, S]_I$. Now by [Nourouzi and Reza 2019, Theorem 5 (ii), (i)] we have $[\mathcal{A}, S]_I \cong [\mathcal{A}]_I \cong C([0, 1], \mathcal{A})$. □

A subset K of real numbers is said to be δ-regular if $K_\delta = \{x^\delta : x \in K\} = [0, 1]$, where $\delta > 0$ is an even integer. For example, the set

$$K = \{\sqrt[\delta]{x} : x \in \mathbb{Q}, 0 \leq x \leq 1\} \cup \{-\sqrt[\delta]{x} : x \in \mathbb{R} - \mathbb{Q}, 0 \leq x \leq 1\}$$

is a δ-regular set for any even integer $\delta > 1$.

Theorem 3 *If K is a δ-regular set and S is a numerical semigroup, then $[\mathcal{A}, \delta S]_K \cong C([0, 1], \mathcal{A})$ for any* C**-algebra* $\mathcal{A}$.

Proof Let $(m_n)_{n=0}^{\infty}$ be an enumeration of the elements of S. If

$$F(Z) = \sum_{n=0}^{\infty} a_n Z^{\delta m_n} \in (\mathcal{A}, \delta S)[Z],$$

then

$$\begin{aligned}\|F(Z)\|_K &= \sup_{t\in K}\left\|\sum_{n=0}^{\infty} a_n t^{\delta m_n}\right\| \\ &= \sup_{t\in[0,1]}\left\|\sum_{n=0}^{\infty} a_n t^{m_n}\right\| = \left\|F^{\delta}(Z)\right\|_{[0,1]},\end{aligned}$$

where $F^{\delta}(Z) = \sum_{n=0}^{\infty} a_n Z^{m_n} \in (\mathcal{A}, S)[Z]$. Note that $F \mapsto F^{\delta}$ is a bijective isometric $*$-homomorphism between $(\mathcal{A}, \delta S)[Z]$ and $(\mathcal{A}, S)[Z]$. Therefore, by Theorem 2, $[\mathcal{A}, \delta S]_K \cong C([0, 1], \mathcal{A})$. □

4 Dirichlet Extension of C*-algebras

Let $\mathcal{A}{<}Z{>}$ be the set of all formal Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-Z}$, where $\mathcal{A}$ is a C*-algebra and each $a_n \in \mathcal{A}$ with $\sum_{n=1}^{\infty}\|a_n\| < \infty$. Then, $\mathcal{A}{<}Z{>}$ is a $*$-algebra with the pointwise addition, scalar multiplication and involution and the Dirichlet product. That is,

$$\sum_{n=1}^{\infty} a_n n^{-Z} + \sum_{n=1}^{\infty} b_n n^{-Z} = \sum_{n=1}^{\infty} (a_n + b_n) n^{-Z},$$
$$\lambda \sum_{n=1}^{\infty} a_n n^{-Z} = \sum_{n=1}^{\infty} \lambda a_n n^{-Z},$$
$$\left(\sum_{n=1}^{\infty} a_n n^{-Z}\right)^* = \sum_{n=1}^{\infty} a_n^* n^{-Z},$$
$$\left(\sum_{n=1}^{\infty} a_n n^{-Z}\right)\left(\sum_{n=1}^{\infty} b_n n^{-Z}\right) = \sum_{n=1}^{\infty}\left(\sum_{n=pq} a_p b_q\right) n^{-Z},$$

where $\sum_{n=1}^{\infty} a_n n^{-Z}, \sum_{n=1}^{\infty} b_n n^{-Z} \in \mathcal{A}<Z>$, and $\lambda \in \mathbb{C}$. If

$$\mathcal{A}_1 = \left\{\sum_{n=1}^{\infty} a_n n^{-Z} \in \mathcal{A}<Z> \ : a_1 = 0\right\},$$

then $\mathcal{A}_1$ is an ideal of $\mathcal{A}<Z>$. Since $1 + 4^{-Z}$ has an inverse

$$\sum_{n=1}^{\infty} (-1)^{n+1} 4^{-(n-1)Z},$$

then -1 is a spectral value of $(2^{-Z})^2$, and therefore, there is no complete C*-norm on $\mathcal{A}<Z>$. We need the following proposition [see (Apostol 1976, Theorem 11.3)].

Proposition 1 *Suppose that $\mathcal{A}$ is a C*-algebra and*

$$F(Z) = \sum_{n=1}^{\infty} a_n n^{-Z} \in \mathcal{A}<Z>.$$

If $(s_m)_{m=1}^{\infty}$ is a sequence of real numbers such that $s_m \to +\infty$ and

$$\sum_{n=1}^{\infty} a_n n^{-s_m} = 0$$

for all m, then $a_n = 0$ for all n.

Define $\|\cdot\|_D$ by

$$\|F(Z)\|_D = \sup_{t\in[0,1]} \|\bar{F}(t)\|,$$

where $F(Z) = \sum_{n=1}^{\infty} a_n n^{-Z} \in \mathcal{A}<Z>, \alpha(t) = 1 - 1/t$ and $\bar{F}(t) = \sum_{n=1}^{\infty} a_n n^{\alpha(t)}$ is a continuous function on [0, 1] with $\bar{F}(0) = a_1$. Note that $\|\cdot\|_D$ is clearly a seminorm on $\mathcal{A}<Z>$ and if $\|F\|_D = 0$, then $F = 0$ by Proposition 1. Therefore, $\|\cdot\|_D$ is a norm on $\mathcal{A}<Z>$. Furthermore,

$$\|FG\|_D \le \|F\|_D \|G\|_D$$
$$\|FF^*\|_D = \|F\|_D^2,$$

for all $F, G \in \mathcal{A}<Z>$. Then, $(\mathcal{A}<Z>, \|\cdot\|_D)$ is a pre-C*-algebra and we call the completion $<\mathcal{A}>_D$ of $\mathcal{A}<Z>$ the Dirichlet extension of $\mathcal{A}$.

In the following theorem, if $f \in C[0,1]$ and $a \in \mathcal{A}$, by the notation fa we mean an $\mathcal{A}$-valued continuous function defined by $(fa)(t) = f(t)a$, for any $t \in [0,1]$.

Theorem 4 *Let $\mathcal{A}$ be a C*-algebra. Then, $<\mathcal{A}>_D \cong C([0,1],\mathcal{A})$.*

Proof First note that any $F = \sum_{n=1}^{\infty} a_n n^{-Z} \in \mathcal{A}<Z>$ induces a continuous function $\bar{F} : [0,1] \to \mathcal{A}$ as

$$\bar{F}(t) = \sum_{n=1}^{\infty} a_n n^{\alpha(t)},$$

where $\alpha(t) = 1 - 1/t$ and $\bar{F}(0) = a_1$. Let

$$\tilde{\mathcal{A}} = \left\{\bar{F} : F = \sum_{n=1}^{\infty} a_n n^{-Z} \in \mathcal{A}<Z>\right\}.$$

For any $a \in \mathcal{A}$ and $\bar{F} \in \tilde{\mathbb{C}}$ we have $\bar{F}a \in \tilde{\mathcal{A}}$, where

$$(\bar{F}a)(t) = \sum_{n=1}^{\infty} \lambda_n a n^{\alpha(t)} \quad (t \in [0,1]).$$

Note that $\tilde{\mathbb{C}}$ is a self-adjoint algebra of complex functions on [0, 1] which separates the points of [0, 1] and never vanishes on it. By the Stone–Weierstrass theorem $\tilde{\mathbb{C}}$ is dense in $C([0,1])$. Let $f \in C([0,1])$. Consider a sequence $(F_m)_{m=1}^{\infty}$ in $\mathbb{C}<Z>$ such that $\bar{F}_m \to f$ in $\|\cdot\|_D$. Since each $\bar{F}_m a \in \tilde{\mathcal{A}} \cong \mathcal{A}<Z>$ and $\bar{F}_m a \to fa$ in $(<\mathcal{A}>_D, \|\cdot\|_D)$, by (Murphy 1990, Lemma 6.4.16) the closed linear span of $\{fb : f \in \mathbb{C}<Z>, b \in \mathcal{A}\}$ is equal to $C([0,1],\mathcal{A})$. That is $<\mathcal{A}>_D \cong C([0,1],\mathcal{A})$. □

5 *M*-Cauchy Envelope of C*-algebras

In this section, we replace the subsemigroups of $\mathbb{N}$ in Sect. 3 by an at most countable commutative monoid M and use the idea of Sect. 2 to obtain new C*-algebras. In fact, by the idea given here we generalize Sects. 2, 3 and 4.

We began with some notations of crossed product of C*-algebras [see e.g., Pedersen (1979) and Williams (2007)]. Let G be a discrete group and $\mathcal{A}$ a C*-algebra. A G-C*-algebra $(\mathcal{A}, \lambda_{\mathcal{A}})$ is a group homomorphism $\lambda_{\mathcal{A}} : G \to \mathrm{Aut}(\mathcal{A})$. We denote by $\mathcal{A} \rtimes_{\lambda_{\mathcal{A}}} G$ the crossed product of $\mathcal{A}$ by G. The $*$-homomorphism $\varphi : \mathcal{A} \to \mathcal{B}$ of C*-algebras is called G-equivariant (or λ_A–λ_B-equivariant) homomorphism if

$$\lambda_{\mathcal{B}}(g) \circ \varphi = \varphi \circ \lambda_{\mathcal{A}}(g) \quad (g \in G).$$

The class G-C*-algebras and G-equivariant homomorphisms form a category which is denoted by G-C***-alg**. We also denote by C***-alg** the category of C*-algebras and $*$-homomorphisms.

Let $\mathcal{A}$, S and δ be a C*-algebra, a numerical semigroup and a positive integer, respectively. If K is a subset of $[-1,1]$ with 0 as limit point then by an argument similar to that of given in Theorem 3, the pre-C*-algebra $(\mathcal{A},\delta S)[Z]$ with the norm $\|\cdot\|_K$ is isometric $*$-isomorphism to the pre-C*-algebra $(\mathcal{A},S)[Z]$ with the norm $\|\cdot\|_{K_\delta}$. Therefore, an argument similar to that of given in Theorem 2 gives that $[\mathcal{A},\delta S]_K \cong [\mathcal{A}]_{K_\delta}$. In particular, we have $[\mathcal{A},\delta S]_I \cong [\mathcal{A}]_I$, where $I=[0,1]$.

Let $\mathcal{A}$ be a C*-algebra and M an at most countable commutative monoid with unit e. Put

$$(\mathcal{A},M)[Z]=\left\{F(Z)=\sum_{s\in M}a_sZ^s:\sum_{s\in M}\|a_s\|<\infty\right\}$$

and define addition, scalar multiplication and involution component-wise on $(\mathcal{A},M)[Z]$. Product is defined by

$$F(Z)G(Z)=\sum_{s\in M}\left(\sum_{s=pq}a_pb_q\right)Z^s,$$

for any $F(Z)=\sum_{s\in M}a_sZ^s$ and $G(Z)=\sum_{s\in M}b_sZ^s$ in $(\mathcal{A},M)[Z]$. Define a norm on $(\mathcal{A},M)[Z]$ by

$$\|F(Z)\|_1=\sum_{s\in M}\|a_s\| \quad \left(F(Z)=\sum_{s\in M}a_sZ^s\in(\mathcal{A},M)[Z]\right).$$

Then $(\mathcal{A},M)[Z]$ is a Banach $*$-algebra with approximate identity. We denote by $[\mathcal{A},M]$ the C*-envelope of $(\mathcal{A},M)[Z]$. We call $[\mathcal{A},M]$ the M-Cauchy envelope of $\mathcal{A}$. Suppose that the map $i:\mathcal{A}\to(\mathcal{A},M)[Z]$ is defined by $a\mapsto F(Z)=aZ^e$ and the map $p:(\mathcal{A},M)[Z]\to\mathcal{A}$ is defined by $\sum_{s\in M}a_sZ^s\mapsto\sum_{s\in M}a_s$. Since $p\circ i=id\mathcal{A}$, we have $\hat{p}\circ\hat{i}=id\mathcal{A}$, and therefore, $\hat{i}$ is an embedding of $\mathcal{A}$ in $[\mathcal{A},M]$. Let π be a nontrivial representation of $(\mathcal{A},M)[Z]$. We denote by $[\mathcal{A},\pi,M]$ the completion of $(\mathcal{A},M)[Z]/\ker\pi$ with norm

$$\|x+\ker\pi\|=\|\pi(x)\|.$$

In particular, if π is the universal representation, then $[\mathcal{A},\pi,M]=[\mathcal{A},M]$. The Banach $*$-algebra $(\mathcal{A},M)[Z]$ is called $*$-semisimple if there is a representation π of $(\mathcal{A},M)[Z]$ with $\ker\pi=0$.

Example 1 In the following all Banach $*$-algebras are assumed to be $*$-semisimple.

(i) If $M=\mathbb{N}$, then, by Theorem 1, $[\mathcal{A},M]\cong C([0,1],\mathcal{A})$;

(ii) Let π be the representation of $(\mathcal{A},\mathbb{N})[Z]$ with norm

$$\|\pi(F)\|=\sup_{t\in K}\left\|\sum_{n=0}^{\infty}a_nt^n\right\|$$

for any $F(Z)=\sum_{n=0}^{\infty}a_nZ^n$ in $(\mathcal{A},\mathbb{N})[Z]$. Then, $[\mathcal{A},\pi,\mathbb{N}]=[\mathcal{A}]_K$;

(iii) Let $\mathbb{N}^*=\{1,2,3,\dots\}$ be the monoid of natural numbers with multiplication as operation. Suppose that π is the representation of $(\mathcal{A},\mathbb{N}^*)[Z]$ with norm

$$\|\pi(F)\|=\sup_{t\in[0,1]}\left\|\sum_{n=1}^{\infty}a_nn^{\alpha(t)}\right\|$$

for any $F(Z)=\sum_{n=1}^{\infty}a_nn^{-Z}$ in $(\mathcal{A},\mathbb{N}^*)[Z]$. Then, $[\mathcal{A},\pi,\mathbb{N}^*]\cong C([0,1],\mathcal{A})$ by Theorem 4;

(iv) Let G be a discrete group such that $a^2=e$ for any $a\in G$. Then, $[\mathcal{A},G]\cong\mathcal{A}\rtimes_tG$ where t is the trivial action on G.

Theorem 5 *For any at most countable commutative monoid M, there exists a discrete group G and a functor* $\boldsymbol{M}:\boldsymbol{C^*}$***-alg*** $\to G$-$\boldsymbol{C^*}$***-alg***.

Proof Let $\varphi:\mathcal{A}\to\mathcal{B}$ be a $*$-homomorphism of C*-algebras. The map

$$\tilde{\varphi}:(\mathcal{A},M)[Z]\to(\mathcal{B},M)[Z]$$

defined by

$$\tilde{\varphi}\left(\sum_{s\in M}a_sZ^s\right)=\sum_{s\in M}\varphi(a_s)Z^s$$

induces a $*$-homomorphism $\hat{\varphi}:[\mathcal{A},M]\to[\mathcal{B},M]$. That is $[-,M]$ is a functor on the category of C*-algebras. Let $f:M\to M$ be an automorphism of monoids. The map $\tilde{f}:(\mathcal{A},M)[Z]\to(\mathcal{A},M)[Z]$ defined by $\sum_{s\in M}a_sZ^s\mapsto\sum_{s\in M}a_{f(s)}Z^{f(s)}$ induces the automorphism $\hat{f}:[\mathcal{A},M]\to[\mathcal{A},M]$ of C*-algebras. Consider the group $\mathrm{Aut}(M)=G^{op}$, where G^{op} is the opposite group of G. Consider G as a discrete group. The map

$$\lambda_{\mathcal{A}}:G\to\mathrm{Aut}([\mathcal{A},M])$$

defined by

$$\lambda_{\mathcal{A}}(g)=\hat{g}_{\mathcal{A}}:[\mathcal{A},M]\to[\mathcal{A},M]$$

is a homomorphism of groups and $\lambda_{\mathcal{A}}$ is an action of G on $[\mathcal{A},M]$. Let $\varphi:\mathcal{A}\to\mathcal{B}$ be a $*$-homomorphism of C*-algebras. Since

$$\tilde{g}_{\mathcal{B}}\circ\tilde{\varphi}=\tilde{\varphi}\circ\tilde{g}_{\mathcal{A}}\quad(g\in G)$$

we have

$$\lambda_{\mathcal{B}}(g)\circ\hat{\varphi}=\hat{\varphi}\circ\lambda_{\mathcal{A}}(g)\quad(g\in G)$$

and $\hat{\varphi}$ is a G-equivariant map. Therefore, $\boldsymbol{M}=[-,M]:$ C*-**alg** $\to G$-C*-**alg** is the desired functor. □

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