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RESEARCH PAPER

Bivariate Dunkl Analogue of Stancu Type q-Szász–Mirakjan–Kantorovich Operators and Rate of Convergence

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Abstract
In the present paper, we prove some results on rate of convergence for Dunkl analogue of Stancu type q-Szász–Mirakjan–Kantorovich operators in terms of second-order modulus of continuity and Lipschitz functions. Further, we construct the bivariate extension of these operators and obtain some approximation results.

Keywords q-Integers · q-Exponential functions · q-Hypergeometric functions · Szász operators · Dunkl analogue of Stancu type q-Szász–Mirakjan–Kantorovich operators · Modulus of continuity · Lipschitz functions · Peetre's K-functional

Mathematics Subject Classification Primary 41A25 · 41A36 · Secondary 33C45

1 Introduction

In 1912, Bernstein (1912) introduced a sequence of operators, known as the Bernstein operators and gave a constructive proof of the Weierstrass Approximation Theorem. Szász (1950) generalized the Bernstein operators on unbounded interval $[0, \infty)$ and obtained the approximation properties of the operators.

The applications of q-calculus are established for last 30 years in the field of approximation theory. Lupaş (1987) introduced the first q-analogue of the Bernstein polynomials. Later on, several authors studied and introduced various operators, q-analogues of several operators and studied their approximation properties (see Acar et al. 2018a, b, c; Cheikh et al. 2014; İçöz and Çekim 2015; Mursaleen and Ansari 2017; Mursaleen and Khan 2013, 2017; Mursaleen et al. 2015a, b, 2016a, b; Mursaleen and Nasiruzzaman 2018; Mohiuddine et al. 2018; Srivastava et al. 2017, 2019; Ulusoy and Acar 2016). For more other details, we can refer Bin Jebreen et al. (2019), Khan and Sharma (2018), Korovkin (1953), Khan et al. (2019) and Mohiuddine et al. (2017).

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First we recall some basic definitions and notations of q-calculus which are used in the present paper.

Definition 1.1 For the given value of $|q| < 1$, the q-integer $[k]_q$ is defined by

$$[k]_q = \begin{cases} \dfrac{1-q^k}{1-q} & (k \in \mathbb{C}) \\ \sum_{l=0}^{n-1} q^l = 1 + q + q^2 + \cdots + q^{n-1} & (k = n \in \mathbb{N}). \end{cases} \tag{1.1}$$

Definition 1.2 For the given value of $|q| < 1$, the q-factorial $[k]_q!$ is defined as

$$[k]_q! = \begin{cases} 1 & (k = 0) \\ \prod_{l=1}^{k} [l]_q & (k \in \mathbb{N}). \end{cases} \tag{1.2}$$

Rosenblum (1994) generalized the exponential function in the following form:

$$e_\mu(y) = \sum_{k=0}^{\infty} \frac{y^k}{\gamma_\mu(k)},$$

where

$$\gamma_\mu(2j) = \frac{2^{2j} j! \Gamma\left(j + \mu + \frac{1}{2}\right)}{\Gamma\left(\mu + \frac{1}{2}\right)},$$

and

$$\gamma_\mu(2j+1)=\frac{2^{2j+1}j!\Gamma\left(j+\mu+\frac{3}{2}\right)}{\Gamma\left(\mu+\frac{1}{2}\right)}.$$

For γ_μ, the recursion formula is given as

$$\gamma_\mu(j+1)=(j+1+2\mu\theta_{j+1})\gamma_\mu(j),\quad \left(j\in\mathbb{N}_0=0,1,2,3\ldots;\ \ \mu>-\frac{1}{2}\right)$$

where

$$\theta_j=\begin{cases}0, & \text{if } j\in 2\mathbb{N}\\ 1, & \text{if } j\in 2\mathbb{N}+1.\end{cases}$$

In 2014, Sucu (2014) introduced the Dunkl analogue of the Szász operators given by

$$S_n^*(f;y):=\frac{1}{e_\mu(ny)}\sum_{k=0}^{\infty}\frac{(ny)^k}{\gamma_\mu(l)}f\left(\frac{k+2\mu\theta_k}{n}\right),\tag{1.3}$$

where $y\geq 0,\mu\geq 0,n\in\mathbb{N}$ and $f\in C[0,\infty)$.

İçöz and Çekim (2016) introduced the Kantorovich integral generalization of q-Szász operators via Dunkl generalization. Dunkl analogue of the q-exponential functions is defined by Cheikh et al. (2014) and their recurrence relations as follows:

$$e_{\mu,q}(y)=\sum_{k=0}^{\infty}\frac{y^k}{\gamma_{\mu,q}(k)},\quad \text{for}\quad y\in[0,\infty),\quad q\in(0,1),\quad \mu>-\frac{1}{2}\tag{1.4}$$

$$E_{\mu,q}(y)=\sum_{k=0}^{\infty}\frac{q^{\frac{k(k-1)}{2}}y^k}{\gamma_{\mu,q}(k)},\quad y\in[0,\infty)\tag{1.5}$$

$$\gamma_{\mu,q}(j+1)=\left(\frac{1-q^{2\mu\theta_{j+1}+(j+1)}}{1-q}\right)\gamma_{\mu,q}(j),\quad j\in\mathbb{N},\tag{1.6}$$

where

$$\theta_j=\begin{cases}0, & \text{if } j\in 2\mathbb{N},\\ 1, & \text{if } j\in 2\mathbb{N}+1.\end{cases}$$

An explicit formula for $\gamma_{\mu,q}(n)$ is defined by

$$\gamma_{\mu,q}(k)=\frac{(q^{2\mu+1},q^2)_{[\frac{k+1}{2}]}(q^2,q^2)_{[\frac{k}{2}]}}{(1-q)^k},\quad k\in\mathbb{N}.$$

In Sect. 2, we will recall the definition and auxiliary results of the Dunkl analogue of Stancu type q-Szász–Mirakjan–Kantorovich operators introduced by Mursaleen and Ahasan (2018). In Sect. 3, we will obtain the rate of convergence for these operators, in terms of the weighted, second-order modulus of continuity and Lipschitz functions. In Sect. 4, we will construct the bivariate form of these operators and obtain the rate of convergence.

2 Dunkl Analogue of Stancu Type *q*-Szász–Mirakjan–Kantorovich Operators

Srivastava et al. (2017) constructed Dunkl generalization of q-Szász–Mirakjan–Kantorovich operators for $n\in\mathbb{N}$, $q\in(0,1)$, $|\mu|\leq\frac{1}{2}$ and $y\in[0,\infty)$. We have

$$\tilde{K_{n,q}}(f;y)=\frac{[n]_q}{E_{\mu,q}([n]_qy)}\sum_{k=0}^{\infty}\frac{([n]_qy)^k}{\gamma_{\mu,q}(k)}q^{\frac{k(k-1)}{2}}\int_{\frac{[k+2\mu\theta_k]_q}{q^{k-2}[n]_q}}^{\frac{[k+1+2\mu\theta_k]_q-1}{q^{k-1}[n]_q}+\frac{1}{[n]_q}}f(t)d_qt,\tag{2.1}$$

where f is a continuously nondecreasing function defined on $[0,\infty)$.

Quite recently, Mursaleen and Ahasan (2018) introduced the Dunkl analogue of Stancu type q-Szász–Mirakjan–Kantorovich operators. For any $n\in\mathbb{N}$, $y\in[0,\infty)$, $0<q<1$ and $|\mu|\leq\frac{1}{2}$, we have

$$K_{n,q}^{\alpha,\beta}(f;y)=\frac{[n]_q}{E_{\mu,q}([n]_qy)}\sum_{k=0}^{\infty}\frac{([n]_qy)^k}{\gamma_{\mu,q}(k)}q^{\frac{k(k-1)}{2}}\int_{\frac{[k+2\mu\theta_k]_q}{q^{k-2}[n]_q}}^{\frac{[k+1+2\mu\theta_k]_q-1}{q^{k-1}[n]_q}+\frac{1}{[n]_q}}f\left(\frac{[n]_qt+\alpha}{[n]_q+\beta}\right)d_qt.\tag{2.2}$$

Remark 2.1 If $\alpha=\beta=0$ in (2.2), then $K_{n,q}^{\alpha,\beta}(f;y)$ reduce to the operators (2.1).

For operators (2.2), we have Mursaleen and Ahasan (2018).

Lemma 2.2 *For each* $y\geq 0$, *we have*

(1) $K_{n,q}^{\alpha,\beta}(1;y)=1,\quad$ if $f(t)=1$

(2) $$K_{n,q}^{\alpha,\beta}(t;y)=\frac{1}{([n]_q+\beta)}\left(\alpha+\frac{1}{[2]_q}+\frac{2q[n]_q}{[2]_q}y\right),\quad \text{if}\ \ f(t)=\frac{[n]_qt+\alpha}{[n]_q+\beta}$$

(3) *If* $f(t)=\left(\frac{[n]_q t+\alpha}{[n]_q+\beta}\right)^2$, *we have*

$$\frac{2\alpha}{[2]_q([n]_q+\beta)^2}(1+2q[n]_q y)+\frac{\alpha^2}{([n]_q+\beta)^2}+\frac{\left(1+3q[n]_q y+3q[n]_q q^{2\mu}[1-2\mu]_q y+3q[n]_q^2 y^2\right)}{[3]_q([n]_q+\beta)^2}$$
$$\leq K_{n,q}^{\alpha,\beta}(t^2;y)\leq+\frac{1}{[3]_q([n]_q+\beta)^2}\left(1+3[n]_q y+3[n]_q[1+2\mu]_q y+3[n]_q^2 y^2\right)+\frac{\alpha^2}{([n]_q+\beta)^2}+\frac{2\alpha}{[2]_q([n]_q+\beta)^2}\left(1+2q[n]_q y\right).$$

Lemma 2.3 *We have the following moments*

(1) $K_{n,q}^{\alpha,\beta}(t-1;y)=\frac{\alpha}{([n]_q+\beta)}+\frac{1}{[2]_q([n]_q+\beta)}\left(1+2q[n]_q y\right)-1,$

(2) $K_{n,q}^{\alpha,\beta}(t-y;y)=\frac{1}{([n]_q+\beta)}\left(\alpha+\frac{1}{[2]_q}\right)+\left(\frac{2q[n]_q}{[2]_q([n]_q+\beta)}-1\right)y,$

(3)

$$\left(\frac{\left(3q[n]_q+3[n]_q q^{2\mu+1}[1-2\mu]_q\right)}{[3]_q([n]_q+\beta)^2}+\frac{2}{([n]_q+\beta)}\left(\frac{2\alpha q[n]_q}{[2]_q([n]_q+\beta)}-\alpha-\frac{1}{[2]_q}\right)\right)y+\left(1+\frac{3q[n]_q^2}{[3]_q([n]_q+\beta)^2}-\frac{4q[n]_q}{[2]_q([n]_q+\beta)}\right)y^2+\frac{1}{([n]_q+\beta)^2}\left(\alpha^2+\frac{1}{[3]_q}+\frac{2\alpha}{[2]_q}\right)$$
$$\leq K_{n,q}^{\alpha,\beta}\left((t-y)^2;y\right)\leq\left(\frac{\left(3[n]_q+3[n]_q[1+2\mu]_q\right)}{[3]_q([n]_q+\beta)^2}+\frac{2}{([n]_q+\beta)}\left(\frac{2\alpha q[n]_q}{[2]_q([n]_q+\beta)}-\alpha-\frac{1}{[2]_q}\right)\right)y+\left(1+\frac{3[n]_q^2}{[3]_q([n]_q+\beta)^2}-\frac{4q[n]_q}{[2]_q([n]_q+\beta)}\right)y^2+\frac{1}{([n]_q+\beta)^2}\left(\alpha^2+\frac{1}{[3]_q}+\frac{2\alpha}{[2]_q}\right).$$

3 Rate of Convergence

Here, we calculate the rate of convergence for the operators (2.2) by using modulus of continuity, Lipschitz-type maximal functions and Peetre's K-functional. Let $f\in C[0,\infty)$, $\omega(f,\delta)$ be the modulus of continuity of f and the maximum oscillation of f for any interval of length does not exceed $\delta>0$ and defined by the following relation:

$$\omega(f,\delta)=\sup_{|w-v|\leq\delta}|f(w)-f(v)|,\quad v,w\in[0,\infty). \tag{3.1}$$

Since $\lim_{\delta\to0+}\omega(f,\delta)=0$, for any $f\in C[0,\infty)$ and $\delta>0$, we have

$$|f(w)-f(v)|\leq\left(\frac{|w-v|}{\delta}+1\right)\omega(f,\delta). \tag{3.2}$$

The following results give the rate of convergence of the operators (2.2) in terms of modulus of continuity and the usual Lipschitz class $\mathrm{Lip}_S(\vartheta)$, respectively, as proved in Mursaleen and Ahasan (2018).

Theorem 3.1 *Let the operators* $K_{n,q}^{\alpha,\beta}(f;y)$ *is defined by* (2.2) *. Then for a given* $q\in(0,1)$, $y\geq0$, *and* $f\in C^*[0,\infty)$, *we have*

$$|K_{n,q}^{\alpha,\beta}(f;y)-f(y)|\leq\left\{1+\sqrt{\phi^*(y)}\right\}\omega\left(f;\frac{1}{\sqrt{([n]_q+\beta)}}\right), \tag{3.3}$$

where

$$\phi^*(y)=\left(\frac{\left(3[n]_q+3[n]_q[1+2\mu]_q\right)}{[3]_q}+2([n]_q+\beta)\left(\frac{2\alpha q[n]_q}{[2]_q([n]_q+\beta)}-\alpha-\frac{1}{[2]_q}\right)\right)y+\left(\alpha^2+\frac{1}{[3]_q}+\frac{2\alpha}{[2]_q}\right)+\left(([n]_q+\beta)^2+\frac{3[n]_q^2}{[3]_q}-\frac{4q[n]_q([n]_q+\beta)}{[2]_q}\right)y^2,$$

$\omega(f,\delta)$ *is defined in* (3.1) *and* (3.2), $C^*[0,\infty)$ *is the space of all uniformly continuous functions defined on* $\mathbb{R}^+$.

Let $f\in C[0,\infty)$, $S>0$, $0<\vartheta\leq1$ and $\mathrm{Lip}_S(\vartheta)$ denote the usual Lipschitz class defined by

$$\mathrm{Lip}_S(\vartheta)=\left\{f:|f(u_1)-f(u_2)|\leq S|u_1-u_2|^{\vartheta},\quad u_1,u_2\in\mathbb{R}^+\right\}. \tag{3.4}$$

Theorem 3.2 *Mursaleen and Ahasan (2018) For each* $f\in\mathrm{Lip}_S(\vartheta)$, $S>0$ *and* $0<\vartheta\leq1$ *satisfying* (3.4), *we have*

$$|K_{n,q}^{\alpha,\beta}(f;y)-f(y)|\leq S(\lambda_n(y))^{\frac{\vartheta}{2}},$$

where $\lambda_n(y)=K_{n,q}^{\alpha,\beta}((t-y)^2;y)$ *is defined in Lemma* 2.3.

The space of all bounded and continuous functions on $\mathbb{R}^+=[0,\infty)$ is denoted by $C_B[0,\infty)$ and is defined as

$$C_B^2(\mathbb{R}^+) = \{h : h \in C_B(\mathbb{R}^+)\, and\, h', h'' \in C_B(\mathbb{R}^+)\}, \tag{3.5}$$

which is equipped with the norm

$$\| h \|_{C_B^2(\mathbb{R}^+)} = \| h \|_{C_B(\mathbb{R}^+)} + \| h' \|_{C_B(\mathbb{R}^+)} + \| h'' \|_{C_B(\mathbb{R}^+)}, \tag{3.6}$$

and

$$\| h \|_{C_B(\mathbb{R}^+)} = \sup_{y \in \mathbb{R}^+} | h(y) | . \tag{3.7}$$

Theorem 3.3 *For any $h \in C_B^2(\mathbb{R}^+)$, we have*

$$\begin{aligned} | K_{n,q}^{\alpha,\beta}(h;y) - h(y) | &\le \left(\frac{\alpha}{([n]_q + \beta)} + \frac{1}{[2]_q([n]_q + \beta)} + \left(\frac{2q[n]_q}{[2]_q([n]_q + \beta)} - 1 \right) y \right) \\ &\times \| h \|_{C_B^2(\mathbb{R}^+)} + \frac{\lambda_n(y)}{2} \| h \|_{C_B^2(\mathbb{R}^+)}, \end{aligned}$$

where $\lambda_n(y)$ is given in Theorem 3.2.

Proof For proving this theorem, we need generalized mean value theorem in the Taylor series expansion and $h \in C_B^2(\mathbb{R}^+)$, we have

$$h(t) = h(y) + h'(y)(t-y) + h''(\psi)\frac{(t-y)^2}{2}, \quad \psi \in (y, t).$$

Using linearity, we have

$$K_{n,q}^{\alpha,\beta}(h;y) - h(y) = h'(y) K_{n,q}^{\alpha,\beta}((t-y);y) + \frac{h''(\psi)}{2} K_{n,q}^{\alpha,\beta}\left((t-y)^2;y\right),$$

which implies that

$$\begin{aligned} | K_{n,q}^{\alpha,\beta}(h;y) - h(y) | &\le \left(\frac{\alpha}{([n]_q + \beta)} + \frac{1}{[2]_q([n]_q + \beta)} + \left(\frac{2q[n]_q}{[2]_q([n]_q + \beta)} - 1 \right) y \right) \| h' \|_{C_B(\mathbb{R}^+)} \\ &+ \left\{ \frac{1}{([n]_q + \beta)^2} \left(\alpha^2 + \frac{1}{[3]_q} + \frac{2\alpha}{[2]_q} \right) \right\} \frac{\| h'' \|_{C_B(\mathbb{R}^+)}}{2} \\ &+ \left\{ \left(\frac{\left(3[n]_q + 3[n]_q[1+2\mu]_q\right)}{[3]_q([n]_q + \beta)^2} + \frac{2}{([n]_q + \beta)} \left(\frac{2\alpha q[n]_q}{[2]_q([n]_q + \beta)} - \alpha - \frac{1}{[2]_q} \right) \right) y \right\} \\ &\times \frac{\| h'' \|_{C_B(\mathbb{R}^+)}}{2} + \left\{ \left(1 + \frac{3[n]_q^2}{[3]_q([n]_q + \beta)^2} - \frac{4q[n]_q}{[2]_q([n]_q + \beta)} \right) y^2 \right\} \frac{\| h'' \|_{C_B(\mathbb{R}^+)}}{2}. \end{aligned}$$

From (3.6) and $\| h' \|_{C_B[0,\infty)} \le \| h \|_{C_B^2[0,\infty)}$, we have

$$\begin{aligned} | K_{n,q}^{\alpha,\beta}(h;y) - h(y) | &\le \left(\frac{\alpha}{([n]_q + \beta)} + \frac{1}{[2]_q([n]_q + \beta)} + \left(\frac{2q[n]_q}{[2]_q([n]_q + \beta)} - 1 \right) y \right) \| h \|_{C_B^2(\mathbb{R}^+)} \\ &+ \left\{ \frac{1}{([n]_q + \beta)^2} \left(\alpha^2 + \frac{1}{[3]_q} + \frac{2\alpha}{[2]_q} \right) \right\} \frac{\| h \|_{C_B^2(\mathbb{R}^+)}}{2} \\ &+ \left\{ \left(\frac{\left(3[n]_q + 3[n]_q[1+2\mu]_q\right)}{[3]_q([n]_q + \beta)^2} + \frac{2}{([n]_q + \beta)} \left(\frac{2\alpha q[n]_q}{[2]_q([n]_q + \beta)} - \alpha - \frac{1}{[2]_q} \right) \right) y \right\} \\ &\times \frac{\| h \|_{C_B^2(\mathbb{R}^+)}}{2} + \left\{ \left(1 + \frac{3[n]_q^2}{[3]_q([n]_q + \beta)^2} - \frac{4q[n]_q}{[2]_q([n]_q + \beta)} \right) y^2 \right\} \frac{\| h \|_{C_B^2(\mathbb{R}^+)}}{2}, \end{aligned}$$

which completes the proof from part (3) of Lemma 2.3. □

Let $K_2(f, \delta)$ denote the Peetre's K-functional which is defined as

$$K_2(f, \delta) = \inf_{C_B^2(\mathbb{R}^+)} \left\{ \left(\| f - h \|_{C_B(\mathbb{R}^+)} + \delta \| h'' \|_{C_B^2(\mathbb{R}^+)} \right) : h \in \mathcal{T}^2 \right\}, \tag{3.8}$$

where

$$\mathcal{T}^2 = \{h : h \in C_B(\mathbb{R}^+) \quad \text{and} \quad h', h'' \in C_B(\mathbb{R}^+)\}. \tag{3.9}$$

Then there exits a constant $A > 0$ such that $K_2(f, \delta) \le A\omega_2(f, \sqrt{\delta}), \delta > 0$, where $\omega_2(f, \sqrt{\delta})$ is the second-order modulus of continuity which is defined as

$$\omega_2(f, \sqrt{\delta}) = \sup_{0 < l < \sqrt{\delta}} \sup_{y \in \mathbb{R}^+} | f(y + 2l) - 2f(y + l) + f(y) | . \tag{3.10}$$

Theorem 3.4 *For $f \in C_B(\mathbb{R}^+)$ and $y \in \mathbb{R}^+$, we have*

$$\begin{aligned} | K_{n,q}^{\alpha,\beta}(f;y) - f(y) | &\le 2S \left\{ \omega_2\left(f; \sqrt{\frac{\frac{2\alpha}{([n]_q+\beta)} + \frac{2}{[2]_q([n]_q+\beta)} + \left(\frac{4q[n]_q}{[2]_q([n]_q+\beta)} - 2 \right) y + \lambda_n(y)}{4}} \right) \right. \\ &\left. + \min\left(1, \frac{\frac{2\alpha}{([n]_q+\beta)} + \frac{2}{[2]_q([n]_q+\beta)} + \left(\frac{4q[n]_q}{[2]_q([n]_q+\beta)} - 2 \right) y + \lambda_n(y)}{4} \right) \| f \|_{C_B(\mathbb{R}^+)} \right\}, \end{aligned}$$

where $\omega_2(f;\delta), \lambda_n(y)$ *are defined in* (3.10) *and in Theorem* 3.2, *respectively, and S is a positive constant.*

Proof For proving this, we need Theorem 3.3. We have

$$\begin{aligned} &\mid K_{n,q}^{\alpha,\beta}(f;y)-f(y)\mid \\ &\quad\le \mid K_{n,q}^{\alpha,\beta}(f-h;y)\mid+\mid K_{n,q}^{\alpha,\beta}(h;y)-h(y)\mid \\ &\qquad+\mid f(y)-h(y)\mid \\ &\quad\le 2\parallel f-h\parallel_{C_B(\mathbb{R}^+)}+\frac{\lambda_n(y)}{2}\parallel h\parallel_{C_B^2(\mathbb{R}^+)} \\ &\qquad+\left(\frac{\alpha}{([n]_q+\beta)}+\frac{1}{[2]_q([n]_q+\beta)}\right. \\ &\qquad\left.+\frac{(2q[n]_q-[2]_q([n]_q+\beta))}{[2]_q([n]_q+\beta)}y\right)\times\parallel h\parallel_{C_B(\mathbb{R}^+)}. \end{aligned}$$

Clearly from (3.6), we get

$$\parallel h\parallel_{C_B[0,\infty)}\le\parallel h\parallel_{C_B^2[0,\infty)}.$$

Hence

$$\begin{aligned} &\mid K_{n,q}^{\alpha,\beta}(f;y)-f(y)\mid\le 2\Bigg(\parallel f-h\parallel_{C_B(\mathbb{R}^+)} \\ &\quad+\frac{\frac{2\alpha}{([n]_q+\beta)}+\frac{2}{[2]_q([n]_q+\beta)}+\left(\frac{4q[n]_q}{[2]_q([n]_q+\beta)}-2\right)y+\lambda_n(y)}{4}\parallel h\parallel_{C_B^2(\mathbb{R}^+)}\Bigg). \end{aligned}$$

Now take the infimum overall $h\in C_B^2(\mathbb{R}^+)$ and by (3.8), we have

$$\begin{aligned} &\mid K_{n,q}^{\alpha,\beta}(f;y)-f(y)\mid \\ &\quad\le 2K_2\left(f;\frac{\frac{2\alpha}{([n]_q+\beta)}+\frac{2}{[2]_q([n]_q+\beta)}+\left(\frac{4q[n]_q}{[2]_q([n]_q+\beta)}-2\right)y+\lambda_n(y)}{4}\right) \end{aligned}$$

By using the relation Ciupa (1995) and an absolute constant $A>0$, we have

$$K_2(f;\delta)\le A\{\omega_2(f;\delta^{\frac{1}{2}})+\min(1,\delta)\parallel f\parallel\}.$$

This completes the proof of the theorem. □

4 Construction of Bivariate *q*-Operators

The aim of this section is to construct a bivariate extension of Dunkl analogue of Stancu type *q*-Szász–Mirakjan–Kantorovich operators defined in (2.2).

Let $\mathbb{R}_+^2=[0,\infty)\times[0,\infty), f: C(\mathbb{R}_+^2)\to\mathbb{R}, 0<q_{n_1},q_{n_2}<1$ and $\max\{\mid\mu_1\mid,\mid\mu_2\mid\}\le\frac{1}{2}$. We define

$$\begin{aligned} &K_{n_1,n_2}^{\alpha,\beta}(f;q_{n_1},q_{n_2};y,z)=\frac{1}{E_{\mu_1,q_{n_1}}([n_1]_{q_{n_1}}y)} \\ &\quad\frac{1}{E_{\mu_2,q_{n_2}}([n_2]_{q_{n_2}}z)}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{([n_1]_{q_{n_1}}y)^{k_1}}{\gamma_{\mu_1,q_{n_1}}(k_1)} \\ &\quad\times\frac{([n_2]_{q_{n_2}}z)^{k_2}}{\gamma_{\mu_2,q_{n_2}}(k_2)}q_{n_1}^{\frac{k_1(k_1-1)}{2}}q_{n_2}^{\frac{k_2(k_2-1)}{2}}\int\int_R f(y,z)d_qz d_qy \end{aligned} \tag{4.1}$$

where

$$\begin{aligned} R&=\left[\frac{[k_1+2\mu_1\theta_{k_1}]_{q_{n_1}}}{q_{n_1}^{k_1-2}[n_1]_{q_{n_1}}},\frac{[k_1+1+2\mu_1\theta_{k_1}]_{q_{n_1}-1}}{q_{n_1}^{k_1-1}[n_1]_{q_{n_1}}}+\frac{1}{[n_1]_{q_{n_1}}}\right] \\ &\quad\times\left[\frac{[k_2+2\mu_2\theta_{k_2}]_{q_{n_2}}}{q_{n_2}^{k_2-2}[n_2]_{q_{n_2}}},\frac{[k_2+1+2\mu_2\theta_{k_2}]_{q_{n_2}-1}}{q_{n_2}^{k_2-1}[n_2]_{q_{n_2}}}+\frac{1}{[n_2]_{q_{n_2}}}\right] \\ &=\Bigg\{(y,z)\in\mathbb{R}^2\quad\text{such that}\quad\frac{[k_1+2\mu_1\theta_{k_1}]_{q_{n_1}}}{q_{n_1}^{k_1-2}[n_1]_{q_{n_1}}}\le y \\ &\quad\le\frac{[k_1+1+2\mu_1\theta_{k_1}]_{q_{n_1}-1}}{q_{n_1}^{k_1-1}[n_1]_{q_{n_1}}}+\frac{1}{[n_1]_{q_{n_1}}}\quad\text{and} \\ &\quad\frac{[k_2+2\mu_2\theta_{k_2}]_{q_{n_2}}}{q_{n_2}^{k_2-2}[n_2]_{q_{n_2}}}\le z\le\frac{[k_2+1+2\mu_2\theta_{k_2}]_{q_{n_2}-1}}{q_{n_2}^{k_1-1}[n_2]_{q_{n_2}}} \\ &\quad+\frac{1}{[n_2]_{q_{n_2}}}\Bigg\}\quad\text{and} \end{aligned}$$

$$E_{\mu_1,q_{n_1}}([n_1]_{q_{n_1}}y)=\sum_{k_1=0}^{\infty}\frac{([n_1]_{q_{n_1}}y)^{k_1}}{\gamma_{\mu_1,q_{n_1}}(k_1)}q_{n_1}^{\frac{k_1(k_1-1)}{2}},$$

$$E_{\mu_2,q_{n_2}}([n_2]_{q_{n_2}}z)=\sum_{k_2=0}^{\infty}\frac{([n_2]_{q_{n_2}}z)^{k_2}}{\gamma_{\mu_2,q_{n_2}}(k_2)}q_{n_2}^{\frac{k_2(k_2-1)}{2}}.$$

Lemma 4.1 *Let the two-dimensional test functions* $e_{i,j}:\mathbb{R}_+^2\to[0,\infty)$ *be defined by* $e_{i,j}=u^iv^j$ $(i,j=0,1,2)$. *Then for the q-bivariate operators defined in* (4.1), *we have*

(1) $K_{n_1,n_2}^{\alpha,\beta}(e_{0,0};q_{n_1},q_{n_2};y,z)=1,$

(2)
$$\begin{aligned} &K_{n_1,n_2}^{\alpha,\beta}(e_{1,0};q_{n_1},q_{n_2};y,z) \\ &\quad=\frac{\alpha}{([n_1]_{q_{n_1}}+\beta)}+\frac{1}{[2]_{q_{n_1}}([n_1]_{q_{n_1}}+\beta)} \\ &\qquad+\frac{2q_{n_1}[n_1]_{q_{n_1}}}{[2]_{q_{n_1}}([n_1]_{q_{n_1}}+\beta)}y, \end{aligned}$$

(3) $$K_{n_1,n_2}^{\alpha,\beta}(e_{0,1};q_{n_1},q_{n_2};y,z) = \frac{\alpha}{([n_2]_{q_{n_2}}+\beta)} + \frac{1}{[2]_{q_{n_2}}([n_2]_{q_{n_2}}+\beta)} + \frac{2q_{n_2}[n_2]_{q_{n_2}}}{[2]_{q_{n_2}}([n_2]_{q_{n_2}}+\beta)}z,$$

(4) $$K_{n_1,n_2}^{\alpha,\beta}(e_{2,0};q_{n_1},q_{n_2};y,z) \leq \frac{\alpha^2}{([n_1]_{q_{n_1}}+\beta)^2} + \frac{2\alpha}{[2]_{q_{n_1}}([n_1]_{q_{n_1}}+\beta)^2}\left(1+2q_{n_1}[n_1]_{q_{n_1}}y\right) + \frac{1}{[3]_{q_{n_1}}([n_1]_{q_{n_1}}+\beta)^2}\left(1+3[n_1]_{q_{n_1}}y + 3[n_1]_{q_{n_1}}[1+2\mu]_{q_{n_1}}y + 3[n_1]_{q_{n_1}}^2 y^2\right),$$

(5) $$K_{n_1,n_2}^{\alpha,\beta}(e_{0,2};q_{n_1},q_{n_2};y,z) \leq \frac{\alpha^2}{([n_2]_{q_{n_2}}+\beta)^2} + \frac{2\alpha}{[2]_{q_{n_2}}([n_2]_{q_{n_2}}+\beta)^2}\left(1+2q_{n_2}[n_2]_{q_{n_2}}z\right) + \frac{1}{[3]_{q_{n_2}}([n_2]_{q_{n_2}}+\beta)^2}\left(1+3[n_2]_{q_{n_2}}z + 3[n_2]_{q_{n_2}}[1+2\mu]_{q_{n_2}}z + 3[n_2]_{q_{n_2}}^2 z^2\right).$$

Lemma 4.2 *For the q-bivariate operators given in (4.1), we have*

(1) $$K_{n_1,n_2}^{\alpha,\beta}(e_{1,0}-y;q_{n_1},q_{n_2};y,z) = \frac{\alpha}{([n_1]_{q_{n_1}}+\beta)} + \frac{1}{[2]_{q_{n_1}}([n_1]_{q_{n_1}}+\beta)} + \left(\frac{2q_{n_1}[n_1]_{q_{n_1}}}{[2]_{q_{n_1}}([n_1]_{q_{n_1}}+\beta)}-1\right)y,$$

(2) $$K_{n_1,n_2}^{\alpha,\beta}(e_{0,1}-z;q_{n_1},q_{n_2};y,z) = \frac{\alpha}{([n_2]_{q_{n_2}}+\beta)} + \frac{1}{[2]_{q_{n_2}}([n_2]_{q_{n_2}}+\beta)} + \left(\frac{2q_{n_2}[n_2]_{q_{n_2}}}{[2]_{q_{n_2}}([n_2]_{q_{n_2}}+\beta)}-1\right)z,$$

(3) $$K_{n_1,n_2}^{\alpha,\beta}\left((e_{1,0}-y)^2;q_{n_1},q_{n_2};y,z\right) \leq \left(\frac{\left(3[n_1]_{q_{n_1}}+3[n_1]_{q_{n_1}}[1+2\mu]_{q_{n_1}}\right)}{[3]_{q_{n_1}}([n_1]_{q_{n_1}}+\beta)^2} + \frac{2}{([n_1]_{q_{n_1}}+\beta)}\left(\frac{2\alpha q_{n_1}[n_1]_{q_{n_1}}}{[2]_{q_{n_1}}([n_1]_{q_{n_1}}+\beta)}-\alpha-\frac{1}{[2]_{q_{n_1}}}\right)\right)y + \frac{1}{([n_1]_{q_{n_1}}+\beta)^2}\left(\alpha^2+\frac{1}{[3]_{q_{n_1}}}+\frac{2\alpha}{[2]_{q_{n_1}}}\right) + \left(1+\frac{3[n_1]_{q_{n_1}^2}}{[3]_{q_{n_1}}([n_1]_{q_{n_1}}+\beta)^2}-\frac{4q_{n_1}[n_1]_{q_{n_1}}}{[2]_{q_{n_1}}([n_1]_{q_{n_1}}+\beta)}\right)y^2,$$

(4) $$K_{n_1,n_2}^{\alpha,\beta}\left((e_{0,1}-z)^2;q_{n_1},q_{n_2};y,z\right) \leq \left(\frac{\left(3[n_2]_{q_{n_2}}+3[n_2]_{q_{n_2}}[1+2\mu]_{q_{n_2}}\right)}{[3]_{q_{n_2}}([n_2]_{q_{n_2}}+\beta)^2} + \frac{2}{([n_2]_{q_{n_2}}+\beta)}\times\left(\frac{2\alpha q_{n_2}[n_2]_{q_{n_2}}}{[2]_{q_{n_2}}([n_2]_{q_{n_2}}+\beta)}-\alpha-\frac{1}{[2]_{q_{n_2}}}\right)\right)z + \frac{1}{([n_2]_{q_{n_2}}+\beta)^2}\left(\alpha^2+\frac{1}{[3]_{q_{n_2}}}+\frac{2\alpha}{[2]_{q_{n_2}}}\right) + \left(1+\frac{3[n_2]_{q_{n_2}^2}}{[3]_{q_{n_2}}([n_2]_{q_{n_2}}+\beta)^2}-\frac{4q_{n_2}[n_2]_{q_{n_2}}}{[2]_{q_{n_2}}([n_2]_{q_{n_2}}+\beta)}\right)z^2.$$

To obtain the convergence results for the operators $K_{n_1,n_2}^{\alpha,\beta}(f;q_{n_1},q_{n_2};y,z)$, we take $q=q_{n_1},q_{n_2}$ where $q_{n_1},q_{n_2}\in(0,1)$ and satisfying

$$\lim_{n_i\to\infty} q_{n_i}=1, \quad i=1,2. \tag{4.2}$$

For $f\in H_\omega(\mathbb{R}_+^2)$. In case of bivariate extension, the modulus of continuity is defined by

$$\omega^*(f;\delta_1,\delta_2)=\sup_{u_1,y\geq 0}\{|f(u_1,u_2)-f(y,z)|, (u_1,u_2) \text{ and } (y,z)\in\mathbb{R}_+^2\}, \tag{4.3}$$

where $|u_1-y|\leq\delta_1, |u_2-z|\leq\delta_2$ and $H_\omega(\mathbb{R}_+)$ denotes the space of all real-valued continuous functions. Then, for all $f\in H_\omega(\mathbb{R}_+)$, $\omega^*(f;\delta_1,\delta_2)$

(1) $\lim_{\delta_1,\delta_2\to 0}\omega^*(f;\delta_1,\delta_2)=0,$

(2) $|f(u_1,u_2)-f(y,z)|\leq\omega^*(f;\delta_1,\delta_2)\left(\frac{|u_1-y|}{\delta_1}+1\right)\left(\frac{|u_2-z|}{\delta_2}+1\right).$

Theorem 4.3 *Let $q=q_{n_1},q_{n_2}$ satisfy (4.2). Then for $(y,z)\in\mathbb{R}_+^2$ and for any function $f\in C^*([0,\infty)\times[0,\infty))$, $0<q_{n_1},q_{n_2}<1$ we have*

$$| K^{\alpha,\beta}_{n_1,n_2}(f;q_{n_1},q_{n_2},y,z) - f(y,z) | \leq \omega\left(f;\frac{1}{\sqrt{([n_1]_{q_{n_1}}+\beta)}},\frac{1}{\sqrt{([n_2]_{q_{n_2}}+\beta)}}\right) \times \left(1+\sqrt{L_1}\right)\left(1+\sqrt{L_2}\right),$$

where

$$L_1 = \left(\frac{\left(3[n_1]_{q_{n_1}}+3[n_1]_{q_{n_1}}[1+2\mu]_{q_{n_1}}\right)}{[3]_{q_{n_1}}} + 2([n_1]_{q_{n_1}} + \beta)\left(\frac{2\alpha q_{n_1}[n_1]_{q_{n_1}}}{[2]_{q_{n_1}}([n_1]_{q_{n_1}}+\beta)} - \alpha - \frac{1}{[2]_{q_{n_1}}}\right)\right)y + \left(\alpha^2 + \frac{1}{[3]_{q_{n_1}}} + \frac{2\alpha}{[2]_{q_{n_1}}}\right) + \left(([n_1]_{q_{n_1}}+\beta)^2 + \frac{3[n_1]^2_{q_{n_1}}}{[3]_{q_{n_1}}} - \frac{4([n_1]_{q_{n_1}}+\beta)q_{n_1}[n]_{q_{n_1}}}{[2]_{q_{n_1}}}\right)y^2,$$

$$L_2 = \left(\frac{\left(3[n_2]_{q_{n_2}}+3[n_2]_{q_{n_2}}[1+2\mu]_{q_{n_2}}\right)}{[3]_{q_{n_2}}} + 2([n_2]_{q_{n_2}} + \beta)\left(\frac{2\alpha q_{n_2}[n_2]_{q_{n_2}}}{[2]_{q_{n_2}}([n_2]_{q_{n_2}}+\beta)} - \alpha - \frac{1}{[2]_{q_{n_2}}}\right)\right)z + \left(\alpha^2 + \frac{1}{[3]_{q_{n_2}}} + \frac{2\alpha}{[2]_{q_{n_2}}}\right) + \left(([n_2]_{q_{n_2}}+\beta)^2 + \frac{3[n_2]^2_{q_{n_2}}}{[3]_{q_{n_2}}} - \frac{4([n_2]_{q_{n_2}}+\beta)q_{n_2}[n]_{q_{n_2}}}{[2]_{q_{n_2}}}\right)z^2,$$

$C^*[0,\infty)$ be the space of uniformly continuous functions on $\mathbb{R}^+$ and $\omega^*(f;\delta_{n_1},\delta_{n_2})$ be the modulus of continuity of $f \in C^*([0,\infty)\times[0,\infty))$ which is given by (4.3).

Proof We obtain it easily by using the Cauchy–Schwarz inequality and choosing

$$\delta_1 = \delta_{n_1} = \frac{1}{\sqrt{([n_1]_{q_{n_1}}+\beta)}} \quad \text{and} \quad \delta_2 = \delta_{n_2} = \frac{1}{\sqrt{([n_2]_{q_{n_2}}+\beta)}}.$$

So we omit the details. □

In terms of elements of the usual Lipschitz class $\mathrm{Lip}_S(v_1,v_2)$, we obtain the rate of convergence for the bivariate q-operators $K^{\alpha,\beta}_{n_1,n_2}(f;q_{n_1},q_{n_2};y,z)$ given in (4.1).

For $S>0$, $f\in C([0,\infty)\times[0,\infty))$ and $v_1,v_2\in(0,1]$, $\mathrm{Lip}_S(v_1,v_2)$ is given by

$$\mathrm{Lip}_S(v_1,v_2) = \{f : | f(u_1,u_2) - f(y,z) | \leq S \mid u_1 - y \mid^{v_1} \mid u_2 - z \mid^{v_2}\}, \tag{4.4}$$

where (u_1,u_2) and $(y,z)\in[0,\infty)\times[0,\infty)$.

Theorem 4.4 *For each $f\in\mathrm{Lip}_S(v_1,v_2)$, $(v_1,v_2)\in(0,1]$ and $S>0$ satisfying (4.4), we have*

$$| K^{\alpha,\beta}_{n_1,n_2}(f;q_{n_1},q_{n_2};y,z) - f(y,z) | \leq S(\lambda_{n_1}(y))^{\frac{v_1}{2}}(\lambda_{n_2}(z))^{\frac{v_2}{2}},$$

where $\lambda_{n_1}(y) = K^{\alpha,\beta}_{n_1,n_2}\left((e_{1,0}-y)^2;q_{n_1},q_{n_2};y,z\right)$ *and* $\lambda_{n_2}(z) = K^{\alpha,\beta}_{n_1,n_2}\left((e_{0,1}-z)^2;q_{n_1},q_{n_2};y,z\right)$.

Proof By Hölder's inequality and $\mathrm{Lip}_S(v_1,v_2)$ defined in (4.4), we easily get the above result. Hence, we omit the details. □

5 Conclusion

In this paper, we have determined the rate of convergence for the operator (2.2) and (4.1) in terms of modulus of continuity and Lipschitz function. Further a bivariate extension of the Dunkl analogue of Stancu type q-Szász–Mirakjan–Kantorovich positive linear operator is introduced, and some approximation results for these operators are obtained.

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