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RESEARCH PAPER

Left Ideal Preserving Maps on Triangular Algebras

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Abstract

Let $\mathcal{A}, \mathcal{B}$ be unital algebras, $\mathcal{M}$ be an $(\mathcal{A}, \mathcal{B})$-bimodule and $\mathcal{T} = \begin{pmatrix} \mathcal{A} & \mathcal{M} \\ 0 & \mathcal{B} \end{pmatrix}$ be the corresponding unital triangular algebra over a commutative unital ring $\mathcal{R}$. In this paper, we study whether every $\mathcal{R}$-linear map on $\mathcal{T}$ that leaves invariant every left ideal of $\mathcal{T}$ is a left multiplier, and give some necessary or sufficient conditions for a triangular algebra to have this property. We also give various examples illustrating limitations on extending some of the theory developed. We then apply our established results to generalized triangular matrix algebras and block upper triangular matrix algebras. Moreover, we introduce some algebras other than triangular algebras on which every $\mathcal{R}$-linear map is a left multiplier.

Keywords Left multiplier · Local left multiplier · Left ideal preserving · Triangular algebra · Generalized triangular matrix algebras · Block upper triangular matrix algebras

Mathematics Subject Classification 15A86 · 16S50 · 16D99 · 16S99

1 Introduction

Throughout this article, $\mathcal{R}$ will denote a commutative ring with unity, and unless otherwise stated, all algebras are associative over $\mathcal{R}$ with unity 1 and all modules are unital. Let $\mathcal{A}$ be an algebra and $\mathcal{X}$ be a right $\mathcal{A}$-module. Recall that an $\mathcal{R}$-linear map $\psi : \mathcal{A} \to \mathcal{X}$ is a *left multiplier* if $\psi(a) = \psi(1)a$ for all $a \in \mathcal{A}$. It is called a *local left multiplier* if for any $a \in \mathcal{A}$ there exists an element $x_a \in \mathcal{X}$ such that $\psi(a) = x_a a$. Clearly, each left multiplier is a local left multiplier. The converse is, in general, not true. Following (Hadwin and Kerr 1997), we say that an $\mathcal{R}$-linear map $\psi : \mathcal{A} \to \mathcal{A}$ is **LIP** (left ideal preserving) if $\psi(\mathcal{J}) \subseteq \mathcal{J}$ for any left ideal $\mathcal{J}$ of $\mathcal{A}$. It is then easily verified that the $\mathcal{R}$-linear map $\psi : \mathcal{A} \to \mathcal{A}$ is **LIP** if and only if ψ is a local left multiplier. So it is clear that any left multiplier $\psi : \mathcal{A} \to \mathcal{A}$ is **LIP** map, but the converse is not necessarily true. (Some counterexamples will be given.) It is natural and interesting to ask for what algebras any **LIP** map is a left multiplier, so we are led to define **SLIP** algebras. The algebra $\mathcal{A}$ is ***SLIP** over $\mathcal{R}$* (In short, **SLIP**), if any **LIP** map on $\mathcal{A}$ is a left multiplier. [The notion **SLIP** has already been used in Hadwin and Kerr (1997).]

In the case that $\mathcal{R}$ is a field, to say that $\mathcal{A}$ is **SLIP** is the same as saying that the algebra of left multipliers on $\mathcal{A}$ is algebraically reflexive (Hadwin 1983). Reflexivity (algebraically or topologically) is an important part of operator theory and has been studied in both ring theory and Banach algebra theory by several authors. Johnson (1968) has shown that if $\mathcal{A}$ is a semisimple Banach algebra with an approximate identity and $\psi : \mathcal{A} \to \mathcal{A}$ is a bounded operator that leaves invariant all closed left ideals of $\mathcal{A}$, then ψ is a left multiplier of $\mathcal{A}$. Hadwin and Li (2004) have shown that Johnson's theorem holds for all CSL algebras. In particular, Hadwin, Li and their collaborators (The Hadwin Lunch Bunch 1994; Hadwin and Kerr 1997; Hadwin and Li 2004, 2008; Li and Pan 2010) have investigated problems of this type in the past twenty years for various reflexive operator algebras. Recently, Katsoulis (2016) has studied the reflexivity of left multipliers over certain operator algebras. In the purely algebraic, Brešar and Šemrl (1993) and Brešar (2007) have investigated local multipliers in various other settings. Also, Hadwin and Kerr (1997) have studied various **SLIP** algebras. The notion of **SLIP** algebras (and reflexivity) are also studied in ring theory by several authors in a number of papers; see Fuller et al.

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(1989, 1991, 1995), Hadwin and Kerr (1988, 1989), Snashall (1992, 1993, 1994). On the other hand, recently there has been a growing interest in the study of preserving linear maps on triangular algebras, for example linear maps that preserve zero products, Jordan products, commutativity, etc. and derivable, Jordan derivable, Lie derivable maps at zero point, etc; see, for instance, An and Hou (2009), Benkovič and Eremita (2004), Liu and Zhang (2016), Zhang et al. (2006), Zhao and Zhu (2010) and the references therein. Motivated by the above investigations, we will study whether a triangular algebra is **SLIP**, and give some sufficient conditions under which a triangular algebra is **SLIP**. Our results are then applied to generalized triangular matrix algebras and block upper triangular matrix algebras. Some other **SLIP** algebras are also studied.

The present article is organized as follows. In section 2, some preliminaries including an introduction to triangular algebras, generalized triangular matrix algebras and block upper triangular matrix algebras are given. In section 3, we firstly study the relation between zero product determined algebras and **SLIP** algebras. By applying our results, we establish characterizations of **SLIP** property for several classes of algebras. Then by considering **LIP** maps on triangular algebras, we obtain a necessary condition and some sufficient conditions for a triangular algebra to be **SLIP**. We also give some examples illustrating limitations on extending some of the theory developed. In section 4, we apply the results obtained in the previous section to generalized triangular matrix algebras and block upper triangular matrix algebras. Indeed, we prove that under certain conditions the generalized triangular matrix algebras are **SLIP** algebras, and we also apply the results to block upper triangular matrix algebras.

2 Preliminaries

Recall that a *triangular algebra* $Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$ is an algebra of the form

$$Tri(\mathcal{A}, \mathcal{M}, \mathcal{B}) := \left\{ \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \middle| a \in \mathcal{A},\, b \in \mathcal{B},\, m \in \mathcal{M} \right\} = \begin{pmatrix} \mathcal{A} & \mathcal{M} \\ 0 & \mathcal{B} \end{pmatrix}$$

under the usual matrix operations, where $\mathcal{A}$ and $\mathcal{B}$ are unital algebras and $\mathcal{M}$ is an $(\mathcal{A}, \mathcal{B})$-bimodule. The most important examples of triangular algebras are upper triangular matrices over an algebra $\mathcal{A}$, block upper triangular matrix algebras, nest algebras over a real or complex Hilbert space $\mathcal{H}$ and generalized triangular matrix algebras.

Let $\mathcal{A}$ be an algebra. Recall that an idempotent $e \in \mathcal{A}$ is *left semicentral* if $\mathcal{A}e = e\mathcal{A}e$ (Birkenmeier 1983). We use $\mathcal{S}_l(\mathcal{A})$ exclusively for the sets of all left semicentral idempotents. As is well known (Chase 1961), a left semicentral idempotent e induces a 2-by-2 triangular matrix representation of $\mathcal{A}$. In fact, $\mathcal{A} \cong Tri(e\mathcal{A}e, e\mathcal{A}(1-e), (1-e)\mathcal{A}(1-e))$, where $e\mathcal{A}e$ and $(1-e)\mathcal{A}(1-e)$ are algebras over $\mathcal{R}$ with the addition and multiplication of $\mathcal{A}$, but different unities (e and $1-e$, respectively) and $e\mathcal{A}(1-e)$ is a unital $(e\mathcal{A}e, (1-e)\mathcal{A}(1-e))$-bimodule. If $\mathcal{S}_l(\mathcal{A}) = \{0, 1\}$, then we say $\mathcal{A}$ is *semicentral reduced*. For more information, we refer to Birkenmeier et al. (2000).

We say $\mathcal{A}$ has a *generalized triangular matrix representation* if there exists an $\mathcal{R}$-algebra isomorphism:

$$\theta : \mathcal{A} \to \begin{pmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} & \cdots & \mathcal{A}_{1n} \\ 0 & \mathcal{A}_{22} & \cdots & \mathcal{A}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathcal{A}_{nn} \end{pmatrix},$$

where each $\mathcal{A}_{ii}$ is an algebra with unity and $\mathcal{A}_{ij}$ is a $(\mathcal{A}_{ii}, \mathcal{A}_{jj})$-bimodule for $i<j$. An ordered set $\{e_1, \ldots, e_n\}$ of nonzero distinct idempotents in $\mathcal{A}$ is called a set of *left triangulating idempotents* of $\mathcal{A}$ if all of the following statements hold:

(i) $e_1 + \cdots + e_n = 1$;
(ii) $e_1 \in \mathcal{S}_l(\mathcal{A})$; and
(iii) $e_{k+1} \in \mathcal{S}_l(f_k \mathcal{A} f_k)$, where $f_k = 1 - (e_1 + \cdots + e_k)$, for $1 \le k \le n-1$ (see Birkenmeier et al. 2000).

Proposition 2.1 (Birkenmeier et al. 2000, Proposition 1.3) *$\mathcal{A}$ has a set of left triangulating idempotents if and only if $\mathcal{A}$ has a generalized triangular matrix representation.*

In fact, by the above proposition if $\mathcal{A}$ has a set of left triangulating idempotents $\{e_1, \ldots, e_n\}$, then we have the following $\mathcal{R}$-algebra isomorphism:

$$\mathcal{A} \cong \begin{pmatrix} e_1\mathcal{A}e_1 & e_1\mathcal{A}e_2 & \cdots & e_1\mathcal{A}e_n \\ 0 & e_2\mathcal{A}e_2 & \cdots & e_2\mathcal{A}e_n \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e_n\mathcal{A}e_n \end{pmatrix}.$$

Conversely, if $\mathcal{A}$ has a generalized triangular matrix representation

$$\theta : \mathcal{A} \to \begin{pmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} & \cdots & \mathcal{A}_{1n} \\ 0 & \mathcal{A}_{22} & \cdots & \mathcal{A}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathcal{A}_{nn} \end{pmatrix},$$

then $\{\theta^{-1}(E_1), \ldots, \theta^{-1}(E_n)\}$ is a set of left triangulating idempotents of $\mathcal{A}$, where E_k is the n-by-n matrix with the unity of $\mathcal{A}_k$ in the (k, k)-position and 0 elsewhere.

Remark 2.2 By the definition of a set of left triangulating idempotents and Proposition 2.1, we see that if $\mathcal{A}$ has a set of left triangulating idempotents $\{e_1, \ldots, e_n\}$, then $\{e_2, \ldots, e_n\}$ is a set of left triangulating idempotents of $f\mathcal{A}f$, where $f = 1 - e_1$. Since $e_1 \in \mathcal{S}_l(\mathcal{A})$, it follows that $\mathcal{A}$ has a triangular matrix representation as $\mathcal{A} \cong Tri(e_1\mathcal{A}e_1, e_1\mathcal{A}f, f\mathcal{A}f)$. In this case, we have the $\mathcal{R}$-algebra isomorphism:

$$f\mathcal{A}f \cong \begin{pmatrix} e_2\mathcal{A}e_2 & \cdots & e_2\mathcal{A}e_n \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e_n\mathcal{A}e_n \end{pmatrix},$$

and the $(e_1\mathcal{A}e_1, f\mathcal{A}f)$-bimodule isomorphism:

$$e_1\mathcal{A}f \cong (e_1\mathcal{A}e_2, \ldots, e_1\mathcal{A}e_n).$$

Also, $\{e_1, \ldots, e_{n-1}\}$ is a set of left triangulating idempotents of $f\mathcal{A}f$, where $f = 1 - e_n$. By (Birkenmeier et al. (2000), Lemma 1.2), $e_j\mathcal{A}e_i = \{0\}$ for all $i < j \leq n$. So $f \in \mathcal{S}_l(\mathcal{A})$, and hence, $\mathcal{A}$ has a triangular matrix representation as $\mathcal{A} \cong Tri(f\mathcal{A}f, f\mathcal{A}e_n, e_n\mathcal{A}e_n)$. In this case, we have the $\mathcal{R}$-algebra isomorphism:

$$f\mathcal{A}f \cong \begin{pmatrix} e_1\mathcal{A}e_1 & \cdots & e_1\mathcal{A}e_{n-1} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e_{n-1}\mathcal{A}e_{n-1} \end{pmatrix},$$

and the $(f\mathcal{A}f, e_n\mathcal{A}e_n)$-bimodule isomorphism:

$$f\mathcal{A}e_n \cong \begin{pmatrix} e_1\mathcal{A}e_n \\ \vdots \\ e_{n-1}\mathcal{A}e_n \end{pmatrix}.$$

Let $M_{k\times m}(\mathcal{A})$ denote the set of all k-by-m matrices over $\mathcal{A}$ (we denote $M_{k\times k}(\mathcal{A})$ by $M_k(\mathcal{A})$). Let $\mathbb{N}$ be the set of all positive integers and let $n \in \mathbb{N}$. For each $m \in \mathbb{N}$ with $m \leq n$, we denote by $\bar{k} = (k_1, \ldots, k_m) \in \mathbb{N}^m$ an ordered m-vector of positive integers with $n = k_1 + \cdots + k_m$. The *block upper triangular matrix algebra* $B_n^{\bar{k}}(\mathcal{A})$ is a subalgebra of $M_n(\mathcal{A})$ of the form

$$B_n^{\bar{k}}(\mathcal{A}) \cong \begin{pmatrix} M_{k_1}(\mathcal{A}) & M_{k_1\times k_2}(\mathcal{A}) & \cdots & M_{k_1\times k_m}(\mathcal{A}) \\ 0 & M_{k_2}(\mathcal{A}) & \cdots & M_{k_2\times k_m}(\mathcal{A}) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_{k_m}(\mathcal{A}) \end{pmatrix}.$$

Note that $M_n(\mathcal{A})$ is a special case of block upper triangular matrix algebras. In particular, $B_n^{\bar{k}}(\mathcal{A}) = M_n(\mathcal{A})$ if and only if $\bar{k} = (k_1)$ with $k_1 = n$.

The block upper triangular matrix algebra $B_n^{\bar{k}}(\mathcal{A})$ has a generalized triangular matrix representation with $\{F_1, \ldots, F_m\}$ as a set of left triangulating idempotents such that $F_1 = \sum_{i=1}^{k_1} E_i$ and $F_j = \sum_{i=1}^{k_j} E_{i+k_1+\cdots+k_{j-1}}$ for $2 \leq j \leq m$, where E_i is the n-by-n matrix with the unity of $\mathcal{A}$ in the (i, i)-position and 0 elsewhere. We have $F_jB_n^{\bar{k}}(\mathcal{A})F_j \cong M_{k_j}(\mathcal{A})$ for any $1 \leq j \leq m$.

Let $T_n(\mathcal{A})$ be the algebra of all n-by-n upper triangular matrices over $\mathcal{A}$. Assume that $\bar{k} = (k_1, \ldots, k_n) \in \mathbb{N}^n$, if $k_j = 1$ for any $1 \leq j \leq n$, then $T_n(\mathcal{A}) = B_n^{\bar{k}}(\mathcal{A})$ is a block upper triangular matrix algebra. In fact, $\{E_1, \ldots, E_n\}$ is a set of left triangulating idempotents of $T_n(\mathcal{A})$, and $\mathcal{A} \cong E_jT_n(\mathcal{A})E_j$ for each $1 \leq j \leq n$.

The following terminology is used throughout this article. Let $\mathcal{A}$ be an algebra and $\mathcal{M}$ be a left $\mathcal{A}$-module. Define the left annihilator of $\mathcal{M}$ by $l.ann_{\mathcal{A}}\mathcal{M} = \{a \in \mathcal{A} : a\mathcal{M} = \{0\}\}$. Also, we employ lowercase letters to denote elements in algebras and modules in the abstract setting, and uppercase letters to denote elements in triangular matrix algebras. I stands for the identity element in matrix algebras, and 1 denotes the unity of algebras in general.

3 LIP Maps on Triangular Algebras

Throughout this section, $Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$ denotes a triangular algebra, where $\mathcal{A}$ and $\mathcal{B}$ are algebras and $\mathcal{M}$ is an $(\mathcal{A}, \mathcal{B})$-bimodule. In this section, we study the **LIP** maps on triangular algebras and obtain a necessary condition and some sufficient conditions for a triangular algebra to be **SLIP** over $\mathcal{R}$. Firstly, we investigate the relation between zero product determined algebras and **SLIP** algebras.

The algebra $\mathcal{A}$ is called a *zero product determined algebra* if for every $\mathcal{R}$-module $\mathcal{X}$ and every $\mathcal{R}$-bilinear map $\phi : \mathcal{A} \times \mathcal{A} \to \mathcal{X}$, the following holds: If $\phi(a, b) = 0$ whenever $ab = 0$, then there exists an $\mathcal{R}$-linear map $L : \mathcal{A}^2 \to \mathcal{X}$ such that $\phi(a, b) = L(ab)$ for all $a, b \in \mathcal{A}$. Note that since $\mathcal{A}$ is unital, it follows that $\mathcal{A}^2 = \mathcal{A}$. The question of characterizing linear maps through zero products, etc. on algebras can be sometimes effectively solved by considering bilinear maps that preserve certain zero product properties (for instance, see Ghahramani 2014 and the references therein). For this reason, Brešar et al. in Brešar et al. (2009) introduced the concept of zero product determined algebras, which can be used to study the linear maps preserving zero product and derivable maps at zero point. We will see that the zero product determined algebras are **SLIP** algebras and more is true. So by applying this result, we characterize various **SLIP** algebras.

Theorem 3.1 *Let $\mathcal{A}$ be a zero product determined algebra. Then for any right $\mathcal{A}$-module $\mathcal{X}$, every local left multiplier $\psi : \mathcal{A} \to \mathcal{X}$ is a left multiplier.*

Proof Define $\phi : \mathcal{A} \times \mathcal{A} \to \mathcal{X}$ by $\phi(a,b) = \psi(a)b$. So ϕ is an $\mathcal{R}$-bilinear map. By the hypothesis for any $a \in \mathcal{A}$, there exists an element $x_a \in \mathcal{X}$ such that $\psi(a) = x_a a$. So for $a, b \in \mathcal{A}$ with $ab = 0$, we have

$$\phi(a,b) = \psi(a)b = x_a ab = 0.$$

Since $\mathcal{A}$ is a zero product determined algebra, it follows that there exists an $\mathcal{R}$-linear map $L : \mathcal{A}^2 \to \mathcal{X}$ such that $\psi(a)b = \phi(a,b) = L(ab)$ for all $a, b \in \mathcal{A}$. Therefore, $\psi(ab) = L(ab) = \psi(a)b$ for all $a, b \in \mathcal{A}$, and hence, ψ is a left multiplier. □

By the preceding theorem, it is clear that any zero product determined algebra is **SLIP**. However, we will see the converse of this statement is not necessarily true.

Brešar showed that an algebra generated by its idempotents is a zero product determined algebra (Bresar 2012, Theorem 4.1). Now, from Theorem 3.1 we have the following theorem.

Theorem 3.2 *Let $\mathcal{A}$ be an algebra which is generated by its idempotents and $\mathcal{X}$ be a right $\mathcal{A}$-module. If $\psi : \mathcal{A} \to \mathcal{X}$ is a local left multiplier, then ψ is a left multiplier. In particular, $\mathcal{A}$ is* **SLIP** *over $\mathcal{R}$.*

In the following corollary, we provide some classes of **SLIP** algebras generated by their idempotents.

Corollary 3.3 *Let $\mathcal{A}$ be any of the following algebras. Then for any right $\mathcal{A}$-module $\mathcal{X}$, every local left multiplier $\psi : \mathcal{A} \to \mathcal{X}$ is a left multiplier. Indeed, $\mathcal{A}$ is* **SLIP** *over $\mathcal{R}$.*

(i) *$\mathcal{A} = M_n(\mathcal{B})$, where $\mathcal{B}$ is an algebra and $n \geq 2$.*
(ii) *$\mathcal{A}$ is a simple algebra containing a non-trivial idempotent.*
(iii) *$\mathcal{A}$ is an algebra containing an idempotent e such that the ideals generated by e and $1-e$, respectively, are both equal to $\mathcal{A}$.*

Proof By Bresar (2007), the algebra $\mathcal{A}$ is generated by its idempotents. The desired conclusion thus follows readily from Theorem 3.2. □

It should be noted that Corollary 3.3(i) generalizes (Hadwin and Kerr 1997, Theorem 3).

Let $Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be a triangular algebra. It is shown in (Ghahramani 2013a, Theorem 2.1) that $Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$ is a zero product determined algebra if and only if $\mathcal{A}$ and $\mathcal{B}$ are zero product determined algebras. From this result and Theorem 3.1, we have the following proposition.

Proposition 3.4 *Let $\mathcal{T} = Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be a triangular algebra.*

(i) *If $\mathcal{A}$ and $\mathcal{B}$ are zero product determined algebras, and $\mathcal{X}$ is a right $\mathcal{T}$-module, then every local left multiplier $\psi : \mathcal{T} \to \mathcal{X}$ is a left multiplier. Therefore, $\mathcal{T}$ is* **SLIP** *over $\mathcal{R}$.*
(ii) *If $\mathcal{T}$ is a zero product determined algebra, and $\mathcal{X}_1$, $\mathcal{X}_2$ are right $\mathcal{A}$-module and right $\mathcal{B}$-module, respectively, then every local left multipliers $\psi_1 : \mathcal{A} \to \mathcal{X}_1$ and $\psi_2 : \mathcal{B} \to \mathcal{X}_2$ are left multipliers. So $\mathcal{A}$ and $\mathcal{B}$ are* **SLIP** *algebras.*

In light of the above proposition, these questions are naturally being raised: If the triangular algebra $Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$ is **SLIP** over $\mathcal{R}$, is it necessarily true that $Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$ is a zero product determined algebra? If the triangular algebra $Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$ is **SLIP** over $\mathcal{R}$, are both of $\mathcal{A}$ and $\mathcal{B}$, **SLIP** over $\mathcal{R}$? If for any right $\mathcal{A}$-module $\mathcal{X}_1$ and right $\mathcal{B}$-module $\mathcal{X}_2$, every local left multipliers $\psi_1 : \mathcal{A} \to \mathcal{X}_1$ and $\psi_2 : \mathcal{B} \to \mathcal{X}_2$ are left multipliers, is $Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$ **SLIP** over $\mathcal{R}$? We will see that if $Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$ is **SLIP**, $\mathcal{A}$ is not necessarily **SLIP** and so we obtain classes of **SLIP** triangular algebras which are not zero product determined algebras. Also we show that if $\mathcal{A}$ is **SLIP** and for any right $\mathcal{B}$-module $\mathcal{X}$, every local left multiplier $\psi : \mathcal{B} \to \mathcal{X}$ is a left multiplier, then $Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$ is **SLIP**. These results extends Proposition 3.4(i) as **SLIP** algebras are not necessarily zero product determined algebras.

In the following lemma, we describe the structure of **LIP** maps on triangular algebras.

Lemma 3.5 *Let $\mathcal{T} = Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be a triangular algebra and $\psi : \mathcal{T} \to \mathcal{T}$ be a* **LIP** *map. Then there are $\mathcal{R}$-linear maps $\alpha : \mathcal{A} \to \mathcal{A}$, $\tau : \mathcal{M} \to \mathcal{M}$, $\beta_1 : \mathcal{B} \to \mathcal{M}$ and $\beta_2 : \mathcal{B} \to \mathcal{B}$ such that*

$$\psi\left(\begin{pmatrix} a & m \\ 0 & b \end{pmatrix}\right) = \begin{pmatrix} \alpha(a) & \beta_1(b) + \tau(m) \\ 0 & \beta_2(b) \end{pmatrix},$$

where α and β_2 are **LIP** *maps, β_1 is a local left multiplier and $\tau(am) = \alpha(a)m$ for all $a \in \mathcal{A}$ and $m \in \mathcal{M}$.*

Proof Since $\begin{pmatrix} \mathcal{A} & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & \mathcal{M} \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & \mathcal{M} \\ 0 & \mathcal{B} \end{pmatrix}$ are left ideals of $\mathcal{T}$, using the hypothesis, for all $a \in \mathcal{A}, b \in \mathcal{B}, m \in \mathcal{M}$ we have

$$\psi\left(\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} \alpha(a) & 0 \\ 0 & 0 \end{pmatrix}, \quad \psi\left(\begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & \tau(m) \\ 0 & 0 \end{pmatrix}$$

and

$$\psi\left(\begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}\right) = \begin{pmatrix} 0 & \beta_1(b) \\ 0 & \beta_2(b) \end{pmatrix},$$

where $\alpha : \mathcal{A} \to \mathcal{A}$, $\tau : \mathcal{M} \to \mathcal{M}$, $\beta_1 : \mathcal{B} \to \mathcal{M}$ and $\beta_2 : \mathcal{B} \to \mathcal{B}$ are $\mathcal{R}$-linear maps. The mapping ψ is a local left multiplier, since it is a **LIP** map. Thus, for $T = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in \mathcal{T}$, there is an element $X_T =$

$\begin{pmatrix} a_T & m_T \\ 0 & b_T \end{pmatrix} \in \mathcal{T}$ such that $\psi(T) = X_T T = \begin{pmatrix} a_T a & m_T b \\ 0 & b_T b \end{pmatrix}$. Hence, $\alpha(a) = a_T a$, $\beta_1(b) = m_T b$ and $\beta_2(b) = b_T b$ for all $a \in \mathcal{A}, b \in \mathcal{B}$, proving that these maps are local left multiplier. Let $a \in \mathcal{A}$, $m \in \mathcal{M}$, and put $T = \begin{pmatrix} a & am \\ 0 & 0 \end{pmatrix}$, $S = \begin{pmatrix} 0 & -m \\ 0 & 1 \end{pmatrix}$. We have $TS = 0$ and there is an element $X_T \in \mathcal{T}$ such that $\psi(T) = X_T T$. Thus,

$$\begin{pmatrix} \alpha(a) & \tau(am) \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & -m \\ 0 & 1 \end{pmatrix} = \psi(T)S = X_T TS = 0.$$

Therefore, $\tau(am) = \alpha(a)m$ for all $a \in \mathcal{A}, m \in \mathcal{M}$. □

In the next theorem, we obtain a necessary condition for triangular algebra $Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$ to be **SLIP** over $\mathcal{R}$.

Theorem 3.6 *Let $\mathcal{T} = Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be* **SLIP** *over $\mathcal{R}$. Then $\mathcal{B}$ is* **SLIP** *over $\mathcal{R}$ and every local left multiplier from $\mathcal{B}$ into $\mathcal{M}$ is a left multiplier.*

Proof Suppose that $\beta_1 : \mathcal{B} \to \mathcal{M}$ and $\beta_2 : \mathcal{B} \to \mathcal{B}$ are local left multipliers and define the $\mathcal{R}$-linear map $\psi : \mathcal{T} \to \mathcal{T}$ by $\psi\left(\begin{pmatrix} a & m \\ 0 & b \end{pmatrix}\right) = \begin{pmatrix} 0 & \beta_1(b) \\ 0 & \beta_2(b) \end{pmatrix}$. For each $b \in \mathcal{B}$, there are elements $c_b \in \mathcal{B}$ and $n_b \in \mathcal{M}$ such that $\beta_1(b) = n_b b$ and $\beta_2(b) = c_b b$. Now for any $T = \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \in \mathcal{T}$, let $X_T = \begin{pmatrix} 0 & n_b \\ 0 & c_b \end{pmatrix}$. We have

$$\psi(T) = \begin{pmatrix} 0 & n_b b \\ 0 & c_b b \end{pmatrix} = X_T T.$$

Hence, ψ is a **LIP** map, and by the hypothesis, it is a left multiplier; i.e., $\psi(T) = \psi(I)T$ for all $T \in \mathcal{T}$. So for all $b \in \mathcal{B}$ we see that

$$\begin{aligned} \begin{pmatrix} 0 & \beta_1(b) \\ 0 & \beta_2(b) \end{pmatrix} &= \psi\left(\begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}\right) \\ &= \begin{pmatrix} 0 & \beta_1(1) \\ 0 & \beta_2(1) \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} 0 & \beta_1(1)b \\ 0 & \beta_2(1)b \end{pmatrix}, \end{aligned}$$

where 1 is the unity of $\mathcal{B}$. Thus, β_1 and β_2 are left multipliers. □

By invoking the above theorem, we get a necessary condition for an idempotent element in an **SLIP** algebra to be left semicentral.

Corollary 3.7 *Suppose $\mathcal{A}$ is a* **SLIP** *algebra.*

(i) *If $e \in \mathcal{A}$ is a non-trivial left semicentral idempotent, then $(1-e)\mathcal{A}(1-e)$ is* **SLIP** *over $\mathcal{R}$.*

(ii) *If for any non-trivial idempotent $e \in \mathcal{A}$, $(1-e)\mathcal{A}(1-e)$ is not* **SLIP** *over $\mathcal{R}$, then $\mathcal{A}$ is semicentral reduced.*

In the following results, we give some sufficient conditions for a triangular algebra to be **SLIP** over $\mathcal{R}$.

Theorem 3.8 *Let $\mathcal{T} = Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be a triangular algebra. Let $l.ann_{\mathcal{A}}\mathcal{M} = \{0\}$, $\mathcal{B}$ be* **SLIP** *over $\mathcal{R}$ and every local left multiplier from $\mathcal{B}$ into $\mathcal{M}$ be a left multiplier. Then $\mathcal{T}$ is* **SLIP** *over $\mathcal{R}$.*

Proof Suppose that $\psi : \mathcal{T} \to \mathcal{T}$ is a **LIP** map. By Lemma 3.5

$$\psi\left(\begin{pmatrix} a & m \\ 0 & b \end{pmatrix}\right) = \begin{pmatrix} \alpha(a) & \beta_1(b) + \tau(m) \\ 0 & \beta_2(b) \end{pmatrix},$$

where $\beta_1 : \mathcal{B} \to \mathcal{M}$ and $\beta_2 : \mathcal{B} \to \mathcal{B}$ are local left multipliers and $\tau(am) = \alpha(a)m$ for all $a \in \mathcal{A}$ and $m \in \mathcal{M}$. By the hypothesis,

$$\beta_1(b) = \beta_1(1)b, \quad \beta_2(b) = \beta_2(1)b \quad (b \in \mathcal{B}).$$

For every $a, a' \in \mathcal{A}$ and $m \in \mathcal{M}$, we have

$$\tau(aa'm) = \alpha(aa')m.$$

On the other hand,

$$\tau(aa'm) = \alpha(a)a'm.$$

Comparing the two above equalities and noting that $l.ann_{\mathcal{A}}\mathcal{M} = \{0\}$, we arrive at $\alpha(aa') = \alpha(a)a'$. So

$$\alpha(a) = \alpha(1)a, \quad \tau(m) = \alpha(1)m \quad (a \in \mathcal{A}, m \in \mathcal{M}).$$

From these equations, we deduce that for all $T \in \mathcal{T}$:

$$\psi(T) = \psi(I)T.$$

□

Let $\mathcal{X}$ be an $\mathcal{R}$-module. It is obvious that each $\mathcal{R}$-linear local left multiplier from $\mathcal{R}$ into $\mathcal{X}$ is a left multiplier. So from Theorem 3.8, we have the next corollary.

Corollary 3.9 *Let $\mathcal{A}$ be an algebra over $\mathcal{R}$. Then $Tri(\mathcal{A}, \mathcal{A}, \mathcal{R})$ is* **SLIP** *over $\mathcal{R}$.*

Let $\mathcal{M}$ be a right $\mathcal{B}$-module. Denote by $End_{\mathcal{B}}(\mathcal{M})$, the algebra of all $\mathcal{B}$-module endomorphisms of $\mathcal{M}$. Let $\mathcal{A} := End_{\mathcal{B}}(\mathcal{M})$. Then $\mathcal{M}$ is an $(\mathcal{A}, \mathcal{B})$-bimodule equipped with $\phi\Delta m := \phi(m)$ $(m \in \mathcal{M}, \phi \in \mathcal{A})$ such that $l.ann_{\mathcal{A}}\mathcal{M} = \{0\}$. So by Theorems 3.6 and 3.8, we obtain the following corollary.

Corollary 3.10 *Let $\mathcal{M}$ be a right $\mathcal{B}$-module. Then $Tri(End_{\mathcal{B}}(\mathcal{M}), \mathcal{M}, \mathcal{B})$ is* **SLIP** *over $\mathcal{R}$ if and only if $\mathcal{B}$ is* **SLIP** *over $\mathcal{R}$ and every local left multiplier from $\mathcal{B}$ into $\mathcal{M}$ is a left multiplier.*

In the following theorem, we do not require the condition $l.ann_{\mathcal{A}}\mathcal{M} = \{0\}$.

Theorem 3.11 *Let* $\mathcal{T} = Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$ *be a triangular algebra. Assume that* $\mathcal{A}$ *and* $\mathcal{B}$ *are* **SLIP** *algebras and every local left multiplier from* $\mathcal{B}$ *into* $\mathcal{M}$ *is a left multiplier. Then* $\mathcal{T}$ *is* **SLIP** *over* $\mathcal{R}$.

Proof Suppose that $\psi : \mathcal{T} \to \mathcal{T}$ is a **LIP** map. By Lemma 3.5,

$$\psi\left(\begin{pmatrix} a & m \\ 0 & b \end{pmatrix}\right) = \begin{pmatrix} \alpha(a) & \beta_1(b) + \tau(m) \\ 0 & \beta_2(b) \end{pmatrix},$$

where $\alpha : \mathcal{A} \to \mathcal{A}$ and $\beta_2 : \mathcal{B} \to \mathcal{B}$ are **LIP** maps, $\beta_1 : \mathcal{B} \to \mathcal{M}$ is a local left multiplier and $\tau : \mathcal{M} \to \mathcal{M}$ satisfies $\tau(am) = \alpha(a)m$ for all $a \in \mathcal{A}$ and $m \in \mathcal{M}$. By the hypothesis,

$$\alpha(a) = \alpha(1)a, \ \beta_1(b) = \beta_1(1)b,$$
$$\beta_2(b) = \beta_2(1)b \text{ and } \tau(m) = \alpha(1)m,$$

for all $a \in \mathcal{A}, b \in \mathcal{B}, m \in \mathcal{M}$, concluding that

$$\psi(T) = \psi(I)T,$$

for all $T \in \mathcal{T}$. □

Now we give some examples illustrating limitations on extending some of the theory developed. The examples show that the classes of triangular algebras satisfying the conditions of Theorem 3.8 are different from those satisfying the assumptions of Theorem 3.11. Thus, we firstly need to provide some algebras which are not **SLIP**.

Remark 3.12 Every division **SLIP** algebra $\mathcal{A}$ is a field. Consider the arbitrary elements $a, b \in \mathcal{A}$ and define the $\mathcal{R}$-linear map $\psi_b(a) = ab$. Since $\mathcal{A}$ is a division algebra, it follows that ψ_b is a **LIP** map. So by the hypothesis that $\mathcal{A}$ is **SLIP**, we have $ab = \psi_b(a) = \psi_b(1)a = ba$. Hence, $\mathcal{A}$ is commutative.

From Remark 3.12, one concludes that the quaternion algebra $\mathbb{H}(\mathbb{R})$ over the real field $\mathbb{R}$ is not **SLIP**. In the following example, a non-division algebra is presented which is not **SLIP**. This example is given in Katsoulis (2016, Example 2.4).

Example 3.13 Let $\mathbb{F}$ be a field. Then

$$\mathcal{U} := \left\{ \begin{pmatrix} \lambda & \mu \\ 0 & \lambda \end{pmatrix} \middle| \lambda, \mu \in \mathbb{F} \right\}$$

is an algebra over $\mathbb{F}$ which is not **SLIP**.

Now by Corollary 3.9 and above examples, we can obtain an **SLIP** triangular algebra $Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$ in which $\mathcal{A}$ is not **SLIP**. Hence, $Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$ is not a zero product determined algebra [by Theorem 3.1 and (Ghahramani 2013a, Theorem 2.1)].

Example 3.14 Let $\mathcal{T}$ be either $Tri(\mathbb{H}(\mathbb{R}), \mathbb{H}(\mathbb{R}), \mathbb{R})$ or $Tri(\mathcal{U}, \mathcal{U}, \mathbb{F})$ where $\mathcal{U}$ is the algebra described in Example 3.13. Then by Corollary 3.9, $\mathcal{T}$ is **SLIP**, whereas $\mathbb{H}(\mathbb{R})$ and $\mathcal{U}$ are not **SLIP** algebras.

In the above example, $\mathcal{T}$ satisfies the conditions of Theorem 3.8 but not those of Theorem 3.11. This example shows also that the converse of Theorem 3.11 is not necessarily true.

In the following, we give an example showing that the converse of Theorem 3.8 is not necessarily valid.

Example 3.15 Assume that $\mathcal{A}$ is a zero product determined algebra. By (Ghahramani (2013b, Proposition 2.8), $\mathcal{A} \times \mathcal{A}$ is a zero product determined algebra. The usual right multiplication of $\mathcal{A}$ and the following left multiplication make $\mathcal{A}$ into an $((\mathcal{A} \times \mathcal{A}), \mathcal{A})$-bimodule:

$$(a, b)x := ax \quad (a, b, x \in \mathcal{A}).$$

By Proposition 3.4(i), $\mathcal{T} = Tri(\mathcal{A} \times \mathcal{A}, \mathcal{A}, \mathcal{A})$ is **SLIP**, while $l.ann_{\mathcal{A}\times\mathcal{A}}\mathcal{A} = \{0\} \times \mathcal{A} \neq \{0\}$.

In Example 3.15, $\mathcal{T}$ satisfies their conditions of Theorem 3.11 but does not satisfy the conditions of Theorem 3.8.

In the following example, we show that the conditions on $\mathcal{A}$ in Theorems 3.8 and 3.11 cannot be dropped.

Example 3.16 Let $\mathcal{A}$ and $\mathcal{B}$ be algebras such that $\mathcal{A}$ is a zero product determined algebra and $\mathcal{B}$ is not **SLIP**. By the following module actions, we turn $\mathcal{A}$ into an $((\mathcal{A} \times \mathcal{B}), \mathcal{A})$-bimodule:

$$(a, b)x := ax \quad (a, x \in \mathcal{A}, b \in \mathcal{B}),$$

and the right multiplication is the usual multiplication of $\mathcal{A}$. We show that the triangular algebra $\mathcal{T} = Tri(\mathcal{A} \times \mathcal{B}, \mathcal{A}, \mathcal{A})$ is not **SLIP**. Since $\mathcal{B}$ is not **SLIP**, there is a local left multiplier $\rho : \mathcal{B} \to \mathcal{B}$ which is not a left multiplier. So for any $b \in \mathcal{B}$ there exists an element $c_b \in \mathcal{B}$ such that $\rho(b) = c_b b$. Define the $\mathcal{R}$-linear map $\psi : \mathcal{T} \to \mathcal{T}$ by

$$\psi\left(\begin{pmatrix} (a_1, b) & a_2 \\ 0 & a_3 \end{pmatrix}\right) = \begin{pmatrix} (0, \rho(b)) & 0 \\ 0 & 0 \end{pmatrix}.$$

For any $T = \begin{pmatrix} (a_1, b) & a_2 \\ 0 & a_3 \end{pmatrix} \in \mathcal{T}$, there is $X_T = \begin{pmatrix} (0, c_b) & 0 \\ 0 & 0 \end{pmatrix}$ such that

$$\psi(T) = X_T T.$$

So ψ is a **LIP** map which is not a left multiplier. If otherwise, $\psi(T) = \psi(I)T$ for all $T \in \mathcal{T}$, concluding that ρ is a left multiplier and this is a contradiction. In this example, $l.ann_{\mathcal{A}\times\mathcal{B}}\mathcal{A} = \{0\} \times \mathcal{B} \neq \{0\}$, and by Hadwin and Kerr (1997, Lemma 5), $\mathcal{A} \times \mathcal{B}$ is not **SLIP**.

In the above example, $\mathcal{A}$ can be assumed to be any of the stated algebras in Corollary 3.3 and $\mathcal{B}$ to be either the quaternion algebra $\mathbb{H}(\mathbb{R})$ or the algebra $\mathcal{U}$ as mentioned in Example 3.13.

4 Applications to Generalized Triangular Matrix Algebras and Block Upper Triangular Matrix Algebras

In this section, we apply our results obtained in the previous section to generalized triangular matrix algebras and block upper triangular matrix algebras.

Theorem 4.1 *Let $\mathcal{A}$ be an algebra having a generalized triangular matrix representation*

$$\mathcal{A}=\begin{pmatrix}\mathcal{A}_{11} & \mathcal{A}_{12} & \cdots & \mathcal{A}_{1n}\\ 0 & \mathcal{A}_{22} & \cdots & \mathcal{A}_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \mathcal{A}_{nn}\end{pmatrix},$$

where each $\mathcal{A}_{ii}$ is an algebra with unity and $\mathcal{A}_{ij}$ is a $(\mathcal{A}_i,\mathcal{A}_j)$-bimodule for $i<j$. If every local left multiplier from $\mathcal{A}_{ii}$ $(1\le i\le n)$ into $\mathcal{A}_{ki}$ $(1\le k\le i)$ is a left multiplier, then $\mathcal{A}$ is **SLIP** *over $\mathcal{R}$.*

Proof We denote the elements of $\mathcal{A}_{ij}$ by a_{ij}, 1_i for the unity of $\mathcal{A}_{ii}$ and $a_{ij}E_{ij}$ for the n-by-n matrix with $a_{ij}\in\mathcal{A}_{ij}$ at (i,j)-entry and 0 in all other entries. Note that E_i denotes the matrix 1_iE_{ii}, and $\{E_1,\ldots,E_n\}$ is a set of left triangulating idempotents of $\mathcal{A}$ (by Proposition 2.1).

The proof is by induction on n. If $n=1$, then $\mathcal{A}=\mathcal{A}_{11}$ and the result is obvious in this case.

Let $n\ge 2$ and assume that for each algebra that has a set of left triangulating idempotents with $n-1$ elements, the result is true.

Let $\{E_1,\ldots,E_n\}$ be a set of left triangulating idempotents of $\mathcal{A}$. By Remark 2.2, $\mathcal{A}\cong Tri(E_1\mathcal{A}E_1,E_1\mathcal{A}F,F\mathcal{A}F)$ and $\{E_2,\ldots,E_n\}$ is a set of left triangulating idempotents of $F\mathcal{A}F$, where $F=I-E_1=\sum_{i=2}^n E_i$ is the unity of $F\mathcal{A}F$. Also we have the $\mathcal{R}$-algebra isomorphisms:

$$E_1\mathcal{A}E_1\cong\mathcal{A}_{11},\quad F\mathcal{A}F\cong\begin{pmatrix}\mathcal{A}_{22} & \cdots & \mathcal{A}_{2n}\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \mathcal{A}_{nn}\end{pmatrix},$$

and the $(\mathcal{A}_{11},F\mathcal{A}F)$-bimodule isomorphism:

$$E_1\mathcal{A}F\cong(\mathcal{A}_{12},\ldots,\mathcal{A}_{1n}).$$

By the hypothesis, $\mathcal{A}_{11}$ is **SLIP** over $\mathcal{R}$. Also, by the induction hypothesis $F\mathcal{A}F$ is **SLIP** over $\mathcal{R}$. Let $\psi: F\mathcal{A}F\to E_1\mathcal{A}F$ be a local left multiplier. We show that ψ is a left multiplier.

Since ψ is an $\mathcal{R}$-linear map, there exist $\mathcal{R}$-linear maps $\phi_{ij}^k:\mathcal{A}_{ij}\to\mathcal{A}_{1k}$ such that

$$\psi(a_{ij}E_{ij})=\sum_{k=2}^n\phi_{ij}^k(a_{ij})E_{1k},$$

where $2\le i\le j\le n$ and $2\le k\le n$.

We complete the proof by checking some steps.

Step 1 $\phi_{ii}^k=0$ for all $2\le i,k\le n$ with $i\ne k$. For each $a_{ii}\in\mathcal{A}_{ii}$ $(2\le i\le n)$, let $T=a_{ii}E_i$. Since ψ is a local left multiplier, there exists $X_T\in E_1\mathcal{A}F$ such that $\psi(T)=X_TT$. Now $T(F-E_i)=0$, and hence, $\psi(T)(F-E_i)=0$. So

$$0=\left(\sum_{k=2}^n\phi_{ii}^k(a_{ii})E_{1k}\right)(F-E_i)=\sum_{k=2}^n\phi_{ii}^k(a_{ii})E_{1k}-\phi_{ii}^i(a_{ii})E_{1i}.$$

Therefore, for all $2\le i,k\le n$ with $i\ne k$, we have $\phi_{ii}^k=0$.

Step 2 $\phi_{ij}^k=0$ for all $2\le i<j\le n$, $2\le k\le n$ with $j\ne k$. For each $a_{ij}\in\mathcal{A}_{ij}$ $(2\le i<j\le n)$, let $T=a_{ij}E_{ij}$. For any $2\le k\le n$ with $j\ne k$, we have $TE_k=0$. So by a similar argument as in Step 1, $\psi(T)E_k=0$. Hence,

$$0=\left(\sum_{l=2}^n\phi_{ij}^l(a_{ij})E_{1l}\right)E_k=\phi_{ij}^k(a_{ij})E_{1k}.$$

The result now follows from the above equation.

Step 3 $\phi_{ij}^j(a_{ij})E_{1j}=\phi_{ii}^i(1_i)E_{1i}a_{ij}E_{ij}$ for all $2\le i<j\le n$ and $a_{ij}\in\mathcal{A}_{ij}$. We have $(E_i+a_{ij}E_{ij})(-a_{ij}E_{ij}+E_j)=0$ for all $a_{ij}\in\mathcal{A}_{ij}$. So $\psi(E_i+a_{ij}E_{ij})(-a_{ij}E_{ij}+E_j)=0$. By Steps 1, 2, we see that $\psi(E_i)=\phi_{ii}^i(1_i)E_{1i}$ and $\psi(a_{ij}E_{ij})=\phi_{ij}^j(a_{ij})E_{1j}$. Thus,

$$(\phi_{ii}^i(1_i)E_{1i}+\phi_{ij}^j(a_{ij})E_{1j})(-a_{ij}E_{ij}+E_j)=0,$$

and hence,

$$\phi_{ij}^j(a_{ij})E_{1j}=\phi_{ii}^i(1_i)E_{1i}a_{ij}E_{ij}.$$

Step 4 $\phi_{ii}^i(a_{ii})E_{1i}=\phi_{ii}^i(1_i)E_{1i}a_{ii}E_i$ for all $2\le i\le n$ and $a_{ii}\in\mathcal{A}_{ii}$. For each $a_{ii}\in\mathcal{A}_{ii}$ $(2\le i\le n)$, let $T=a_{ii}E_i$. Since ψ is a local left multiplier, there exists $X_T=\sum_{k=2}^n x_{1k}E_{1k}$ such that $\psi(T)=X_TT$. So by Step 1,

$$\phi_{ii}^i(a_{ii})E_{1i}=x_{1i}a_{ii}E_{1i},$$

and hence, ϕ_{ii}^i is a local left multiplier. By the hypothesis, $\phi_{ii}^i(a_{ii})=\phi_{ii}^i(1_i)a_{ii}$, and hence,

$$\phi_{ii}^i(a_{ii})E_{1i}=\phi_{ii}^i(1_i)E_{1i}a_{ii}E_i.$$

From Steps 1–4, it follows that $\psi(T)=\psi(F)T$ for all $T\in F\mathcal{A}F$. So ψ is a left multiplier.

We now apply Theorem 3.11 to deduce that $\mathcal{A}$ is **SLIP** over $\mathcal{R}$. □

As we will see in Theorems 4.2 and 4.3, being $\mathcal{A}_{kk}$ $(1 \le k \le n-1)$ **SLIP** algebras is not necessary condition for $\mathcal{A}$ to be **SLIP**.

Theorem 4.2 *Suppose that the algebra $\mathcal{A}$ has the generalized triangular matrix representation*

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} & \cdots & \mathcal{A}_{1n} \\ 0 & \mathcal{A}_{22} & \cdots & \mathcal{A}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathcal{A}_{nn} \end{pmatrix}.$$

Suppose further that any local left multiplier from $\mathcal{A}_{ii}$ $(2 \le i \le n)$ into $\mathcal{A}_{ki}$ $(1 \le k \le i)$ is a left multiplier and $l.ann_{\mathcal{A}_{11}}\mathcal{A}_{1k} = \{0\}$ for some $2 \le k \le n$. Then $\mathcal{A}$ is **SLIP** *over $\mathcal{R}$.*

Proof Let $\mathcal{A}$ have a set of left triangulating idempotents $\{E_1, \ldots, E_n\}$. By Remark 2.2, $\mathcal{A} \cong Tri(E_1\mathcal{A}E_1, E_1\mathcal{A}F, F\mathcal{A}F)$ and $\{E_2, \ldots, E_n\}$ is a set of left triangulating idempotents of $F\mathcal{A}F$, where $F = I - E_1$. By the hypothesis, the algebra

$$F\mathcal{A}F \cong \begin{pmatrix} \mathcal{A}_{22} & \cdots & \mathcal{A}_{2n} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathcal{A}_{nn} \end{pmatrix}$$

satisfies the conditions of Theorem 4.1, and hence, it is **SLIP**. A similar proof to that of Theorem 4.1 shows that every local left multiplier from $F\mathcal{A}F$ into $E_1\mathcal{A}F$ is a left multiplier. Since $l.ann_{\mathcal{A}_{11}}\mathcal{A}_{1k} = \{0\}$ for some $2 \le k \le n$, it follows that $l.ann_{\mathcal{A}_{11}}E_1\mathcal{A}F = \{0\}$. Therefore, $\mathcal{A}$ is **SLIP**, by Theorem 3.8. □

Theorem 4.3 *Let $\mathcal{A}$ have a set of left triangulating idempotents $\{E_1, \ldots, E_n\}$ with the generalized triangular matrix representation*

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} & \cdots & \mathcal{A}_{1n} \\ 0 & \mathcal{A}_{22} & \cdots & \mathcal{A}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathcal{A}_{nn} \end{pmatrix}.$$

Suppose that $l.ann_{F\mathcal{A}F}F\mathcal{A}E_n = \{0\}$, where $F = I - E_n$ and any local left multiplier from $\mathcal{A}_{nn}$ into $\mathcal{A}_{kn}$ $(1 \le k \le n)$ is a left multiplier. Then $\mathcal{A}$ is **SLIP** *over $\mathcal{R}$.*

Proof We use the same notations as those in the proof of Theorem 4.1. Let $F = I - E_n$. By Remark 2.2, $\mathcal{A} \cong Tri(F\mathcal{A}F, F\mathcal{A}E_n, E_n\mathcal{A}E_n)$. In this case, E_n is the unity of $\mathcal{A}_{nn}$ and we have the $\mathcal{R}$-algebra isomorphism:

$$E_n\mathcal{A}E_n \cong \mathcal{A}_{nn},$$

and the $(F\mathcal{A}F, \mathcal{A}_{nn})$-bimodule isomorphism:

$$F\mathcal{A}E_n \cong \begin{pmatrix} \mathcal{A}_{1n} \\ \vdots \\ \mathcal{A}_{n-1,n} \end{pmatrix}.$$

By the hypothesis, $\mathcal{A}_{nn}$ is **SLIP**. Let $\psi : \mathcal{A}_{nn} \to F\mathcal{A}E_n$ be a local left multiplier. We show that ψ is a left multiplier.

Since ψ is an $\mathcal{R}$-linear map, there exist $\mathcal{R}$-linear maps $\phi_n^k : \mathcal{A}_{nn} \to \mathcal{A}_{kn}$ $(1 \le k \le n-1)$ such that

$$\psi(a_{nn}E_n) = \sum_{k=1}^{n-1} \phi_n^k(a_{nn})E_{kn}.$$

For each $a_{nn} \in \mathcal{A}_{nn}$, put $T = a_{nn}E_n$. Since ψ is a local left multiplier, there is $X_T = \sum_{k=1}^{n-1} x_{kn}E_{kn} \in F\mathcal{A}E_n$ such that $\psi(T) = X_T T$. So

$$\sum_{k=1}^{n-1} \phi_n^k(a_{nn})E_{kn} = \sum_{k=1}^{n-1} x_{kn}a_{nn}E_{kn},$$

and hence, any ϕ_n^k is a local left multiplier. By the hypothesis, each ϕ_n^k $(1 \le k \le n-1)$ satisfies

$$\phi_n^k(a_{nn}) = \phi_n^k(1_n)a_{nn} \quad (a_{nn} \in \mathcal{A}_{nn}).$$

So

$$\psi(a_{nn}E_n) = \sum_{k=1}^{n-1} \phi_n^k(1_n)a_{nn}E_{kn} = \psi(E_n)a_{nn}E_n,$$

for all $a_{nn} \in \mathcal{A}_{nn}$. So ψ is a left multiplier. Now from hypothesis and Theorem 3.8, it follows that $\mathcal{A}$ is **SLIP**. □

The following proposition shows that being $E_n\mathcal{A}E_n$ an **SLIP** algebra is a necessary condition for $\mathcal{A}$ with a set of left triangulating idempotents $\{E_1, \ldots, E_n\}$ to be an **SLIP** algebra.

Proposition 4.4 *Let $\mathcal{A}$ be* **SLIP** *over $\mathcal{R}$ with a set of left triangulating idempotents $\{E_1, \ldots, E_n\}$. Then $E_n\mathcal{A}E_n$ is* **SLIP** *over $\mathcal{R}$.*

Proof Let $F = I - E_n$. By Remark 2.2, $\mathcal{A} \cong Tri(F\mathcal{A}F, F\mathcal{A}E_n, E_n\mathcal{A}E_n)$. Now, from Theorem 3.6, it follows that $E_n\mathcal{A}E_n$ is **SLIP** over $\mathcal{R}$. □

In continuation, we apply our results to block upper triangular matrix algebras. In order to prove Theorem 4.6, we need the following lemma.

Lemma 4.5 *Let $\mathcal{A}$ be* **SLIP** *over $\mathcal{R}$ and $M_{r\times s}(\mathcal{A})$ be the right $\mathcal{A}$-module of the set of all r-by-s matrices over $\mathcal{A}$. Then any local left multiplier from $\mathcal{A}$ into $M_{r\times s}(\mathcal{A})$ is a left multiplier.*

Proof We use the same notations as those in the proof of Theorem 4.1. Let $\psi : \mathcal{A} \to M_{r\times s}(\mathcal{A})$ be a local left multiplier. Then there exist $\mathcal{R}$-linear maps $\phi^{ij} : \mathcal{A} \to \mathcal{A}E_{ij}$ $(1 \le i \le r, 1 \le j \le s)$ such that

$$\psi(a) = \sum_{i=1}^{r}\sum_{j=1}^{s} \phi^{ij}(a)E_{ij}.$$

Since ψ is a local left multiplier, it is easily checked that each ϕ^{ij} is a local left multiplier. So by the hypothesis and the fact that $\mathcal{A}E_{ij} \cong \mathcal{A}$ (as right $\mathcal{A}$-modules), we have

$$\phi^{ij}(a) = \phi^{ij}(1)aE_{ij},$$

for all $1 \le i \le r, 1 \le j \le s$. Thus, $\psi(a) = \psi(1)a$ for all $a \in \mathcal{A}$; i.e., ψ is a left multiplier. □

Theorem 4.6 *Let $B_n^{\bar{k}}(\mathcal{A})$ $(n \ge 1)$ be a block upper triangular matrix algebra, where $\bar{k} = (k_1, \ldots, k_m) \in \mathbb{N}^m$. Then*

(i) *if $k_m \ge 2$, then $B_n^{\bar{k}}(\mathcal{A})$ $(n \ge 2)$ is* **SLIP** *over $\mathcal{R}$.*
(ii) *if $k_m = 1$, then $B_n^{\bar{k}}(\mathcal{A})$ is* **SLIP** *over $\mathcal{R}$ if and only if $\mathcal{A}$ is* **SLIP** *over $\mathcal{R}$.*

Proof Let $\{F_1, \ldots, F_m\}$ be a set of left triangulating idempotents of $B_n^{\bar{k}}(\mathcal{A})$ such that $F_1 = \sum_{i=1}^{k_1} E_i$ and $F_j = \sum_{i=1}^{k_j} E_{i+k_1+\cdots+k_{j-1}}$ for $2 \le j \le m$. Suppose that $F = I - F_m$ and $l = k_1 + \ldots + k_{m-1}$. Then $FB_n^{\bar{k}}(\mathcal{A})F$ is a subalgebra of $M_l(\mathcal{A})$ and $FB_n^{\bar{k}}(\mathcal{A})F_m \cong M_{l\times k_m}(\mathcal{A})$. Since $l.ann_{M_l(\mathcal{A})}M_{l\times k_m}(\mathcal{A}) = \{0\}$, it follows that $l.ann_{FB_n^{\bar{k}}(\mathcal{A})F}FB_n^{\bar{k}}(\mathcal{A})F_m = \{0\}$.

(*i*) We have $F_mB_n^{\bar{k}}(\mathcal{A})F_m \cong M_{k_m}(\mathcal{A})$ $(k_m \ge 2)$. By Corollary 3.3(i), any local left multiplier from $F_mB_n^{\bar{k}}(\mathcal{A})F_m$ into any right $F_mB_n^{\bar{k}}(\mathcal{A})F_m$-module is a left multiplier. We deduce from Theorem 4.3 that $\mathcal{A}$ is **SLIP** over $\mathcal{R}$.

(*ii*) Let $B_n^{\bar{k}}(\mathcal{A})$ be **SLIP** over $\mathcal{R}$. By Proposition 4.4, $F_mB_n^{\bar{k}}(\mathcal{A})F_m \cong M_{k_m}(\mathcal{A}) = \mathcal{A}$ (since $k_m = 1$) is **SLIP** over $\mathcal{R}$.

Conversely, assume that $\mathcal{A}$ is **SLIP** over $\mathcal{R}$. We have $FB_n^{\bar{k}}(\mathcal{A})F_m \cong M_{l\times 1}(\mathcal{A})$. (In this case, $F_m = E_n$.) Now by Lemma 4.5, each of Theorems 3.8 and 4.3 implies that $B_n^{\bar{k}}(\mathcal{A})$ is **SLIP** over $\mathcal{R}$. □

The n-by-n upper triangular matrices $T_n(\mathcal{A})$ $(n \ge 1)$ are the block upper triangular matrix algebra $B_n^{\bar{k}}(\mathcal{A})$, whenever $\bar{k} = (k_1, \ldots, k_n) \in \mathbb{N}^n$ with $k_j = 1$ for any $1 \le j \le n$. Therefore, by Theorem 4.6, we obtain the following corollary.

Corollary 4.7 *The algebra of upper triangular matrices $T_n(\mathcal{A})$ $(n \ge 1)$ is* **SLIP** *over $\mathcal{R}$ if and only if $\mathcal{A}$ is* **SLIP** *over $\mathcal{R}$.*

Since each $\mathcal{R}$-linear local left multiplier from $\mathcal{R}$ into any right $\mathcal{R}$-module $\mathcal{X}$ is a left multiplier, in view of Theorem 4.6, the next corollary is immediate.

Corollary 4.8 *The block upper triangular matrix algebra $B_n^{\bar{k}}(\mathcal{R})$ is* **SLIP** *over $\mathcal{R}$ for every $n \ge 1$. Particularly, $T_n(\mathcal{R})$ $(n \ge 1)$ is* **SLIP** *over $\mathcal{R}$.*

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