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RESEARCH PAPER

The Prediction Performance of the Alternative Biased Estimators for the Distributed Lag Models

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Abstract

The finite distributed lag models include highly correlated variables, lagged and unlagged values of the same variables. Some problems are faced for this model when applying the ordinary least squares method or econometric models such as Almon models. Gültay and Kaçıranlar (J Math Stat 44:1215–1233, 2015) compared the performance of the alternative biased estimators to the Almon estimator in terms of the mean square error. The primary aim of this study is to evaluate the predictive performance of the alternative biased estimators to the Almon estimator according to the prediction mean square error criterion under the target function. We use the Almon (Econometrica 178–196, 1965) data to illustrate our theoretical results.

Keywords Finite distributed lag model · Almon estimator · Ridge estimator · Liu estimator · Prediction mean square error

1 Introduction

Consider the finite distributed lag model,

$$y_t = \beta_0 x_t + \beta_1 x_{t-1} + \cdots + \beta_p x_{t-p} + \varepsilon_t, \quad t = p+1, \ldots, T$$
$$= \sum_{i=0}^{p} \beta_i x_{t-i} + \varepsilon_t \tag{1}$$

where ε_t is $IN(0, \sigma^2)$. The coefficients β_i are called lag weights. Model in Eq. (1) can be written in the matrix notation as

$$y = X\beta + \varepsilon \tag{2}$$

where

$$y = \begin{bmatrix} y_{p+1} \\ y_{p+2} \\ \vdots \\ y_T \end{bmatrix}, \quad \beta = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_p \end{bmatrix},$$
$$X = \begin{bmatrix} x_{p+1} & x_p & \cdots & x_1 \\ x_{p+2} & x_{p+1} & \cdots & x_2 \\ \vdots & \vdots & \ddots & \vdots \\ x_T & x_{T-1} & \cdots & x_{T-p} \end{bmatrix}, \quad \varepsilon = \begin{bmatrix} \varepsilon_{p+1} \\ \varepsilon_{p+2} \\ \vdots \\ \varepsilon_T \end{bmatrix}.$$

Some kind of distributed lag models have been introduced to be able to estimate the parameters using some prior information about the behavior of the β's in (1) such as the Almon models.Fisher (1937) initially introduced nonstochastic smoothness prior information of the following type:

$$\begin{aligned} \beta_i &= (p+1-i)\gamma \quad & 0 \le i \le p \\ &= 0 & i > p \end{aligned} \tag{3}$$

where γ is any unknown parameter.Then, Almon (1965) proposed the polynomial lag weights of the rth degree

$$\beta_i = \gamma_0 + \gamma_1 i + \gamma_2 i^2 + \cdots + \gamma_r i^r \quad p \ge r \ge 0. \tag{4}$$

Equation (4) can be written in the matrix notation as

$$\beta = A\gamma \tag{5}$$

where

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$$A = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2^2 & \cdots & 2^r \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & p & p^2 & \vdots & p \end{bmatrix}, \quad \gamma = \begin{bmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_r \end{bmatrix}.$$

are $A : (p+1) \times (r+1)$ matrix and $\gamma : (r+1) \times 1$ vector. The ranks of matrices X and A are assumed to be $(p+1) < (T-p)$ and $(r+1) < (p+1)$, respectively. If $r < p$, then the rank of A is $r+1$. We estimate β in (2), under the nonstochastic prior information on β which is given by (5), using Almon estimation method. By substituting (5) in (2),

$$\begin{aligned} y &= XA\gamma + \varepsilon \\ &= W\gamma + \varepsilon, \quad \varepsilon \sim N(0, \sigma^2 I) \end{aligned} \tag{6}$$

is obtained. This model can be called a linear Almon distributed lag model.Then, the ordinary least squares (OLS) estimator of γ in model (6) is

$$\hat{\gamma}_A = (W'W)^{-1}W'y = (A'X'XA)^{-1}A'X'y. \tag{7}$$

In this case,

$$\hat{\beta}_A = A\hat{\gamma}_A \tag{8}$$

is the Almon estimator (AE) of β. $\hat{\beta}_A$ is the best linear unbiased estimator (BLUE) (see, also Vinod and Ullah (1981)).

In case of estimating model (1) by OLS, multicollinearity problem among the explanatory variables may be occurred because there are p lags of the same variables in the model. To overcome the multicollinearity problem, the following alternative biased estimator to the AE for the distributed lag model is introduced.

- Following Hoerl and Kennard's (1970a) method for defining ridge regression estimator, the Almon-ridge estimator (ARE) of γ in model (6) is defined as follows,

  $$\begin{aligned} \hat{\gamma}_k &= (W'W + kI)^{-1}W'y \\ &= (A'SA + kI)^{-1}A'X'y, \quad k > 0 \end{aligned} \tag{9}$$

  where $S = X'X$. Thus

  $$\hat{\beta}_k = A\hat{\gamma}_k \tag{10}$$

  is the ARE for model (2), (see, Maddala (1974), Vinod and Ullah (1981), Chanda and Maddala (1984) and Yeo and Trivedi (1989)).Then, Gültay and Kaçıranlar (2015) also introduced the following three other alternative estimators to the AE for the distributed lag model in order to overcome the multicollinearity problem.
- Following Swindel's (1976) method, the Almon-modified ridge estimator (AMRE) of γ in model (6) is defined as follows,

  $$\begin{aligned} \hat{\gamma}_m(k) &= (W'W + kI)^{-1}(W'y + kb_0) \\ &= T_k\hat{\gamma}_A + (I - T_k)\hat{\gamma}_k \end{aligned} \tag{11}$$

  where $b_0 = \hat{\gamma}_k$ and $T_k = (W'W + kI)^{-1}W'W$. Thus, AMRE of β in model (2) is $\hat{\beta}_m(k) = A\hat{\gamma}_m(k)$.
- Following Liu's (1993) method for defining the estimator which is called Liu estimator in Akdeniz and Kaciranlar (1995), the Almon-Liu estimator (ALE) of γ in model (6) is defined as follows,

  $$\begin{aligned} \hat{\gamma}_d &= (W'W + I)^{-1}(W'y + d\hat{\gamma}_A) \\ &= (A'SA + I)^{-1}(A'X'y + d\hat{\gamma}_A) \\ &= (A'SA + I)^{-1}(A'SA + dI)\hat{\gamma}_A \\ &= F_d\hat{\gamma}_A \end{aligned} \tag{12}$$

  where $F_d = (W'W + I)^{-1}(W'W + dI)$. Thus, the ALE of β is $\hat{\beta}_d = A\hat{\gamma}_d$. The comparison of $\hat{\gamma}_A$ with $\hat{\gamma}_d$ and the selection of d are given in Kaçıranlar (2010).
- Following Li and Yang's (2012) method, the Almon-modified Liu estimator (AMLE) of γ in model (6) is defined as follows,

  $$\begin{aligned} \hat{\gamma}_m(d) &= (W'W + I)^{-1}(W'W + dI)\hat{\gamma}_A + (1-d)(W'W + I)^{-1}b_0 \\ &= F_d\hat{\gamma}_A + (I - F_d)\hat{\gamma}_d \end{aligned} \tag{13}$$

  where $b_0 = \hat{\gamma}_d$. Thus, AMLE of β in model (2) is $\hat{\beta}_m(d) = A\hat{\gamma}_m(d)$.

2 Prediction Mean Squared Error under the Target Function

In this section, we will introduce the prediction mean square error (PMSE) under the target function.

Generally predictions from a linear regression model are made either for the actual values of the study variable or for the average values at a time. However, situations may occur in which one may be required to consider the predictions of both the actual and average values simultaneously.

If $\tilde{\beta}$ denotes an estimator of β, then the predictor for the values of study variable is generally formulated as $\hat{Y} = X\tilde{\beta}$ which is used for predicting either the actual values y or the average values $E(y) = X\beta$ at a time. When the situation demands prediction of both the actual and average values together, the target function is defined as follows,

$$T(y) = ty + (1-t)E(y) = Y^* \tag{14}$$

and use $\hat{Y} = X\tilde{\beta}$ for predicting it where $0 \le t \le 1$ is a nonstochastic scalar specifying the weightage to be assigned to the prediction of actual and average values of the study variable, see, Shalabh (1995). Toutenburg and Shalabh (1996) analyzed the performance properties of predictors arising from the methods of restricted regression and mixed regression besides least squares according to the target function. Then Toutenburg and Shalabh (2000) improved predictions in linear regression models with stochastic linear constraints in terms of the target function. Also, Shalabh et al. (2009) introduced the extended balanced loss function (EBLF) under the target function and discussed the stein rule estimation. In addition, Chaturvedi and Shalabh (2014) discussed the Bayesian estimation of regression coefficients under EBLF.

Gunst and Mason (1979) compared OLS, principal components and ridge regression estimators in terms of the integrated MSE using models with two explanatory variables. Friedman and Montgomery (1985) adopted the similar approach by focusing on the prediction of a new response y based on PMSE under linear regression model. They considered the predictive ability of the estimators evaluated at a particular observation. Then Özbey and Kaçıranlar (2015) also used the same criterion to evaluate the predictive performance of the Liu estimator.

Now, from Eq. (14), the target function at the point $x_0' = [1, x_{01}, x_{02}, \ldots, x_{0k}]$ is defined as follows:

$$y_0^* = t\,y_0 + (1-t)\,E(y_0) \tag{15}$$

where $y_0 = x_0'\beta + \varepsilon_0$.

Therefore, the predictive measure at the point $x_0' = [1, x_{01}, x_{02}, \ldots, x_{0k}]$ is

$$y_0^* - \hat{y}_0, \tag{16}$$

where $\hat{y}_0 = x_0'\tilde{\beta}$.

So, the PMSE which is a measure of the closeness of a predictor to the response being predicted under the target function is defined as follows:

$$\text{PMSE} = E(y_0^* - \hat{y}_0)^2. \tag{17}$$

Let J represents the PMSE. J is the sum of the variance (V) and the squared bias (B):

$$J = V + B \tag{18}$$

If y_0^* is the value to be predicted, and $\hat{y}_0$ is the prediction of that value, then the variance and the bias of the prediction error are

$$V(y_0^* - \hat{y}_0) = V(y_0^*) + V(\hat{y}_0) \tag{19}$$

and

$$\text{Bias} = E(y_0^* - \hat{y}_0). \tag{20}$$

3 Evaluations of Prediction Mean Squared Errors Under the Target Function

In this section, we will obtain the PMSEs of AE, ARE, ALE, AMRE and AMLE.For convenience, the canonical form of (6)

$$y = Z\alpha + \varepsilon, \quad \varepsilon \sim N(0, \sigma^2 \mathrm{I}) \tag{21}$$

will be used where $Z = WU$, $\alpha = U'\gamma$ and U is the orthogonal matrix whose columns constitute the eigenvectors of $W'W$. Then

$$Z'Z = U'W'WU = \Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{r+1}) \tag{22}$$

where $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{r+1} > 0$ are ordered eigenvalues of $W'W$. The AE of α in (21) is

$$\hat{\alpha}_A = (Z'Z)^{-1}Z'y = \Lambda^{-1}Z'y. \tag{23}$$

If z_0 is the orthonormalized point at which the prediction $\hat{y}_0$ is made. The variance of the prediction error of the AE is

$$\begin{aligned} V_A(y_0^* - \hat{y}_0) &= V(y_0^*) + V_A(\hat{y}_0) \\ &= \sigma^2 t^2 + V(z_0'\hat{\alpha}_A) \\ &= \sigma^2\left(t^2 + \sum_{i=1}^{r+1} \frac{z_{0i}^2}{\lambda_i}\right). \end{aligned} \tag{24}$$

Note that, since the AE is unbiased, its PMSE is equal to its prediction variance

$$J_A = V_A. \tag{25}$$

The ARE of α in (21) is

$$\hat{\alpha}_k = (Z'Z + kI)^{-1}Z'y = (\Lambda + k\mathrm{I})^{-1}Z'y, \quad k \geq 0. \tag{26}$$

The variance of the prediction error of the ARE is

$$\begin{aligned} V_k(y_0^* - \hat{y}_0) &= V(y_0^*) + V_k(\hat{y}_0) \\ &= \sigma^2 t^2 + V(z_0'\hat{\alpha}_k) \\ &= \sigma^2\left(t^2 + \sum_{i=1}^{r+1} \frac{z_{0i}^2\lambda_i}{a_i^2}\right). \end{aligned} \tag{27}$$

where $a_i = \lambda_i + k$. The bias of the prediction error of the ARE is

$$\begin{aligned} \text{Bias}_k = E(y_0^* - \hat{y}_0) &= z_0'\alpha - z_0'E(\hat{\alpha}_k) \\ &= k\sum_{i=1}^{r+1} \frac{z_{oi}\alpha_i}{a_i} \end{aligned} \tag{28}$$

so, the squared bias is

$$B_k = \text{Bias}_k^2 = k^2\left(\sum_{i=1}^{r+1}\frac{z_{oi}\alpha_i}{a_i}\right)^2. \tag{29}$$

By summing up the variance and the squared bias of the ARE, we obtain

$$\begin{aligned} J_k &= V_k + B_k \\ &= \sigma^2\left(1+\sum_{i=1}^{r+1}\frac{z_{0i}^2\lambda_i}{a_i^2}\right)+k^2\left(\sum_{i=1}^{r+1}\frac{z_{oi}\alpha_i}{a_i}\right)^2. \end{aligned} \tag{30}$$

The ALE of α in (21) is

$$\begin{aligned} \hat{\alpha}_d &= (Z'Z+\mathrm{I})^{-1}(Z'y+d\hat{\alpha}) \\ &= (\Lambda+\mathrm{I})^{-1}(\Lambda+dI)\hat{\alpha}, \quad 0<d<1 \end{aligned} \tag{31}$$

The variance of the prediction error of the ALE is

$$\begin{aligned} V_d(y_0^*-\hat{y}_0) &= V(y_0^*)+V_d(\hat{y}_0) \\ &= \sigma^2 t^2 + V(z_0'\,\hat{\alpha}_d) \\ &= \sigma^2\left(t^2+\sum_{i=1}^{r+1}\frac{z_{0i}^2 c_i^2}{\lambda_i b_i^2}\right). \end{aligned} \tag{32}$$

where $b_i = \lambda_i + 1$ and $c_i = \lambda_i + d$. The bias of the prediction error of the ALE is

$$\begin{aligned} \text{Bias}_d = E(y_0^*-\hat{y}_0) &= z_0'\,\alpha - z_0'\,E(\hat{\alpha}_d) \\ &= (1-d)\sum_{i=1}^{r+1}\frac{z_{oi}\alpha_i}{b_i} \end{aligned} \tag{33}$$

so, the squared bias is

$$B_d = \text{Bias}_d^2 = (1-d)^2\left(\sum_{i=1}^{r+1}\frac{z_{oi}\alpha_i}{b_i}\right)^2. \tag{34}$$

By summing up the variance and the squared bias of the ARE, we obtain

$$\begin{aligned} J_d &= V_d + B_d \\ &= \sigma^2\left(t^2+\sum_{i=1}^{r+1}\frac{z_{0i}^2 c_i^2}{\lambda_i b_i^2}\right)+(1-d)^2\left(\sum_{i=1}^{r+1}\frac{z_{oi}\alpha_i}{b_i}\right)^2. \end{aligned} \tag{35}$$

The AMRE of α in (21) is

$$\begin{aligned} \hat{\alpha}_m(k) &= (\Lambda+k\mathrm{I})^{-1}(Z'y+k\hat{\alpha}_k) \\ &= (\Lambda+k\mathrm{I})^{-1}\Lambda\hat{\alpha}_A + k\,(\Lambda+k\mathrm{I})^{-1}\hat{\alpha}_k \\ &= \left[(\Lambda+k\mathrm{I})^{-1}+k\,(\Lambda+k\mathrm{I})^{-2}\right]Z'\,y. \end{aligned} \tag{36}$$

The variance of the prediction error of the AMRE is

$$\begin{aligned} V_{mk}(y_0^*-\hat{y}_0) &= V(y_0^*)+V_{mk}(\hat{y}_0) \\ &= \sigma^2 t^2 + V(z_0'\,\hat{\alpha}_m(k)) \\ &= \sigma^2\left(t^2+\sum_{i=1}^{r+1}z_{0i}^2\frac{\lambda_i(a_i+k)^2}{a_i^4}\right). \end{aligned} \tag{37}$$

The bias of the prediction error of the AMRE is

$$\begin{aligned} \text{Bias}_{mk} = E(y_0^*-\hat{y}_0) &= z_0'\,\alpha - z_0'\,E(\hat{\alpha}_m(k)) \\ &= k^2\sum_{i=1}^{r+1}\frac{z_{oi}\alpha_i}{a_i^2} \end{aligned} \tag{38}$$

so, the squared bias is

$$B_{mk} = Bias_{mk}^2 = k^4\left(\sum_{i=1}^{r+1}\frac{z_{oi}\alpha_i}{a_i^2}\right)^2. \tag{39}$$

By summing up the variance and the squared bias of the AMRE, we obtain

$$\begin{aligned} J_{mk} &= V_{mk} + B_{mk} \\ &= \sigma^2\left(t^2+\sum_{i=1}^{r+1}z_{0i}^2\frac{\lambda_i(a_i+k)^2}{a_i^4}\right)+k^4\left(\sum_{i=1}^{r+1}\frac{z_{oi}\alpha_i}{a_i^2}\right)^2. \end{aligned} \tag{40}$$

The AMLE of α in (21) is

$$\begin{aligned} \hat{\alpha}_m(k) &= \left[(\Lambda+I)^{-1}(\Lambda+dI)\right]\hat{\alpha}_A + \left[I-(\Lambda+I)^{-1}(\Lambda+dI)\right]\hat{\alpha}_d \\ &= L_d\,\hat{\alpha}_A + (I-L_d)\,\hat{\alpha}_d \\ &= (2L_d - L_d^2)\,\hat{\alpha}_A \\ &= (2L_d - L_d^2)\,\Lambda^{-1}Z'\,y. \end{aligned} \tag{41}$$

The variance of the prediction error of the AMLE is

$$\begin{aligned} V_{md}(y_0^*-\hat{y}_0) &= V(y_0^*)+V_{md}(\hat{y}_0) \\ &= \sigma^2 t^2 + V(z_0'\,\hat{\alpha}_m(d)) \\ &= \sigma^2\left(t^2+\sum_{i=1}^{r+1}z_{0i}^2\frac{c_i^2(c_i-2b_i)^2}{\lambda_i\,b_i^4}\right). \end{aligned} \tag{42}$$

The bias of the prediction error of the AMLE is

$$\begin{aligned} \text{Bias}_{md} = E(y_0^*-\hat{y}_0) &= z_0'\,\alpha - z_0'\,E(\hat{\alpha}_m(d)) \\ &= (d-1)^2\sum_{i=1}^{r+1}\frac{z_{oi}\alpha_i}{b_i^2} \end{aligned} \tag{43}$$

so, the squared bias is

$$B_{mk} = \text{Bias}_{mk}^2 = (d-1)^4\left(\sum_{i=1}^{r+1}\frac{z_{oi}\alpha_i}{b_i^2}\right)^2. \tag{44}$$

By summing up the variance and the squared bias of the AMLE, we obtain

$$J_{md} = V_{md} + B_{md}$$
$$= \sigma^2\left(t^2 + \sum_{i=1}^{r+1} z_{0i}^2 \frac{c_i^2(c_i - 2b_i)^2}{\lambda_i b_i^4}\right) + (d-1)^4\left(\sum_{i=1}^{r+1}\frac{z_{oi}\alpha_i}{b_i^2}\right)^2. \tag{45}$$

4 Superiority of the Biased Estimators Under the PMSE Criterion with the Target Function

Since AMRE and AMLE are biased alternatives to the AE in the presence of multicollinearity, we will discuss the predictive performance of the AMRE and AMLE. Following Friedman and Montgomery's (1985) and Özbey and Kaçıranlar's (2015) method for making comparisons among estimators using the two-dimensional spaces, we will focus on obtaining the ratio z_{02}^2/z_{01}^2 to use it as the reference point in our comparisons as well as α_1^2 will be set to zero because nonzero values of α_1^2 increase only the intercept values for J_k, J_d, J_{mk} and J_{md} but leave the curve for J_A unchanged. In the following five subsections, we will compare the AMRE with the AE and the ARE. Also, AMLE is compared to the AE and the ALE. In addition to these, AMLE and AMRE are compared in terms of PMSE criterion under the target function. Furthermore, in the last subsection, we will give the method for choosing the biasing parameters k and d for the above mentioned estimators.

4.1 The Comparison of AMRE and AE

In this subsection, we will discuss the superiority of the AMRE over the AE in terms of PMSE criterion under the target function.

Theorem 1

1. *If* $\alpha_2^2 < \frac{\sigma^2(2\lambda_2^2+4k\lambda_2+k^2)}{\lambda_2 k^2}$, *then* $J_{mk} < J_A$.
2. *If* $\alpha_2^2 > \frac{\sigma^2(2\lambda_2^2+4k\lambda_2+k^2)}{\lambda_2 k^2}$, *then* $J_{mk} < J_A$ iff $\frac{z_{02}^2}{z_{01}^2} < f_1(\alpha_2^2)$. *where*

$$f_1(\alpha_2^2) = \frac{\sigma^2\left(\frac{1}{\lambda_1} - \frac{\lambda_1(a_1+k)^2}{a_1^4}\right)}{\left(\frac{\sigma^2\lambda_2(a_2+k)^2}{a_2^4} + \frac{k^4\alpha_2^2}{a_2^4} - \frac{\sigma^2}{\lambda_2}\right)}. \tag{46}$$

Proof If the AMRE is superior to the AE in term of PMSE criterion, we have $J_{mk} < J_A$. That is, $\sigma^2 t^2 + \sigma^2\left(\frac{\lambda_1(a_1+k)^2 z_{01}^2}{a_1^4} + \frac{\lambda_2(a_2+k)^2 z_{02}^2}{a_2^4}\right) + \frac{k^4\alpha_2^2 z_{02}^2}{a_2^4} < \sigma^2 t^2 + \sigma^2 \left(\frac{z_{01}^2}{\lambda_1} + \frac{z_{02}^2}{\lambda_2}\right)$. Rearranging this inequality, we will obtain

$$z_{02}^2\left(\frac{\sigma^2\lambda_2(a_2+k)^2}{a_2^4} + \frac{k^4\alpha_2^2}{a_2^4} - \frac{\sigma^2}{\lambda_2}\right) < z_{01}^2\sigma^2\left(\frac{1}{\lambda_1} - \frac{\lambda_1(a_1+k)^2}{a_1^4}\right).$$

Here, z_{01}^2, z_{02}^2 and $\sigma^2\left(\frac{1}{\lambda_1} - \frac{\lambda_1(a_1+k)^2}{a_1^4}\right)$ are positive, but the sign of $\left(\frac{\sigma^2\lambda_2(a_2+k)^2}{a_2^4} + \frac{k^4\alpha_2^2}{a_2^4} - \frac{\sigma^2}{\lambda_2}\right)$ depends on the value of α_2^2. Let's define $\sigma^2\left(\frac{1}{\lambda_1} - \frac{\lambda_1(a_1+k)^2}{a_1^4}\right) \Big/ \left(\frac{\sigma^2\lambda_2(a_2+k)^2}{a_2^4} + \frac{k^4\alpha_2^2}{a_2^4} - \frac{\sigma^2}{\lambda_2}\right)$ as a function of α_2^2 and denote it by $f_1(\alpha_2^2)$. The function $f_1(\alpha_2^2)$ has a vertical asymptote at the point

$$\left(\frac{\sigma^2\lambda_2(a_2+k)^2}{a_2^4} + \frac{k^4\alpha_2^2}{a_2^4} - \frac{\sigma^2}{\lambda_2}\right). \tag{47}$$

From this Equation, we get

$$\alpha_2^2 = \frac{\sigma^2(2\lambda_2^2 + 4k\lambda_2 + k^2)}{\lambda_2 k^2}. \tag{48}$$

Thus,

1. If
$$\alpha_2^2 < \frac{\sigma^2(2\lambda_2^2 + 4k\lambda_2 + k^2)}{\lambda_2 k^2} \tag{49}$$
we get
$$\frac{z_{02}^2}{z_{01}^2} > f_1(\alpha_2^2). \tag{50}$$
Because z_{02}^2/z_{01}^2 is always positive, and for the condition given in (49) $f_1(\alpha_2^2)$ is always negative, the AMRE is uniformly superior to the AE.
2. If
$$\alpha_2^2 > \frac{\sigma^2(2\lambda_2^2 + 4k\lambda_2 + k^2)}{\lambda_2 k^2} \tag{51}$$
we get
$$\frac{z_{02}^2}{z_{01}^2} < f_1(\alpha_2^2). \tag{52}$$
Because $f_1(\alpha_2^2)$ is positive for the condition given in (51), the AMRE is uniformly superior to the AE when (52) is valid. □

4.2 The Comparison of AMRE and ARE

In this subsection, we will discuss the superiority of the AMRE over the ARE in terms of PMSE criterion under the target function.

Theorem 2

a. *If* $\alpha_2^2 > \frac{\sigma^2(2\lambda_2+3k)}{k(\lambda_2+2k)}$, *then* $J_{mk} < J_k$.

b. *If* $\alpha_2^2 < \frac{\sigma^2(2\lambda_2+3k)}{k(\lambda_2+2k)}$, *then* $J_{mk} < J_k$ iff $\frac{z_{02}^2}{z_{01}^2} < f_2(\alpha_2^2)$.
where

$$f_2(\alpha_2^2) = \frac{\sigma^2\left(\frac{\lambda_1}{a_1^2} - \frac{\lambda_1(a_1+k)^2}{a_1^4}\right)}{\left(\frac{\sigma^2\lambda_2(a_2+k)^2}{a_2^4} + \frac{k^4\alpha_2^2}{a_2^4} - \frac{\sigma^2\lambda_2}{a_2^2} - \frac{k^2\alpha_2^2}{a_2^2}\right)}. \quad (53)$$

Proof If the AMRE is superior to the ARE in term of PMSE criterion, we have $J_{mk} < J_k$. That is,

$$\sigma^2 t^2 + \sigma^2\left(\frac{\lambda_1(a_1+k)^2 z_{01}^2}{a_1^4} + \frac{\lambda_2(a_2+k)^2 z_{02}^2}{a_2^4}\right) + \frac{k^4\alpha_2^2 z_{02}^2}{a_2^4} < \sigma^2 t^2 + \sigma^2\left(\frac{\lambda_1 z_{01}^2}{a_1^2} + \frac{\lambda_2 z_{02}^2}{a_2^2}\right) + \frac{k^2\alpha_2^2}{a_2^2}.$$

Rearranging this inequality, we will obtain$z_{02}^2\left(\frac{\sigma^2\lambda_2(a_2+k)^2}{a_2^4} + \frac{k^4\alpha_2^2}{a_2^4} - \frac{\sigma^2\lambda_2}{a_2^2} - \frac{k^2\alpha_2^2}{a_2^2}\right) < z_{01}^2\sigma^2\left(\frac{\lambda_1}{a_1^2} - \frac{\lambda_1(a_1+k)^2}{a_1^4}\right)$.

Here, z_{01}^2, z_{02}^2 are positive and $\sigma^2\left(\frac{\lambda_1}{a_1^2} - \frac{\lambda_1(a_1+k)^2}{a_1^4}\right)$ is always negative, but the sign of $\left(\frac{\sigma^2\lambda_2(a_2+k)^2}{a_2^4} + \frac{k^4\alpha_2^2}{a_2^4} - \frac{\sigma^2\lambda_2}{a_2^2} - \frac{k^2\alpha_2^2}{a_2^2}\right)$ depends on the value of α_2^2. Let's define $\sigma^2\left(\frac{\lambda_1}{a_1^2} - \frac{\lambda_1(a_1+k)^2}{a_1^4}\right) \Big/ \left(\frac{\sigma^2\lambda_2(a_2+k)^2}{a_2^4} + \frac{k^4\alpha_2^2}{a_2^4} - \frac{\sigma^2\lambda_2}{a_2^2} - \frac{k^2\alpha_2^2}{a_2^2}\right)$ as a function of α_2^2 and denote it by $f_2(\alpha_2^2)$. The function $f_2(\alpha_2^2)$ has a vertical asymptote at the point

$$\left(\frac{\sigma^2\lambda_2(a_2+k)^2}{a_2^4} + \frac{k^4\alpha_2^2}{a_2^4} - \frac{\sigma^2\lambda_2}{a_2^2} - \frac{k^2\alpha_2^2}{a_2^2}\right). \quad (54)$$

From this Equation, we get

$$\alpha_2^2 = \frac{\sigma^2(2\lambda_2+3k)}{k(\lambda_2+2k)}. \quad (55)$$

Thus,

1. If

$$\alpha_2^2 > \frac{\sigma^2(2\lambda_2+3k)}{k(\lambda_2+2k)} \quad (56)$$

we get

$$\frac{z_{02}^2}{z_{01}^2} > f_2(\alpha_2^2). \quad (57)$$

Because z_{02}^2/z_{01}^2 is always positive, and for the condition given in (56) $f_2(\alpha_2^2)$ is always negative, the AMRE is uniformly superior to the ARE.

2. If

$$\alpha_2^2 < \frac{\sigma^2(2\lambda_2+3k)}{k(\lambda_2+2k)} \quad (58)$$

we get

$$\frac{z_{02}^2}{z_{01}^2} < f_2(\alpha_2^2). \quad (59)$$

Because $f_2(\alpha_2^2)$ is positive for the condition given in (58), the AMRE is uniformly superior to the AE when (59) is valid. □

4.3 The Comparison of AMLE and AE

In this subsection, we will discuss the superiority of the AMLE over the AE in terms of PMSE criterion under the target function.

Theorem 3

a. *If* $\alpha_2^2 > \frac{\sigma^2\left(b_2^2(b_2^2-4c_2^2)+c_2^3(4b_2-c_2)\right)}{\lambda_2(d-1)^4}$, *then* $J_{md} < J_A$ *for* $b_1^2(b_1^2 - 4c_1^2) < c_1^3(c_1 - 4b_1)$ *for* $b_1^2(b_1^2 - 4c_1^2) > c_1^3(c_1 - 4b_1)$ *iff* $\frac{z_{02}^2}{z_{01}^2} < f_3(\alpha_2^2)$.

b. *If* $\alpha_2^2 < \frac{\sigma^2\left(b_2^2(b_2^2-4c_2^2)+c_2^3(4b_2-c_2)\right)}{\lambda_2(d-1)^4}$, *then* $J_{md} < J_A$ *for* $b_1^2(b_1^2 - 4c_1^2) > c_1^3(c_1 - 4b_1)$ *for* $b_1^2(b_1^2 - 4c_1^2) < c_1^3(c_1 - 4b_1)$ *iff* $\frac{z_{02}^2}{z_{01}^2} < f_3(\alpha_2^2)$.
where

$$f_3(\alpha_2^2) = \frac{\sigma^2\left(\frac{1}{\lambda_1} - \frac{c_1^2(c_1-2b_1)^2}{\lambda_1 b_1^4}\right)}{\left(\frac{\sigma^2 c_2^2(c_2-2b_2)^2}{\lambda_2 b_2^4} + \frac{(d-1)^4\alpha_2^2}{b_2^4} - \frac{\sigma^2}{\lambda_2}\right)}. \quad (60)$$

Proof If the AMLE is superior to the AE in term of PMSE criterion, we have $J_{md} < J_A$. That is,

$$\sigma^2 t^2 + \sigma^2\left(\frac{c_1^2(c_1-2b_1)^2 z_{01}^2}{\lambda_1 b_1^4} + \frac{c_2^2(c_2-2b_2)^2 z_{02}^2}{\lambda_2 b_2^4}\right) + \frac{(d-1)^4\alpha_2^2}{b_2^4} < \sigma^2 t^2 + \sigma^2\left(\frac{z_{01}^2}{\lambda_1} + \frac{z_{02}^2}{\lambda_2}\right).$$

Rearranging this inequality, we will obtain $z_{02}^2\left(\frac{\sigma^2 c_2^2(c_2-2b_2)^2}{\lambda_2 b_2^4} + \frac{(d-1)^4\alpha_2^2}{b_2^4} - \frac{\sigma^2}{\lambda_2}\right) < z_{01}^2\sigma^2\left(\frac{1}{\lambda_1} - \frac{c_1^2(c_1-2b_1)^2}{\lambda_1 c_1^4}\right)$.

If both

$$\frac{\sigma^2 c_2^2(c_2-2b_2)^2}{\lambda_2 b_2^4} + \frac{(d-1)^4\alpha_2^2}{b_2^4} - \frac{\sigma^2}{\lambda_2} \quad (61)$$

and

$$\frac{1}{\lambda_1} - \frac{c_1^2(c_1-2b_1)^2}{\lambda_1 c_1^4} \quad (62)$$

have the same signs, the condition for superiority of the AMLE over the AE is

$$\frac{z_{02}^2}{z_{01}^2} < f_3(\alpha_2^2). \tag{63}$$

If (61) and (62) have opposite signs, the condition for superiority of the AMLE over the AE is

$$\frac{z_{02}^2}{z_{01}^2} > f_3(\alpha_2^2). \tag{64}$$

It is obvious that if (61) and (62) have opposite signs, the right hand side of (64) is negative, thus (64) always holds. Consequently, at that region the AMLE is uniformly superior to the AE. The condition for positiveness of (61) can be written as

$$\alpha_2^2 > \frac{\sigma^2\left(b_2^2(b_2^2-4c_2^2)+c_2^3(4b_2-c_2)\right)}{\lambda_2(d-1)^4} \tag{65}$$

and the condition for positiveness of (62) can be written as

$$b_1^2(b_1^2-4c_1^2) > c_1^3(c_1-4b_1). \tag{66}$$

Of course, the opposite conditions are needed for the negativeness of (61) and (62). The vertical asymptote of the hyperbola $f_3(\alpha_2^2)$ is at the point

$$\alpha_2^2 = \frac{\sigma^2\left(b_2^2(b_2^2-4c_2^2)+c_2^3(4b_2-c_2)\right)}{\lambda_2(d-1)^4}. \tag{67}$$

□

4.4 The Comparison of AMLE and ALE

In this subsection, we will discuss the superiority of the AMLE over the ALE in terms of PMSE criterion under the target function.

Theorem 4

a. *If* $\alpha_2^2 > \frac{\sigma^2\left(c_2^3(4b_2-c_2)-3b_2^2c_2^2\right)}{\lambda_2(d-1)^2\left[(d-1)^2-b_2^2\right]}$, *then* $J_{md} < J_d$ *for* $c_1(4b_1-c_1) < 3b_1^2$ *for* $c_1(4b_1-c_1) > 3b_1^2$ *iff* $\frac{z_{02}^2}{z_{01}^2} < f_4(\alpha_2^2)$.

b. *If* $\alpha_2^2 < \frac{\sigma^2\left(c_2^3(4b_2-c_2)-3b_2^2c_2^2\right)}{\lambda_2(d-1)^2\left[(d-1)^2-b_2^2\right]}$, *then* $J_{md} < J_d$ *for* $c_1(4b_1-c_1) > 3b_1^2$ *for* $c_1(4b_1-c_1) < 3b_1^2$ iff $\frac{z_{02}^2}{z_{01}^2} < f_4(\alpha_2^2)$.

where

$$f_4(\alpha_2^2) = \frac{\sigma^2\left(\frac{c_1^2}{\lambda_1 b_1^2}-\frac{c_1^2(c_1-2b_1)^2}{\lambda_1 c_1^4}\right)}{\left(\frac{\sigma^2c_2^2(c_2-2b_2)^2}{\lambda_2 b_2^4}+\frac{(d-1)^4\alpha_2^2}{b_2^4}-\frac{\sigma^2c_2^2}{\lambda_2 b_2^2}-\frac{(d-1)^2\alpha_2^2}{b_2^2}\right)}. \tag{68}$$

Proof If the AMLE is superior to the ALE in term of PMSE criterion, we have $J_{md} < J_d$. That is,

$$\sigma^2t^2+\sigma^2\left(\frac{c_1^2(c_1-2b_1)^2z_{01}^2}{\lambda_1 b_1^4}+\frac{c_2^2(c_2-2b_2)^2z_{02}^2}{\lambda_2 b_2^4}\right)+\frac{(d-1)^4\alpha_2^2}{b_2^4} < \sigma^2t^2+\sigma^2\left(\frac{c_1^2z_{01}^2}{\lambda_1 b_1^2}+\frac{c_2^2z_{02}^2}{\lambda_2 b_2^2}\right)+\frac{(d-1)^2\alpha_2^2}{b_2^2}$$

Rearranging this inequality, we will obtain

$$z_{02}^2\left(\frac{\sigma^2c_2^2(c_2-2b_2)^2}{\lambda_2 b_2^4}+\frac{(d-1)^4\alpha_2^2}{b_2^4}-\frac{\sigma^2c_2^2}{\lambda_2 b_2^2}-\frac{(d-1)^2\alpha_2^2}{b_2^2}\right) < z_{01}^2\sigma^2\left(\frac{c_1^2}{\lambda_1 b_2^2}-\frac{c_1^2(c_1-2b_1)^2}{\lambda_1 b_1^4}\right).$$

If both

$$\frac{\sigma^2c_2^2(c_2-2b_2)^2}{\lambda_2 b_2^4}+\frac{(d-1)^4\alpha_2^2}{b_2^4}-\frac{\sigma^2c_2^2}{\lambda_2 b_2^2}-\frac{(d-1)^2\alpha_2^2}{b_2^2} \tag{69}$$

and

$$\frac{c_1^2}{\lambda_1 b_2^2}-\frac{c_1^2(c_1-2b_1)^2}{\lambda_1 b_1^4} \tag{70}$$

have the same signs, the condition for superiority of the AMLE over the ALE is

$$\frac{z_{02}^2}{z_{01}^2} < f_4(\alpha_2^2). \tag{71}$$

If (69) and (70) have opposite signs, the condition for superiority of the AMLE over the ALE is

$$\frac{z_{02}^2}{z_{01}^2} > f_4(\alpha_2^2). \tag{72}$$

It is obvious that if (69) and (70) have opposite signs, the right hand side of (72) is negative, thus (72) always holds. Consequently, at that region the AMLE is uniformly superior to the ALE. The condition for positiveness of (69) can be written as

$$\alpha_2^2 > \frac{\sigma^2\left(c_2^3(4b_2-c_2)-3b_2^2c_2^2\right)}{\lambda_2(d-1)^2\left[(d-1)^2-b_2^2\right]} \tag{73}$$

and the condition for positiveness of (70) can be written as

$$c_1(4b_1-c_1) > 3b_1^2. \tag{74}$$

Of course, the opposite conditions are needed for the negativeness of (69) and (70). The vertical asymptote of the hyperbola $f_4(\alpha_2^2)$ is at the point

$$\alpha_2^2 = \frac{\sigma^2\left(c_2^3(4b_2-c_2)-3b_2^2c_2^2\right)}{\lambda_2(d-1)^2\left[(d-1)^2-b_2^2\right]}. \tag{75}$$

□

4.5 The Comparison of AMLE and AMRE

In this subsection, we will discuss the superiority of the AMLE over the AMRE in terms of PMSE criterion under the target function.

Theorem 5

a. *If* $\alpha_2^2 > \frac{\sigma^2\left(b_2^4\lambda_2^2(a_2+k)^2 - a_2^4c_2^2(2b_2-c_2)^2\right)}{\lambda_2\left[(d-1)^4a_2^4 - k^4b_2^4\right]}$, *then* $J_{md} < J_{mk}$ *for* $b_1^4\lambda_1^2(a_1+k)^2 < a_1^4c_1^2(2b_1-c_1)^2$ *for* $b_1^4\lambda_1^2(a_1+k)^2 > a_1^4c_1^2(2b_1-c_1)^2$ *iff* $\frac{z_{02}^2}{z_{01}^2} < f_5(\alpha_2^2)$.

b. *If* $\alpha_2^2 < \frac{\sigma^2\left(b_2^4\lambda_2^2(a_2+k)^2 - a_2^4c_2^2(2b_2-c_2)^2\right)}{\lambda_2\left[(d-1)^4a_2^4 - k^4b_2^4\right]}$, *then* $J_{md} < J_{mk}$ *for* $b_1^4\lambda_1^2(a_1+k)^2 > a_1^4c_1^2(2b_1-c_1)^2$ *for* $b_1^4\lambda_1^2(a_1+k)^2 < a_1^4c_1^2(2b_1-c_1)^2$ *iff* $\frac{z_{02}^2}{z_{01}^2} < f_5(\alpha_2^2)$.

where

$$f_5(\alpha_2^2) = \frac{\sigma^2\left(\frac{\lambda_1(a_1+k)^2}{a_1^4} - \frac{c_1^2(c_1-2b_1)^2}{\lambda_1b_1^4}\right)}{\left(\frac{\sigma^2c_2^2(c_2-2b_2)^2}{\lambda_2b_2^4} + \frac{(d-1)^4\alpha_2^2}{b_2^4} - \frac{\sigma^2\lambda_2(a_2+k)^2}{a_2^4} - \frac{k^4\alpha_2^2}{a_2^4}\right)}. \tag{76}$$

Proof If the AMLE is superior to the AMRE in term of PMSE criterion, we have $J_{md} < J_{mk}$. That is, $\sigma^2t^2 + \sigma^2\left(\frac{c_1^2(c_1-2b_1)^2z_{01}^2}{\lambda_1b_1^4} + \frac{c_2^2(c_2-2b_2)^2z_{02}^2}{\lambda_2b_2^4}\right) + \frac{(d-1)^4\alpha_2^2}{b_2^4} < \sigma^2t^2 + \sigma^2\left(\frac{\lambda_1(a_1+k)^2}{a_1^4} + \frac{\lambda_2(a_2+k)^2}{a_2^4}\right) + \frac{k^4\alpha_2^2}{a_2^4}$. Rearranging this inequality, we will obtain

$$z_{02}^2\left(\frac{\sigma^2c_2^2(c_2-2b_2)^2}{\lambda_2b_2^4} + \frac{(d-1)^4\alpha_2^2}{b_2^4} - \frac{\sigma^2\lambda_2(a_2+k)^2}{a_2^4} - \frac{k^4\alpha_2^2}{a_2^4}\right) < z_{01}^2\sigma^2\left(\frac{\lambda_1(a_1+k)^2}{a_1^4} - \frac{c_1^2(c_1-2b_1)^2}{\lambda_1b_1^4}\right).$$

If both

$$\frac{\sigma^2c_2^2(c_2-2b_2)^2}{\lambda_2b_2^4} + \frac{(d-1)^4\alpha_2^2}{b_2^4} - \frac{\sigma^2\lambda_2(a_2+k)^2}{a_2^4} - \frac{k^4\alpha_2^2}{a_2^4} \tag{77}$$

and

$$\frac{\lambda_1(a_1+k)^2}{a_1^4} - \frac{c_1^2(c_1-2b_1)^2}{\lambda_1b_1^4} \tag{78}$$

have the same signs, the condition for superiority of the AMLE over the AMRE is

$$\frac{z_{02}^2}{z_{01}^2} < f_4(\alpha_2^2). \tag{79}$$

If (77) and (78) have opposite signs, the condition for superiority of the AMLE over the AMRE is

$$\frac{z_{02}^2}{z_{01}^2} > f_4(\alpha_2^2). \tag{80}$$

It is obvious that if (77) and (78) have opposite signs, the right hand side of (80) is negative, thus (80) always holds. Consequently, at that region the AMLE is uniformly superior to the AMRE. The condition for positiveness of (77) can be written as

$$\alpha_2^2 > \frac{\sigma^2\left(b_2^4\lambda_2^2(a_2+k)^2 - a_2^4c_2^2(2b_2-c_2)^2\right)}{\lambda_2\left[(d-1)^4a_2^4 - k^4b_2^4\right]} \tag{81}$$

and the condition for positiveness of (78) can be written as

$$b_1^4\lambda_1^2(a_1+k)^2 > a_1^4c_1^2(2b_1-c_1)^2. \tag{82}$$

Of course, the opposite conditions are needed for the negativeness of (77) and (78). The vertical asymptote of the hyperbola $f_4(\alpha_2^2)$ is at the point

$$\alpha_2^2 = \frac{\sigma^2\left(b_2^4\lambda_2^2(a_2+k)^2 - a_2^4c_2^2(2b_2-c_2)^2\right)}{\lambda_2\left[(d-1)^4a_2^4 - k^4b_2^4\right]}. \tag{83}$$

□

4.6 The Method for Choosing the Biasing Parameters *k* and *d* in the Above Mentioned Estimators

Now, a very important issue in the study of ridge regression is how to find an appropriate biasing parameter k. Hoerl and Kennard (1970a, b), Hoerl et al. (1975) and Lawless and Wang (1976) suggested the following ridge parameters, that we can estimate for model (21), respectively;

$$\hat{k}_{HK} = \frac{\hat{\sigma}^2}{\sum_{i=1}^{r+1}\hat{\alpha}_i^2}, \tag{84}$$

$$\hat{k}_{HKB} = \frac{(r+1)\hat{\sigma}^2}{\sum_{i=1}^{r+1}\hat{\alpha}_i^2}, \tag{85}$$

$$\hat{k}_{LW} = \frac{(r+1)\,\hat{\sigma}^2}{\sum_{i=1}^{r+1}\lambda_i\hat{\alpha}_i^2} \tag{86}$$

where $\hat{\alpha}$ and $\hat{\sigma}^2$ are the OLS estimates of α and σ^2, respectively. On the other hand Liu (1993) gave the some estimates of d by analogy with the estimate of k in ridge estimate. Two of these estimates are defined as for model (21):

$$\hat{d}_{mm} = 1 - \hat{\sigma}^2 \left[\sum_{i=1}^{r+1} \frac{1}{\lambda_i(\lambda_i + 1)} \Bigg/ \sum_{i=1}^{r+1} \frac{\hat{\alpha}_i^2}{(\lambda_i + 1)^2} \right], \quad (87)$$

$$\hat{d}_{\mathrm{CL}} = 1 - \hat{\sigma}^2 \left[\sum_{i=1}^{r+1} \frac{1}{\lambda_i + 1} \Bigg/ \sum_{i=1}^{r+1} \frac{\lambda_i \hat{\alpha}_i^2}{(\lambda_i + 1)^2} \right]. \quad (88)$$

5 An Illustrative Example with Almon Data

To illustrate our theoretical results, we now consider a dataset due to Almon (1965). These data was taken in the years 1953–1967 using quarterly data where independent variable is appropriations and dependent variable is expenditures. Gültay and Kaçıranlar (2015) found the following results:

a. The eigenvalues of $W'W$: (2.9359, 0.0634, 0.0007),
b. The Almon estimates of $\gamma : (\hat{\gamma}_A)' = (-0.0052, 0.0320, 0.0962)$,
c. The estimates of $\sigma^2 : \hat{\sigma}^2 = 0.0164$.

The 3×3 matrix U is the matrix of normalized eigenvectors, Λ is a 3×3 diagonal matrix of eigenvalues of $W'W$ such that $W'W = U\Lambda U'$. Then, $Z = WU$ and $\alpha = U'\gamma$ so that $y = W\gamma + \varepsilon = Z\alpha + \varepsilon$ where

$$Z'Z = \Lambda = \begin{pmatrix} 2.9359 & 0 & 0 \\ 0 & 0.0634 & 0 \\ 0 & 0 & 0.0007 \end{pmatrix}. \quad (89)$$

In orthogonal coordinates, the OLS-estimator of the regression coefficients obtained is

$$\hat{\alpha} = \Lambda^{-1} Z'y = [0.5580, \ -1.0754, \ 1.2297]'. \quad (90)$$

Also, $k_{\mathrm{HKB}} = 0.0165$ and $d_{\mathrm{CL}} = 0.712$ are obtained by Gültay and Kaçıranlar (2015).

Now, we will illustrate our theoretical results using the given results above.

Firstly, let us consider the predictive performances of AMRE and AE.

- If $\lambda_1 = 2.9359$, $\lambda_2 = 0.0634$, from (46), we get

$$f_1(\alpha_2^2) = \frac{0.0000003489}{0.001818648\,\alpha_2^2 - 0.02159} \quad (91)$$

which is a hyperbola with a vertical asymptote at

$$\alpha_2^2 \cong 11.8727. \quad (92)$$

Because both z_{02}^2/z_{01}^2 and α_2^2 are positive, we are interested only in the points which lie in quadrant I. Figure 1 illustrates this situation. For values of α_2^2 smaller than 11.8727, the AMRE is uniformly superior to the AE. For larger values of α_2^2, there is a trade-off between these two estimators. If the value of the ratio z_{02}^2/z_{01}^2 is smaller than the value of $f_1(\alpha_2^2)$, then the AMRE is superior to the AE, otherwise the AE is superior to the AMRE. As Friedman and Montgomery (1985) and Özbey and Kaçıranlar (2015) pointed out, the ratio z_{02}^2/z_{01}^2 defines the subspace of observation to be predicted.

- If $\lambda_1 = 2.9359$, $\lambda_2 = 0.0007$, from (46), we get

$$f_1(\alpha_2^2) = \frac{0.0000003489}{0.84688022\,\alpha_2^2 - 23.2796}. \quad (93)$$

- If $\lambda_1 = 0.0634$, $\lambda_2 = 0.0007$, from (46), we get

$$f_1(\alpha_2^2) = \frac{0.021592274}{0.84688022\,\alpha_2^2 - 23.2796} \quad (94)$$

which are the hyperbolas with the same vertical asymptote at

$$\alpha_2^2 \cong 27.48866. \quad (95)$$

Figures 2 and 3 illustrate this situation, respectively. For values of α_2^2 smaller than 27.48866, the AMRE is uniformly superior to the AE. For larger values of α_2^2,

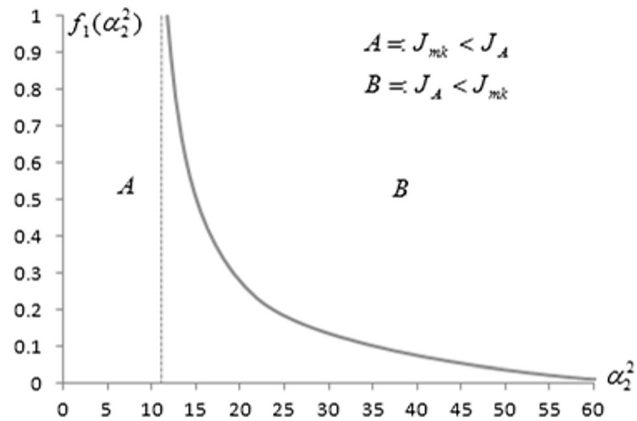

Fig. 1 The predictive performances of AMRE and AE for the first case

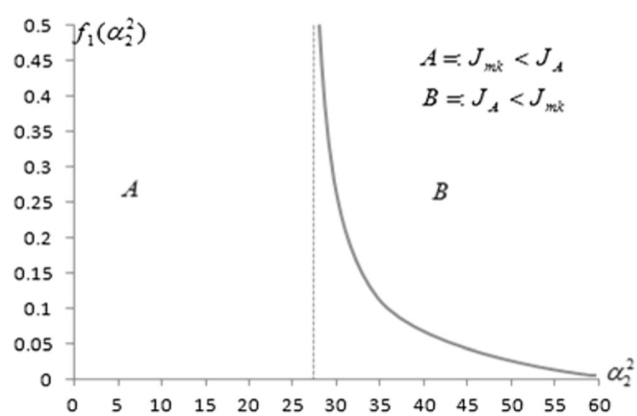

Fig. 2 The predictive performances of AMRE and AE for the second case

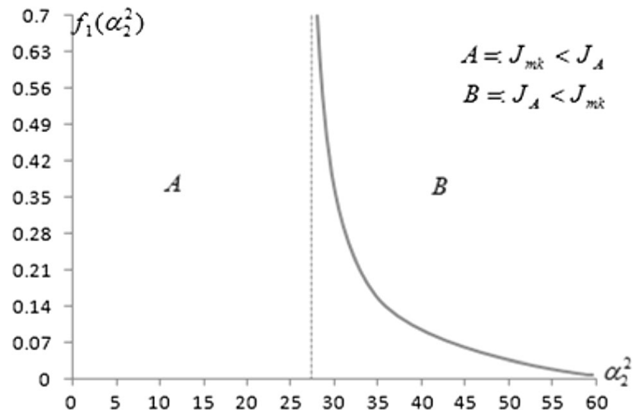

Fig. 3 The predictive performances of AMRE and AE for the third case

there is a trade-off between these two estimators. If the value of the ratio z_{02}^2/z_{01}^2 is smaller than the value of $f_1(\alpha_2^2)$, then the AMRE is superior to the AE, otherwise the AE is superior to the AMRE.

Therefore, we conclude that the intersection region among all the three defined hyperbolas which have the same side in Eqs. (91), (93) and (94) where the AMRE is uniformly superior to AE is the defined hyperbola in Eq. (91) with a vertical asymptote at $\alpha_2^2 \cong 11.8727$. Since $\hat{\alpha}_2 \cong -1.0754$ and $\hat{\alpha}_2^2 \cong 1.156485$ which is lower than the vertical asymptote $\alpha_2^2 \cong 11.8727$. That means, whatever the value of z_{01}^2 and z_{02}^2, AMRE is always uniformly superior to the AE.

Secondly, let us consider the predictive performances of AMRE and ARE

- If $\lambda_1 = 2.9359$, $\lambda_2 = 0.0634$, from (53), we get

$$f_2(\alpha_2^2) = \frac{0.0000619135}{0.04083\,\alpha_2^2 - 0.074213} \tag{96}$$

  which is a hyperbola with a vertical asymptote at

$$\alpha_2^2 \cong 1.817754. \tag{97}$$

  Figure 4 illustrates this situation. For values of α_2^2 smaller than 1.817754, the AMRE is uniformly superior to the ARE. For larger values of α_2^2, there is a trade-off between these two estimators. If the value of the ratio z_{02}^2/z_{01}^2 is smaller than the value of $f_2(\alpha_2^2)$, then the AMRE is superior to the ARE, otherwise the ARE is superior to the AMRE.
- If $\lambda_1 = 2.9359$, $\lambda_2 = 0.0007$, from (53), we get

$$f_2(\alpha_2^2) = \frac{0.0000619135}{0.07338\,\alpha_2^2 - 0.110161} \tag{98}$$

- If $\lambda_1 = 0.0634$, $\lambda_2 = 0.0007$, from (53), we get

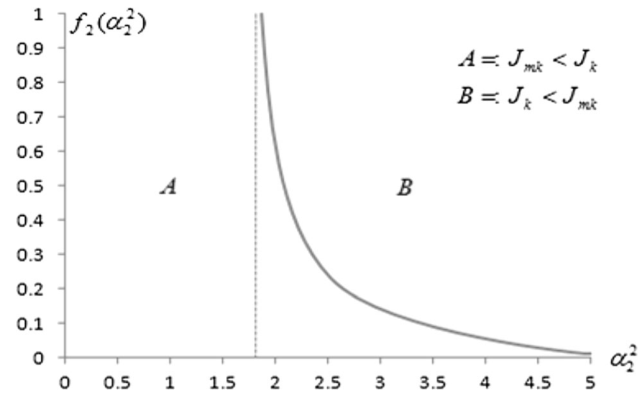

Fig. 4 The predictive performances of AMRE and ARE for the first case

$$f_2(\alpha_2^2) = \frac{0.07421339}{0.07338\,\alpha_2^2 - 0.110161} \tag{99}$$

  which are the hyperbolas with the same vertical asymptote at

$$\alpha_2^2 \cong 1.501232. \tag{100}$$

  Figures 5 and 6 illustrate this situation, respectively. For values of α_2^2 smaller than 1.501232, the AMRE is uniformly superior to the ARE. For larger values of α_2^2, there is a trade-off between these two estimators. If the value of the ratio z_{02}^2/z_{01}^2 is smaller than the value of $f_2(\alpha_2^2)$, then the AMRE is superior to the ARE, otherwise the ARE is superior to the AMRE.

Therefore, we conclude that the intersection region among all the three defined hyperbolas which have the same side in Eqs. (96), (98) and (99) where the AMRE is uniformaly superior to ARE is the defined hyperbola in Eq. (98) with a vertical asymptote at $\alpha_2^2 \cong 1.501232$. Since $\hat{\alpha}_2 \cong 1.2297$ and $\hat{\alpha}_2^2 \cong 1.512162$ which is not lower than the vertical asymptote $\alpha_2^2 \cong 1.501232$. That means, if the

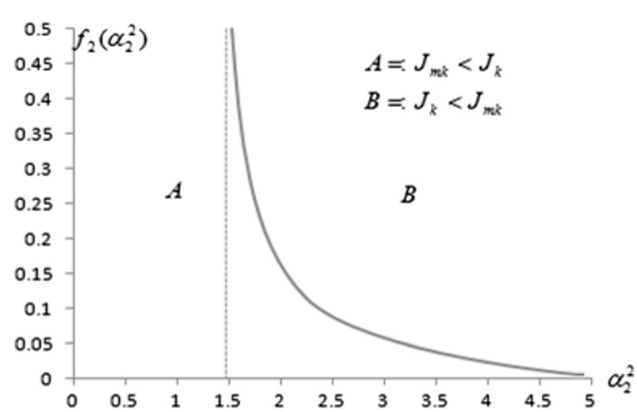

Fig. 5 The predictive performances of AMRE and ARE for the second case

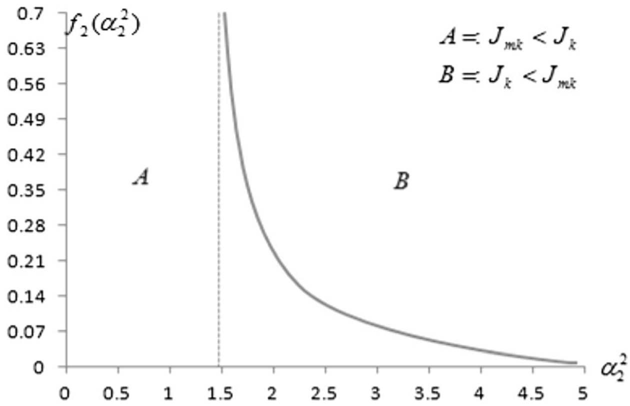

Fig. 6 The predictive performances of AMRE and ARE for the third case

value of z_{01}^2 is closer to the value of z_{02}^2, AMRE is uniformly superior to the ARE, otherwise ARE is better than AMRE.

Thirdly, let us consider the predictive performances of AMLE and AE.

- If $\lambda_1 = 2.9359$, $\lambda_2 = 0.0634$, from (60), we get

$$f_3(\alpha_2^2) = \frac{0.0000597}{0.00538\,\alpha_2^2 - 0.03656} \tag{101}$$

which is a hyperbola with a vertical asymptote at

$$\alpha_2^2 \cong 6.794634. \tag{102}$$

Figure 7 illustrates this situation. For values of α_2^2 smaller than 6.794634, the AMLE is uniformly superior to the AE. For larger values of α_2^2, there is a trade-off between these two estimators. If the value of the ratio z_{02}^2/z_{01}^2 is smaller than the value of $f_3(\alpha_2^2)$, then the AMLE is superior to the AE, otherwise the AE is superior to the AMLE.

- If $\lambda_1 = 2.9359$, $\lambda_2 = 0.0007$, from (60), we get

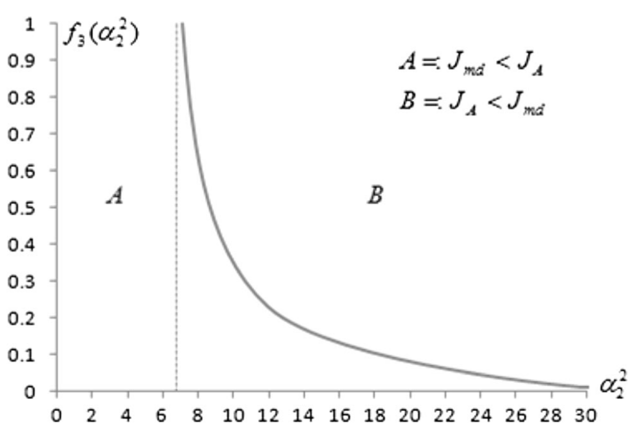

Fig. 7 The predictive performances of AMLE and AE for the first case

$$f_3(\alpha_2^2) = \frac{0.0000597}{0.00686\,\alpha_2^2 - 3.7204}. \tag{103}$$

- If $\lambda_1 = 0.0634$, $\lambda_2 = 0.0007$, from (60), we get

$$f_3(\alpha_2^2) = \frac{0.036555}{0.00686\,\alpha_2^2 - 3.7204} \tag{104}$$

which are the hyperbolas with the same vertical asymptote at

$$\alpha_2^2 \cong 542.288. \tag{105}$$

Figures 8 and 9 illustrate this situation, respectively. For values of α_2^2 smaller than 542.288, the AMLE is uniformly superior to the AE. For larger values of α_2^2, there is a trade-off between these two estimators. If the value of the ratio z_{02}^2/z_{01}^2 is smaller than the value of $f_3(\alpha_2^2)$, then the AMLE is superior to the AE, otherwise the AE is superior to the AMLE.

Therefore, we conclude that the intersection region among all the three defined hyperbolas which have the

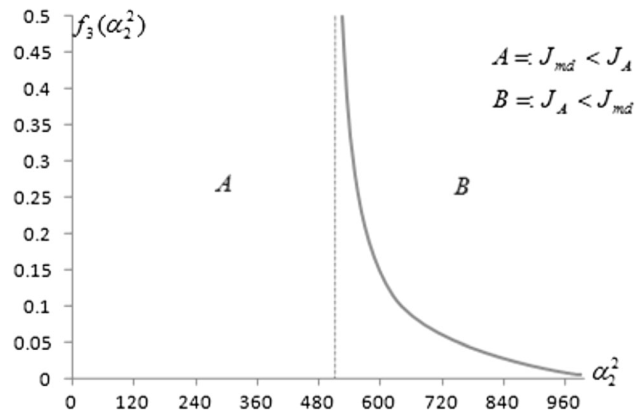

Fig. 8 The predictive performances of AMLE and AE for the second case

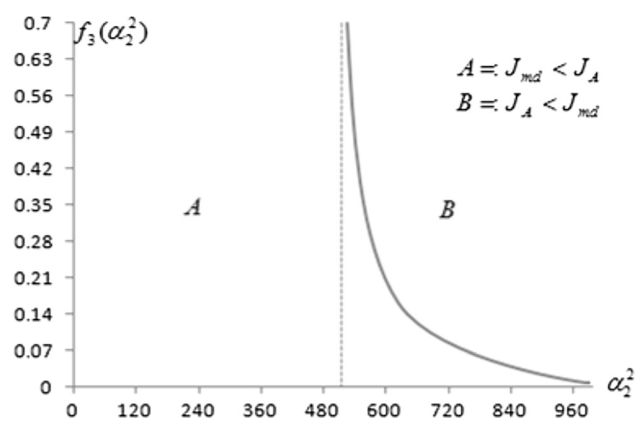

Fig. 9 The predictive performances of AMLE and AE for the third case

same side in Eqs. (101), (103) and (104) where the AMLE is uniformaly superior to AE is the defined hyperbola in Eq. (101) with a vertical asymptote at $\alpha_2^2 \cong 6.794634$. Since $\hat{\alpha}_2 \cong -1.0754$ and $\hat{\alpha}_2^2 \cong 1.156485$ which is lower than the vertical asymptote $\alpha_2^2 \cong 6.794634$. That means, whatever the value of z_{01}^2 and z_{02}^2, AMLE is always uniformly superior to the AE.

Fourthly, let us consider the predictive performances of AMLE and ALE.

- If $\lambda_1 = 2.9359$, $\lambda_2 = 0.0634$, from (68), we get

$$f_4(\alpha_2^2) = \frac{0.00073}{0.06797\,\alpha_2^2 - 0.08459} \tag{106}$$

which is a hyperbola with a vertical asymptote at

$$\alpha_2^2 \cong 1.24447. \tag{107}$$

Figure 10 illustrates this situation. For values of α_2^2 smaller than 1.24447, the AMLE is uniformly superior to the ALE. For larger values of α_2^2, there is a trade-off between these two estimators. If the value of the ratio z_{02}^2/z_{01}^2 is smaller than the value of $f_4(\alpha_2^2)$, then the AMLE is superior to the ALE, otherwise the ALE is superior to the AMLE.

- If $\lambda_1 = 2.9359$, $\lambda_2 = 0.0007$, from (68), we get

$$f_4(\alpha_2^2) = \frac{0.00073}{0.07597\,\alpha_2^2 - 7.8245}. \tag{108}$$

- If $\lambda_1 = 0.0634$, $\lambda_2 = 0.0007$, from (68), we get

$$f_4(\alpha_2^2) = \frac{0.08458}{0.07597\,\alpha_2^2 - 7.8245} \tag{109}$$

which are the hyperbolas with the same vertical asymptote at

$$\alpha_2^2 \cong 102.998 \tag{110}$$

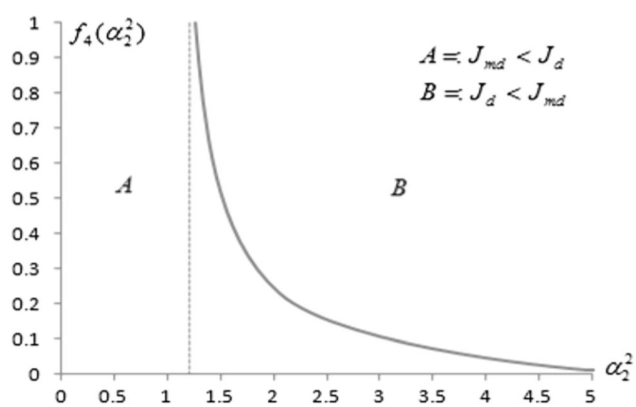

Fig. 10 The predictive performances of AMLE and ALE for the first case

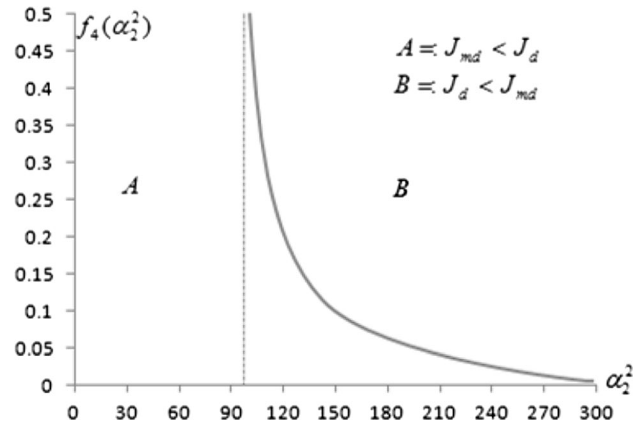

Fig. 11 The predictive performances of AMLE and ALE for the second case

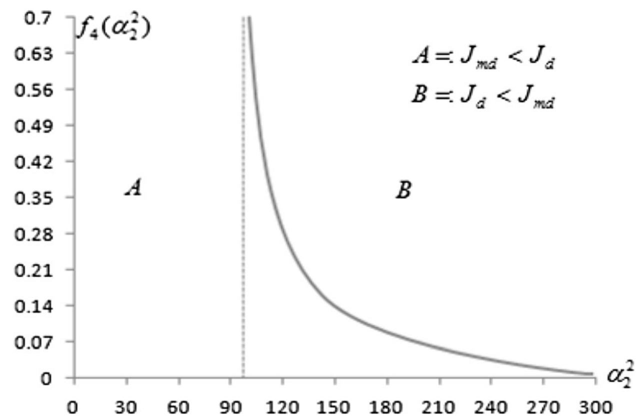

Fig. 12 The predictive performances of AMLE and ALE for the third case

Figures 11 and 12 illustrate this situation, respectively. For values of α_2^2 smaller than 102.998, the AMLE is uniformly superior to the ALE. For larger values of α_2^2, there is a trade-off between these two estimators. If the value of the ratio z_{02}^2/z_{01}^2 is smaller than the value of $f_4(\alpha_2^2)$, then the AMLE is superior to the ALE, otherwise the AE is superior to the AMLE.

Therefore, we conclude that the intersection region among all the three defined hyperbolas which have the same side in Eqs. (106), (108) and (109) where the AMLE is uniformaly superior to ALE is the defined hyperbola in Eq. (106) with a vertical asymptote at $\alpha_2^2 \cong 1.24447$. Since $\hat{\alpha}_2 \cong 1.2297$ and $\hat{\alpha}_2^2 \cong 1.512162$ which is not lower than the vertical asymptote $\alpha_2^2 \cong 1.24447$. That means, if the value of z_{01}^2 is closer to the value of z_{02}^2, AMLE is uniformly superior to the ALE, otherwise ALE is better than AMLE.

Finally, let us consider the predictive performances of AMLE and AMRE.

- If $\lambda_1 = 2.9359$, $\lambda_2 = 0.0634$, from (76), we get

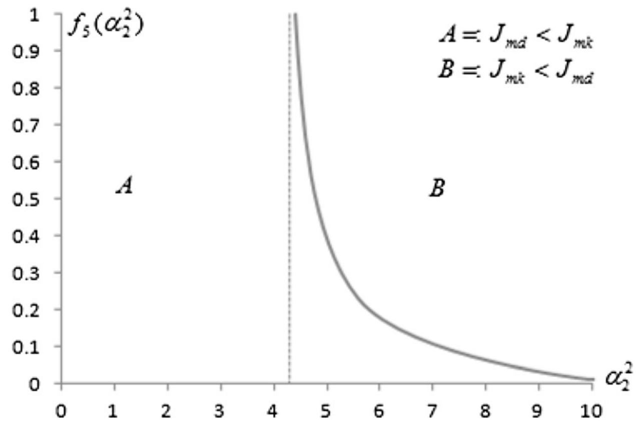

Fig. 13 The predictive performances of AMLE and AMRE for the first case

$$f_5(\alpha_2^2) = \frac{0.0000593}{0.00356\,\alpha_2^2 - 0.01496} \tag{111}$$

which is a hyperbola with a vertical asymptote at

$$\alpha_2^2 \cong 4.20146. \tag{112}$$

Figure 13 illustrates this situation. For values of α_2^2 smaller than 4.20146, the AMLE is uniformly superior to the AMRE. For larger values of α_2^2, there is a trade-off between these two estimators. If the value of the ratio z_{02}^2/z_{01}^2 is smaller than the value of $f_5(\alpha_2^2)$, then the AMLE is superior to the AMRE, otherwise the AMRE is superior to the AMLE.

- If $\lambda_1 = 2.9359$, $\lambda_2 = 0.0007$, from (76), we get

$$f_5(\alpha_2^2) = \frac{0.0000593}{19.55923 - 0.84002\,\alpha_2^2}. \tag{113}$$

- If $\lambda_1 = 0.0634$, $\lambda_2 = 0.0007$, from (76), we get

$$f_5(\alpha_2^2) = \frac{0.014963}{19.55923 - 0.84002\,\alpha_2^2} \tag{114}$$

which are the hyperbolas with the same vertical asymptote at

$$\alpha_2^2 \cong 23.2843. \tag{115}$$

Figures 14 and 15 illustrate this situation, respectively. For values of α_2^2 larger than 23.2843, the AMLE is uniformly superior to the AMRE. For smaller values of α_2^2, there is a trade-off between these two estimators. If the value of the ratio z_{02}^2/z_{01}^2 is smaller than the value of $f_5(\alpha_2^2)$, then the AMLE is superior to the AMRE, otherwise the AMRE estimator is superior to the AMLE.

Therefore, we conclude that the intersection region among all the three defined hyperbolas which have different sides in Eqs. (111), (113) and (114) where the

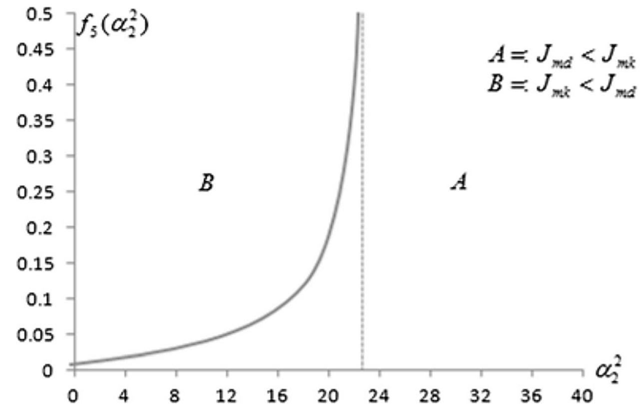

Fig. 14 The predictive performances of AMLE and AMRE for the second case

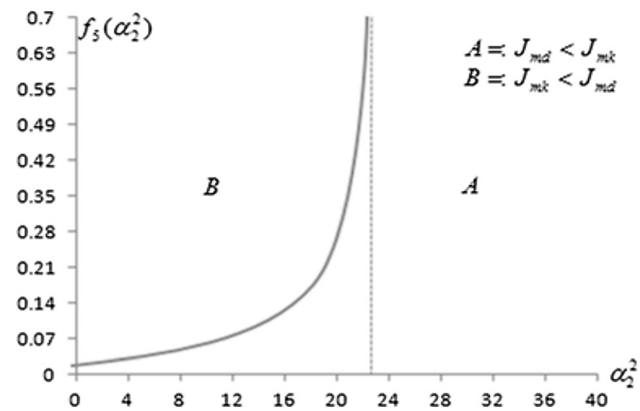

Fig. 15 The predictive performances of AMLE and AMRE for the third case

AMLE is uniformaly superior to AMRE is the intersection region between the defined hyperbolas which have different sides in Eqs. (111) and (113) where the AMLE is uniformaly superior to AMRE over α_2^2 axis and α_2^2 from 0 to 23.2843. Since $\hat{\alpha}_2 \cong 1.2297$ and $\hat{\alpha}_2^2 \cong 1.512162$ for Eq. (111) and $\hat{\alpha}_2 \cong 1.2297$ and $\hat{\alpha}_2^2 \cong 1.512162$ for Eqs. (113) and (114) which lie in the interval 0 and 23.2843 values of α_2^2. That means, if the value of z_{01}^2 is so closer to the value of z_{02}^2, AMLE is uniformly superior to the AMRE, otherwise AMRE is better than AMLE.

6 Conclusions

This paper investigates the predictive performance of the AMRE compared to the AE and the ARE as well as the AMLE compared to the AE, ALE and AMRE. The comparisons of these estimators are in terms of the PMSE under the target function at a specific point in two-dimensional regressor variable spaces. In this context, the PMSE under the target function of the AMRE and AMLE estimators are developed and five theorems are given. The theoretical consequences are illustrated by a numerical example with Almon data, and the regions are assigned for superiority of the given estimators. For some values of α_2^2, there are trade-offs between the relative effectiveness of the estimators. The AE is effective only when the value of α_2^2 is small compared to AMRE and AMLE. The effectiveness of these techniques is also affected by the location of the prediction point. Hence, the choice of the estimator may depend on the location of the point to be predicted. In the numerical example, a region is established where the AMRE and AMLE estimators are uniformly superior to the above mentioned estimators. This implies that it is theoretically possible to determine such a region.

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