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Three-Manifolds with Bounded Curvature and Uniformly Positive Scalar Curvature

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Abstract
In this note, we prove that for a complete noncompact three-dimensional Riemannian manifold with bounded sectional curvature, if it has uniformly positive scalar curvature, then there is a uniform lower bound on the injectivity radius.

Keywords 3-Manifolds · Positive scalar curvature · Collapsing · Injectivity radius

Mathematics Subject Classification 53C21

1 Introduction

A well-known problem posed by Yau (see Problem Section in [29]) is how to classify 3-manifolds admitting complete Riemannian metrics of positive scalar curvature up to diffeomorphism. In the case of closed 3-manifolds, it has been resolved by Schoen–Yau [24], Gromov–Lawson [16], Hamilton [17] and Perelman [20–22]. But for open 3-manifolds, this problem remains wide open. Some recent progress were made under the assumption of uniformly positive scalar curvature. Particularly, Chang–Weinberger–Yu [5] classified complete 3-manifolds with uniformly positive scalar curvature and finitely generated fundamental group. Bessieres–Besson–Maillot [2] classified complete 3-manifolds with uniformly positive scalar curvature and bounded geometry. Here the bounded geometry means the sectional curvature is bounded and the injectivity radius is bounded away from zero.

In this note, we make further progress towards Yau's problem and show that bounded curvature and uniformly positive scalar curvature on complete open 3-manifolds are enough to derive a lower bound on the injectivity radius.

Theorem 1.1 *Assume (M^3, g) is a smooth three-dimensional complete noncompact Riemannian manifold. If the sectional curvature is bounded by $|K_g| \leq \Lambda$ and the*

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scalar curvature is bounded below by $R_g \geq 1$, then the injectivity radius has a uniform lower bound. In other words, there exists a uniform positive number $r_0 = r_0(\Lambda) > 0$, depending only on Λ, such that for any $x \in M$, the injectivity radius at x is bounded from below by r_0.

This theorem together with the main result in [2] gives the following corollary.

Corollary 1.2 *Let M be a connected orientable 3-manifold which carries a complete metric of bounded sectional curvature and uniformly positive scalar curvature, then there is a collection $\mathcal{F}$ of spherical space forms with finitely many diffeomorphim type such that M is a (possibly infinite) connected sum of copies of $S^2 \times S^1$ with members of $\mathcal{F}$.*

Remark 1.1 Jian Wang recently claimed a result on classification of open three manifolds carrying uniformly positive scalar curvature in [27].

Also, we have the following improvement of the main result in [3].

Corollary 1.3 *The moduli space of complete Riemannian metrics of bounded curvature and uniformly positive scalar curvature on an orientable 3-manifold is path-connected or empty.*

By regularity results in [1], the main theorem implies the following compactness result.

Corollary 1.4 *For any sequence of pointed complete noncompact Riemannian 3-manifolds (M_i^3, g_i, p_i) with uniformly bounded curvature and uniformly positive scalar curvature, up to a subsequence, they converge locally in $C^{1,\alpha}$-topology for any $0 < \alpha < 1$.*

Another corollary is that 3-manifolds with uniformly positive scalar curvature and bounded curvature have at least linear volume growth.

Corollary 1.5 *Assume (M^3, g) is a complete noncompact 3-manifold satisfying that the sectional curvature is bounded by $|K_g| \leq \Lambda$ and the scalar curvature is bounded below by $R_g \geq 1$. Then there is a positive constant $c = c(\Lambda) > 0$, depending only on Λ, such that for any point $p \in M$,*

$$\liminf_{r \to \infty} \frac{Vol_g B(p,r)}{r} \geq c.$$

Proof For any $r \gg 1$, and $q \in \partial B(p, r)$, choose a geodesic segment $\gamma : [0, r] \to M$ between p and q. Let r_0 be the lower bound on the injectivity radius given in Theorem 1.1. Choose a partition of the interval $[0, r]$ by $0 = t_0 < t_1 < \cdots < t_{k+1} \leq r$ with $t_{i+1} = t_i + 3r_0$ for $0 \leq i \leq k$ and $r < t_{k+1} + 3r_0$. Then the geodesic balls $\{B(\gamma(t_i), r_0)\}_{i=0}^k$ are disjoint and all included in $B(p, r)$. Note that $k \geq \frac{r}{3r_0}$.

For any point $x \in M$, since the injective radius at x is bigger than r_0 and $K_g \leq \Lambda$, for $r_1 := \frac{1}{2}\min\{r_0, \frac{\pi}{\sqrt{\Lambda}}\}$, we have a polar coordinate such that $g = dr^2 + g_r$ in $B(x, r_1)$.

By the Hessian comparison theorem [23, Chapter 6, Corollary 2.4], in $B(x, r_1)$ we have

$$(g_r(\partial_i, \partial_j)) \geq \frac{1}{\Lambda} \sin^2 \sqrt{\Lambda} r \cdot I_2$$

as 2×2-matrix, where $\partial_i = d \exp_x(\partial\theta_i)$ for a coordinate $\{\theta_i\}$ of the unit sphere in $T_x M$. Denote by $g_\Lambda = dr^2 + \frac{1}{\Lambda} \sin^2 \sqrt{\Lambda} r ds_2^2$ the standard metric on the 3-sphere with constant curvature Λ. So

$$\mathrm{Vol}_g B(x, r_0) \geq \mathrm{Vol}_g B(x, r_1) \geq \mathrm{Vol}_{g_\Lambda} B(r_1) =: v_\Lambda(r_0).$$

Together with above arguments, we have

$$\mathrm{Vol}_g B(p, r) \geq \sum_{i=0}^{k} \mathrm{Vol}_g B(\gamma(t_i), r_0) \geq r \cdot \frac{v_\Lambda(r_0)}{3r_0}.$$

It's done by taking $c = \frac{v_\Lambda(r_0)}{3r_0}$. □

Remark 1.2 Yau [28] proved that any complete noncompact manifolds with nonnegative Ricci curvature have at least linear volume growth. In general, the volume growth is not uniform. Munteanu–Wang [18] proved that for a complete 3-manifold with nonnegative Ricci curvature and uniformly positive scalar curvature, the volume has at most uniform linear growth. So in the case of a complete noncompact 3-manifold (M^3, g) with $0 \leq Ric_g \leq \Lambda$ and $R_g \geq 1$, the volume of a geodesic ball has uniformly linear growth. That is, for some uniform constants $c_1, c_2 > 0$, as $r \to \infty$,

$$c_1 r \leq \mathrm{Vol}_g B(p, r) \leq c_2 r.$$

Now we explain the ideas in the proof of the main theorem and give some remarks. The proof is by contradiction, and involves two main ingredients: the structure of collapsing manifolds with bounded curvature, which has been intensively studied, see works [8–11]; and Gromov's width inequality [14, 15].

Let us assume by contradiction that, up to a double covering of orientations, there is a sequence of three dimensional complete noncompact orientable Riemannian manifolds (M_i^3, g_i, p_i) with uniformly bounded curvature and uniformly positive scalar curvature, which, up to a subsequence, converges to a complete Alexandrov space (X, d, o) in the pointed Gromov-Hausdorff topology, and the injectivity radius $\mathrm{inj}(p_i)$ at p_i converges to 0.

By our assumptions, the sequence is collapsing and X can't be a point, i.e. $\dim X = 1$ or 2. The theory of collapsing with bounded curvature gives a symmetric structure around sufficiently collapsed part. Roughly speaking, if $\dim X = 2$, there is a wide Seifert fibered space with boundary around p_i in M_i, see Proposition 2.2; if $\dim X = 1$, there is a long torical band in M_i, see Proposition 2.3. The noncompact assumption ensures that we can find a Seifert fibered space with any arbitrary width or a torical band whose length is large enough.

But Gromov's width inequality tells us that uniformly positive scalar curvature on a Riemannian band imposes a uniform upper bound on the width, except that there are spheres separating this band, see Propositions 3.2 and 3.3. When $\dim X = 1$, this immediately gives a contradiction. When $\dim X = 2$, note that a Seifert fibered space with boundary is S^2-irreducible, so spheres can not separate the wide Seifert fibered space in the sufficiently collapsed part, which also gives a contradiction (see Sect. 3 for more details).

Remark 1.3 The following example shows that the assumption on the upper bound of sectional curvature is necessary.

Example 1.1 Consider the warped product metric $dr^2 + \rho(r)^2 ds_2^2$ on $\mathbb{R} \times S^2$. Then the scalar curvature is given by

$$R = -\frac{4\ddot{\rho}}{\rho} + \frac{2 - 2\dot{\rho}^2}{\rho^2},$$

and the sectional curvature K is between the values $-\frac{\ddot{\rho}}{\rho}, \frac{1-\dot{\rho}^2}{\rho^2}$. See [23] for computations. Define

$$\rho(r) = \begin{cases} f(r), & |r| \leq R_0 \\ \frac{1}{|r|}, & |r| \geq R_0, \end{cases}$$

where

$$f(r) = \frac{3}{8R_0^5} r^4 - \frac{5}{4R_0^3} r^2 + \frac{15}{8R_0}.$$

Then ρ is a positive C^2-function, and for a fixed $R_0 \geq 100$, it's easy to see that $R > 1$, and $K \geq -\frac{C}{R_0^2}$. A smooth modification of ρ gives a metric with sectional curvature bounded from below and uniformly positive scalar curvature on $\mathbb{R} \times S^2$. But the injectivity radius converges to 0 and the curvature upper bound blows up as $|r| \to \infty$.

Remark 1.4 The main theorem still holds for closed 3-manifolds with big diameter by the same proof. That is, there exists a uniform constant $D_0 > 0$, coming from Gromov's width inequality (e.g. $D_0 = \frac{4\sqrt{6}\pi}{3} + 1$), such that if (M^3, g) is a closed 3-manifold satisfying $|K_g| \leq \Lambda$, $R_g \geq 1$ and $\mathrm{diam}(M, g) \geq D_0$, then there exists a uniform positive number $r_0 = r_0(\Lambda)$ depending only on Λ such that the injectivity radius of (M, g) is bounded from below by r_0.

In general, if we drop the diameter assumption, there exists closed 3-manifolds with uniformly bounded sectional curvature and uniformly positive scalar curvature, but arbitrarily small injectivity radius. Some examples are given by product metrics $ds_2^2 + \varepsilon^2 ds_1^2$ on $S^2 \times S^1$ with small $\varepsilon > 0$, and collapsing Berger 3-spheres.

Remark 1.5 The main result does not hold in higher dimensions. For example, we can consider $S^2 \times \mathbb{R}^2$ with product metric $g = g_{S^2} + g_{\mathbb{R}^2}$, where $(\mathbb{R}^2, g_{\mathbb{R}^2})$ is a complete metric with bounded curvature, and the injectivity radius goes to zero at infinity in $\mathbb{R}^2$, and (S^2, g_{S^2}) is a constant curvature metric. We can choose g_{S^2} with big enough curvature such that g has uniformly positive scalar curvature and bounded sectional curvature, but the injectivity radius does not have a uniform lower bound.

2 Collapsing with Bounded Curvature

In this section, following [10, 11], we talk about the structure around sufficiently collapsing part of three manifolds with bounded curvature.

Assume there is a sequence of complete noncompact orientable three manifolds (M_i, g_i, p_i) with sectional curvature uniformly bounded $|K_{g_i}| \leq \Lambda$ and injectivity radius $\text{inj}(p_i)$ satisfying

$$\lim_{i\to\infty} \text{inj}(p_i) = 0.$$

By Cheeger-Gromov compactness theorem [6, 13], we can take a convergent subsequence in the pointed Gromov-Hausdorff topology. For simplicity, we abuse the notation and write (M_i, d_i, p_i) for a convergent subsequence, that is

$$(M_i, d_i, p_i) \to (X, d, o),$$

where (X, d) is a complete Alexandrov space with curvature bounded below and $0 \leq \dim X \leq 3$. By results in [6, 7], the assumption that $\text{inj}(p_i) \to 0$ implies the sequence is collapsing, i.e. $\dim X < 3$; the noncompactness assumption implies that $\dim X > 0$. Namely, $\dim X = 1$ or 2.

Proposition 2.1 *If* $\dim X = 2$*, then* (X, d) *is a Riemannian orbifold without boundary.*

Remark 2.1 If X has bounded diameter and comes from codimension one collapsing, then the fact that it is a Riemannian orbifold was known in the literature, see Proposition 11.5 in [12], and Proposition 8.1 in [26]. See also [19] for the general collapsing case. For reader's convenience, we reproduce the proof and add some details here.

Proof For any $q_0 \in X$, take $q_i \in M_i$ with $q_i \to q_0$. For a small ball $B(0, \varepsilon) \subset \mathbb{R}^3 \cong T_{q_i} M_i$ with center 0 and a very small radius $\varepsilon \ll \frac{\pi}{\sqrt{\Lambda}}$, consider the pull back metrics $\exp^*_{q_i} g_i$, which we still denote by g_i, then $(B(0, \varepsilon), g_i)$ has a uniform lower bound on injectivity radius. By Cheeger-Gromov compactness theorem, up to a subsequence we have a metric g_0 on $B(0, \varepsilon)$ such that

$$(B(0, \varepsilon), g_i) \to (B(0, \varepsilon), g_0)$$

in $C^{1,\alpha}$-topology. Take G_i to be the local fundamental group

$$G_i = G(q_i, \varepsilon) := \{\gamma : \gamma \text{ is a geodesic loop at } q_i \text{ with length } < \varepsilon\}.$$

Then G_i is a finite set and acts on $B(0,\varepsilon)$ by free local isometries and $B(0,\varepsilon)/G_i = B(q_i,\varepsilon)$. The local group structure $(G_i, *)$ is defined as follows: for $\gamma_1, \gamma_2, \gamma_3 \in G_i$, we put $\gamma_1 * \gamma_2 = \gamma_3$ if $\gamma_1 * \gamma_2$ is well defined and coincides with γ_3, where $\gamma_1 * \gamma_2$ is the Gromov's product defined as the unique geodesic loop in the short homotopy class of $\gamma_1 \cdot \gamma_2$, see [4] for more details. Put the set of maps

$$L = \{\gamma \in C(B(0,\varepsilon), B(0,2\varepsilon)) : \frac{1}{2} \le \frac{d_0(\gamma(x),\gamma(y))}{d_0(x,y)} \le 2,\ \forall x, y \in B(0,\varepsilon)\},$$

where d_0 is the metric associated to g_0. Note that by Arzelà-Ascoli theorem, L is a compact set. For large enough i, we have $G_i \subset L$. By taking a subsequence, there exists a closed subset $G \subset L$ such that $G_i \to G$, and G acts isometrically on $(B(0,\varepsilon), g_0)$. It was proved in Section 3 of [11] that G is a Lie group germ, which means that G is locally isomorphic to a Lie group and its action on $B(0,\varepsilon)$ is smooth.

Then passing to a subsequence, there is an equivariant convergent sequence

$$(B(0,\varepsilon), g_i, G_i) \to (B(0,\varepsilon), g_0, G),$$

and

$$(B(0,\varepsilon), g_0)/G = B(q_0,\varepsilon).$$

In the case $\dim X = 2$, we know $\dim G = 1$.

Let H_0 be the isotropy sub-local group of G at 0, that is

$$H_0 := \{\gamma \in G : \gamma(0) = 0\}.$$

Then H_0 is in fact a group. To show this fact, for any $\gamma_1, \gamma_2 \in H_0$, it's enough to prove $\gamma_1 * \gamma_2$ is well defined and lies in H_0. Assume $\gamma_1^i \to \gamma_1, \gamma_2^i \to \gamma_2$ with $\gamma_1^i, \gamma_2^i \in G_i$. Since $d_0(\gamma_1(0), 0) = d_0(\gamma_2(0), 0) = 0$, we know $d_i(\gamma_1^i(0), 0), d_i(\gamma_2^i(0), 0) \to 0$. In particular, the total length of γ_1^i and γ_2^i is smaller than ε for all large i. Then $\gamma_1^i * \gamma_2^i$ is well defined and lies in G_i. Moreover, $d_i(\gamma_1^i * \gamma_2^i(0), 0) \le d_i(\gamma_1^i(0), 0) + d_i(\gamma_2^i(0), 0)$. By taking a subsequence, $\gamma_1^i * \gamma_2^i \to \gamma_1 * \gamma_2 \in H_0$.

Let H_0', G' be the identity components of H_0, G respectively. From Section 5 of [11], we know for ε small enough, $B(q_0,\varepsilon)$ is isometric to $B(0,\varepsilon)/H_0G'$. □

Claim *H_0 is discrete.*

Proof of the claim We argue it by contradiction and assume that H_0 is not discrete. Then H_0 contains a one-dimensional Lie group germ. Since $\dim G = 1$, H_0 and G has the same Lie group germ at identity. Hence there exists a small neighborhood U of the identity e in G such that $\forall \alpha \in U, \alpha(0) = 0$.

Notice that there is a small $\delta_0 > 0$ such that $\{\alpha \in G : d_0(0, \alpha(0)) \le \delta_0\} \subset H_0$. If not, then there exist $\delta_j \to 0$ and a sequence $\alpha_j \in G$ such that $d_0(0, \alpha_j(0)) \le \delta_j$ but $\alpha_j(0) \ne 0$. Then up to a subsequence, assume $\alpha_j \to \alpha_\infty \in G$. So $\alpha_\infty(0) = 0$. Note $\alpha_\infty^{-1} \cdot \alpha_j \in U$ for large j implies $\alpha_\infty^{-1} \cdot \alpha_j(0) = 0$. So $\alpha_j(0) = \alpha_\infty(\alpha_\infty^{-1} \cdot \alpha_j(0)) = \alpha_\infty(0) = 0$, a contradiction.

Similarly, for any small $0 < \delta_1 < \delta_0$, we have that for any $\gamma_i \in G_i$, if $d_i(0, \gamma_i(0)) \leq \delta_0$, then $d_i(0, \gamma_i(0)) < \delta_1$ for all large $i \geq i_0$ depending on δ_0, δ_1. If not, then there is a sequence γ_i satisfying $\delta_1 \leq d_i(0, \gamma_i(0)) \leq \delta_0$. Up to a subsequence, assume $\gamma_i \to \gamma_0 \in G$. Then $\delta_1 \leq d_0(0, \gamma_0(0)) \leq \delta_0$. But we just proved that $d_0(0, \gamma_0(0)) \leq \delta_0$ implies $\gamma_0(0) = 0$, which is a contradiction.

Now consider the local group

$$G_i(\delta_0) := \{\gamma \in G_i : d_i(0, \gamma(0)) \leq \delta_0\}.$$

There are two cases:

(1) $G_i(\delta_0) = \{e\}$. In this case, $B(q_0, \delta_0) = (B(0, \delta_0), g_0)$ has dimension three and is thus noncollapsing, a contradiction.
(2) $G_i(\delta_0) \neq \{e\}$. Assume that there is a non-identity element $\gamma_i \in G_i(\delta_0)$. Denote the uniform lower bound of injectivity radius on $(B(0, \varepsilon), g_i)$ by r_0, and choose $\delta_1 := \min\{\frac{1}{2}\delta_0, \frac{1}{2}r_0, \frac{\pi}{4\sqrt{\Lambda}}\}$. Let i_0 be a large number depending on δ_0, δ_1 in previous paragraph and $i \geq i_0$. Inductively, for any $n \in \mathbb{Z}$ and $n \geq 2$, assume $d_i(0, \gamma_i^k(0)) \leq \delta_0$ for all $1 \leq k \leq n-1$. Then $d_i(0, \gamma_i^{n-1}(0)) \leq \delta_0$ implies that

$$d_i(0, \gamma_i^{n-1}(0)) < \delta_1 \leq \frac{1}{2}\delta_0.$$

By triangle inequality, we have

$$\begin{aligned} d_i(0, \gamma_i^n(0)) &\leq d_i(0, \gamma_i^{n-1}(0)) + d_i(\gamma_i^{n-1}(0), \gamma_i^n(0)) \\ &= d_i(0, \gamma_i^{n-1}(0)) + d_i(0, \gamma_i(0)) \\ &\leq \frac{1}{2}\delta_0 + \frac{1}{2}\delta_0 \\ &= \delta_0. \end{aligned}$$

So for any $n \in \mathbb{Z}_+$, $\gamma_i^n \in G_i(\delta_0)$. But $G_i(\delta_0)$ is a finite set, so there exists an n_i such that $\gamma_i^{n_i} = e$.

Then we can use the technique of center of mass to show that $\gamma_i = e$. By our choice of δ_1, from Section 8 of [4], there is a unique minimum point q_c of $P(x) = \frac{1}{2}\sum_{j=1}^{n_i} d_i(x, \gamma_i^j(0))$ in $B(0, \delta_1)$. Since γ_i is an isometry and $\gamma_i^{n_i} = e$, $P(q_c) = P(\gamma_i(q_c))$. From the uniqueness of q_c, we know $\gamma_i(q_c) = q_c$. Note γ_i behaves like deck transformation, so γ_i has a fixed point only when $\gamma_i = e$. This gives a contradiction and completes the proof of the claim. □

Note the fact that H_0 is a closed subset of L and thus compact. From the claim, we know H_0 is a finite group. Then the orbit $G\cdot 0 \cong G/H_0$ is one dimensional. By standard slice theorem, we know $B(q_0, \varepsilon)$ is isometric to S_0/H_0, where $S_0 = \exp_0(V)$ for some small ball $V \subset \mathbb{R}^2 \cong (T_0(G\cdot 0))^\perp \subset T_0B(0, \varepsilon)$, and via $\exp_0$, H_0 acts isometrically on V. Note that M_i are orientable and G_i preserve orientations, so G also preserves orientation, which implies that $H_0 \subset SO(2)$, i.e. H_0 is a finite cyclic group. This implies that q_0 is either a regular point or an interior Riemannian orbifold point. □

Proposition 2.2 *If* $\dim X = 2$, *then for any fixed number* $r > 1$, *for all sufficiently large* i, *there exists a Seifert fibered space* Ω_i *with smooth boundaries* $\partial\Omega_i = \partial_- \sqcup \partial_+$ *such that*

$$d_{g_i}(\partial_-, \partial_+) \geq r.$$

Proof Under the same notations as above, if $\dim X = 2$, then from Proposition 2.1 we have a decomposition $S_1(X) \subset X$, where $S_1(X)$ consists of discrete orbifold singularities and $X \setminus S_1(X)$ is a smooth surface.

By [11, Theorem 0.12, Theorem 10.1], there is a continuous map

$$f_i : B(p_i, 4r) \to X$$

such that $f_i(p_i) = o$ and

$$|d_i(x, y) - d(f_i(x), f_i(y))| \leq C(1 + r)\varepsilon_i,$$

where $C > 0$ is a uniform constant and $\varepsilon_i \to 0$ as $i \to \infty$. Moreover, the restriction of f_i on $f_i^{-1}(X \backslash S_1(X))$ is a smooth fiber bundle with fiber diffeomorphic to S^1, and $\forall p \in S_1(X) \cap B(o, 4r)$, $f_i^{-1}(p)$ is diffeomorphic to S^1/H_p for some cyclic group $H_p = \mathbb{Z}_m$. Then for any small disk $D \subset B(o, 4r)$ with $D \cap S_1(X) = \{p\}$, an m-fold covering of $f_i^{-1}(D)$ is diffeomorphic to $D \times S^1$. So

$$\Omega_i := f_i^{-1}(B(o, 3r) \setminus B(o, r))$$

is a Seifert fibered space over orbifold $B(o, 3r) \setminus B(o, r)$.

Set $\partial_- := f_i^{-1}(\partial B(o, r))$ and $\partial_+ := f_i^{-1}(\partial B(o, 3r))$. Perturbing $B(o, 3r) \backslash B(o, r)$ by $B(o, 3r + \varepsilon) \setminus B(o, r - \varepsilon)$ for some $0 < \varepsilon \ll 1$ if necessary, we can assume $\partial(B(o, 3r) \backslash B(o, r))$ consists of no orbifold singularities such that $\partial\Omega_i = \partial_- \sqcup \partial_+$ is smooth. Since

$$d(o, f_i(\partial_-)) \leq r, \; d(o, f_i(\partial_+)) \geq 3r,$$

we know for all large enough i,

$$d_i(\partial_-, \partial_+) \geq 2r - C(1 + r)\varepsilon_i \geq r.$$

□

Proposition 2.3 *If* $\dim X = 1$, *then for any fixed number* $r > 1$, *there is a submanifold* $T_i \times (-1, 1) \subset M_i$ *with* $d_i(T_i \times \{-1\}, T_i \times \{1\}) \geq r$, *where* T_i *is diffeomorphic to a torus* T^2 *or a Klein bottle* K^2.

Proof If $\dim X = 1$, then $X = \mathbb{R}$ or $[0, \infty)$. In both cases, we can choose $q \in X$ with $d(o, q) = r + 1$ and $B(q, r) \subset X$ consists of regular points. Then by [11, Theorem 0.12, Theorem 10.1], for large enough i, there is a map

$$f_i : B(p_i, 2r + 2) \to X$$

such that the restriction on $f_i^{-1}(B(q, r))$ is a fiber bundle, whose fiber is diffeomorphic to a torus or Klein bottle with arbitrary small diameter and f_i is an almost Riemannian submersion. So

$$\Omega_i = f_i^{-1}(B(q, r)) \cong T_i \times (-1, 1)$$

is a desired submanifold. Note $\lim_{i\to\infty} d_i(T_i \times \{-1\}, T_i \times \{1\}) = 2r$. □

3 Proof of the Theorem

Recall that a Riemannian band is a Riemannian manifold $(Y, \partial_\pm)$ with two distinguished disjoint non-empty subsets in the boundary ∂Y, denoted by ∂_- and ∂_+. A band is called proper if $\partial_\pm$ are unions of connected components of ∂Y and $\partial_- \cup \partial_+ = \partial Y$. The width of a Riemannian band $(Y, \partial_\pm)$ is defined as $d(\partial_-, \partial_+)$. Gromov proved the following width inequalities in [15] by using μ-bubbles.

Proposition 3.1 *[$\frac{2\pi}{n}$-Inequality] Let $(Y^n, \partial_\pm)$ be a proper compact Riemannian band of dimension $n \leq 7$ with scalar curvature $R \geq 1$. If no closed hypersurface in Y which separates ∂_- from ∂_+ admits a metric with positive scalar curvature, then*

$$d(\partial_-, \partial_+) \leq \frac{2\pi}{n} \cdot \sqrt{n(n-1)}.$$

As a corollary, width inequalities hold for torical bands, see also [14].

Proposition 3.2 *For a Riemannian torical band $T^{n-1} \times [-1, 1]$ with dimension $n \leq 7$ and scalar curvature $R \geq 1$,*

$$d(T^{n-1} \times \{-1\}, T^{n-1} \times \{1\}) \leq \frac{2\pi}{n} \cdot \sqrt{n(n-1)}.$$

Note that a closed orientable surface which admits a metric with positive scalar curvature is diffeomorphic to a 2-sphere S^2. Taking $n = 3$, we have the following corollary.

Proposition 3.3 *Let $(Y^3, \partial_\pm)$ be a three-dimensional proper compact orientable Riemannian band with scalar curvature $R \geq 1$. There exists a positve number $C_0 = \frac{2\sqrt{6}\pi}{3}$ such that if $d(\partial_-, \partial_+) \geq C_0$, then there exist 2-spheres separating ∂_- from ∂_+.*

Now we can prove the main theorem.

Proof of Theorem 1.1 Assume by contradiction that there exists a sequence of complete noncompact three manifolds (M_i, g_i, p_i), satisfying $|K_{g_i}| \leq \Lambda$ and $R_{g_i} \geq 1$, such that $\mathrm{inj}(p_i) \to 0$. Taking a double covering if necessary, we assume M_i are orientable. As in Sect. 2, up to a subsequence, we assume $(M_i, g_i, p_i) \to (X, d, o)$ in the pointed Gromov-Hausdorff topology, with $\dim X = 1$ or 2.

If $\dim X = 2$, from Proposition 2.2, we know for a big fixed radius $r > C_0 + 100$, where C_0 is the constant in Proposition 3.3, there exists a Seifert fibered space Ω_i with two disjoint non-empty smooth boundaries $\partial\Omega_i = \partial_- \sqcup \partial_+$ such that

$$d_{g_i}(\partial_-, \partial_+) \geq r.$$

Applying Proposition 3.3 to $(\Omega_i, \partial_\pm)$, by our choice of r, we know there are 2-spheres $\{S_k^2\}_{k=1}^N$ in Ω_i separating ∂_- from ∂_+. In particular, we have $[\cup_{k=1}^N S_k^2] = [\partial_-] \neq 0$ in $H_2(\Omega_i)$.

On the other hand, by [25, Lemma 3.1], the universal covering of Ω_i is $\mathbb{R}^3$, so Ω_i is S^2-irreducible and $[\cup_{k=1}^N S_k^2] = 0$ in $H_2(\Omega_i)$, which is a contradiction. This completes the proof of the case when $\dim X = 2$.

If $\dim X = 1$, from Proposition 2.3, for any $r > 0$, we know there is a torical band $T \times (-1, 1)$ with width bigger than r admitting a positive scalar curvature $R_g \geq 1$, where T is either a torus or Klein bottle. Up to a double covering of $T \times (-1, 1)$, we can assume T is a torus. Choosing r big enough gives a contradiction to Proposition 3.2. □

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References

1. Anderson, M.T.: Convergence and rigidity of manifolds under Ricci curvature bounds. Invent. Math. **102**(2), 429–446 (1990)
2. Bessieres, L., Besson, G., Maillot, S.: Ricci flow on open 3-manifolds and positive scalar curvature. Geom. Topol. **15**(2), 927–975 (2011)
3. Bessières, L., Besson, G., Maillot, S., Marques, F.C.: Deforming 3-manifolds of bounded geometry and uniformly positive scalar curvature. J. Eur. Math. Soc. **23**(1), 153–184 (2020)
4. Buser, P., Karcher, H.: Gromov's almost flat manifolds. Number 81 in Astérisque. Société mathématique de France (1981)
5. Chang, S., Weinberger, S., Guoliang, Yu.: Taming 3-manifolds using scalar curvature. Geometriae Dedicata **148**(1), 3–14 (2010)
6. Cheeger, J.: Finiteness theorems for Riemannian manifolds. Am. J. Math. **92**(1), 61–74 (1970)
7. Cheeger, J., Colding, T.H.: On the structure of spaces with Ricci curvature bounded below. J. Differ. Geom. **46**(3), 406–480 (1997)
8. Cheeger, J., Gromov, M.: Collapsing Riemannian manifolds while keeping their curvature bounded. J. Differ. Geom. **23**, 309–346 (1986)
9. Cheeger, J., Fukaya, K., Gromov, M.: Nilpotent structures and invariant metrics on collapsed manifolds. J. Am. Math. Soc. **5**(2), 327–372 (1992)
10. Fukaya, K.: Collapsing Riemannian manifolds to ones of lower dimensions. J. Differ. Geom. **25**, 139–156 (1987)

11. Fukaya, K.: A boundary of the set of Riemannian manifolds with bounded curvatures and diameters. J. Differ. Geom. **28**, 1–21 (1988)
12. Fukaya, K.: Hausdorff convergence of Riemannian manifolds and its applications. In: Recent Topics in Differential and Analytic Geometry, pp. 143–238. Elsevier, New York (1990)
13. Gromov, M.: Metric Structures for Riemannian and Non-Riemannian Spaces. Springer, New York (2007)
14. Gromov, M.: Metric inequalities with scalar curvature. Geom. Funct. Anal. **28**(3), 645–726 (2018)
15. Gromov, M.: Four lectures on scalar curvature (2019). arXiv:1908.10612
16. Gromov, M., Lawson, H.B.: Positive scalar curvature and the Dirac operator on complete Riemannian manifolds. Publ. Math. l'IHÉS 58:83–196 (1983)
17. Hamilton, R.S.: Three-manifolds with positive Ricci curvature. J. Differ. Geom. 17(2):255–306 (1982)
18. Munteanu, O., Wang, J.: Geometry of three-dimensional manifolds with scalar curvature lower bound (2022). arXiv:2201.05595
19. Naber, A., Tian, G.: Geometric structures of collapsing Riemannian manifolds I (2008). arXiv:0804.2275
20. Perelman, G.: The entropy formula for the Ricci flow and its geometric applications (2002). arXiv:math/0211159
21. Perelman, G.: Finite extinction time for the solutions to the Ricci flow on certain three-manifolds (2003). arXiv:math/0307245
22. Perelman, G.: Ricci flow with surgery on three-manifolds (2003). arXiv:math/0303109
23. Petersen, P.: Riemannian Geometry, 1st edn. Springer, New York (1998)
24. Schoen, R., Yau, S.-T.: Existence of incompressible minimal surfaces and the topology of three dimensional manifolds with non-negative scalar curvature. Ann. Math. **110**(1), 127–142 (1979)
25. Scott, P.: The geometries of 3-manifolds. Bull. Lond. Math. Soc. **15**, 401–487 (1983)
26. Shioya, T., Yamaguchi, T.: Collapsing three-manifolds under a lower curvature bound. J. Differ. Geom. **56**, 1–66 (2000)
27. Wang, J.: Topology of 3-manifolds with uniformly positive scalar curvature (2022). arXiv:2212.14383
28. Yau, S.-T.: Some function-theoretic properties of complete Riemannian manifold and their applications to geometry. Indiana Univ. Math. J. **25**(7), 659–670 (1976)
29. Yau, S.-T.: Seminar on differential geometry. Number 102, Princeton University Press, Princeton (1982)