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A Meyer–Vietoris Formula for the Determinant of the Dirichlet-to-Neumann Operator on Riemann Surfaces

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Abstract
This paper presents a Meyer–Vietoris type gluing formula for a conformal invariant of a Riemannian surface with boundary that is defined by the determinant of the Dirichlet-to-Neumann operator. The formula is used to bound the asymptotics of the invariant under degeneration. It is shown that the associated height function on the moduli space of hyperbolic surfaces with geodesic boundary is proper only in genus zero. Properness implies a compactness theorem for Steklov isospectral metrics in the case of genus zero. The formula also provides asymptotics for the determinant of the Laplacian with Dirichlet or Neumann boundary conditions. For the proof, we derive an extension of Kirchhoff's weighted matrix tree theorem for graph Laplacians with an external potential.

Keywords Dirichlet-to-Neumann operator · Steklov eigenvalue · Determinants of elliptic operators · Graph Laplacian

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1 Statement of Results

Let (M, g) be a connected compact oriented Riemannian surface with nonempty boundary of total length $\ell(\partial M)$. The *Dirichlet-to-Neumann operator* $\mathsf{DN}(M)$ is a self-adjoint pseudo-differential operator of order 1 in $L^2(\partial M)$, which contains a great deal of information about M. In fact, $\mathsf{DN}(M)$ essentially determines the conformal class of the metric g, and hence the Riemann surface structure of M (see [21], and also

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[13]). The spectrum of $\mathsf{DN}(M)$ consists of the *Steklov eigenvalues*, which have been studied by many authors. For a survey, see [10, 11], or [4], and the references therein.

Because $\mathsf{DN}(M)$ is a self-adjoint elliptic operator of positive order, the formal product of the Steklov eigenvalues can be defined via the zeta-regularized determinant $\mathrm{Det}^*(\mathsf{DN}(M))$, according to the Ray-Singer procedure [23] (the asterisk indicates that the zero eigenvalue has been omitted). Determinants of elliptic operators often package global information through heat invariants in a way that reflects the geometry more transparently than the individual eigenvalues themselves. The determinant of the Dirichlet-to-Neumann operator for planar domains was first considered by Edward and Wu [7], who showed that for a simply connected bounded domain $\Omega \subset \mathbb{C}$ with smooth boundary, $\mathrm{Det}^*(\mathsf{DN}(\Omega)) = \ell(\partial\Omega)$. More generally, in [12, Theorem 1.1], Guillarmou and Guillopé proved that for any compact Riemannian surface M with boundary,

$$\mathscr{I}(M) := \frac{\mathrm{Det}^*(\mathsf{DN}(M))}{\ell(\partial M)} \tag{1}$$

is a conformal invariant of the metric on M. Equation (1) therefore defines a positive, real-valued function of the underlying biholomorphic equivalence class of the Riemann surface structure. We shall refer to $\mathscr{I}(M)$ as the *DN-invariant* of a Riemann surface M with boundary.

What kind of information does the DN-invariant provide about M? To give some idea, in this paper we prove a Meyer–Vietoris type expression for $\mathscr{I}(M)$ similar to the gluing formulas obtained in [3, 8, 31]. Using this we show that $\mathscr{I}(M)$ detects the existence of long thin separating cylinders in M (or in terms of hyperbolic geometry, short geodesics), at least when there are relatively few "isolated" components. This in turn leads to a compactness theorem for families of genus zero hyperbolic surfaces with geodesic boundary and the same Steklov eigenvalue spectrum. We make the observation that the invariant $\mathscr{I}(M)$ may be expressed as a *ratio* of determinants of Laplace operators on M with Neumann and Dirichlet conditions (see Proposition 2.2 below), whereas by doubling the surface we have an expression for the *product* of these determinants. Therefore, combining the asymptotic properties arising from the Meyer–Vietoris theorem in this paper with the results of Wolpert for closed hyperbolic surfaces (see [32]), we obtain bounds on the asymptotic behavior of the determinant of the Laplacian with Dirichlet and Neumann boundary conditions.

Let us briefly explain the gluing formula. Suppose $\Gamma \subset M$ is a disjoint collection of oriented closed smoothly embedded curves $\gamma : S^1 \hookrightarrow M$ not meeting ∂M. Let M_Γ denote the (possibly disconnected) surface with boundary obtained from $M \setminus \Gamma$ by adjoining two boundary components for every component of Γ. By $\mathscr{I}(M_\Gamma)$ we mean the product of the DN-invariants over the connected components of M_Γ. Let $\mathcal{N}(M, \Gamma)$ denote the Neumann jump operator in $L^2(\Gamma)$ (cf. [3]). This is simply the pairwise sum of the DN operators associated to M_Γ for each component in Γ. We emphasize that in defining $\mathcal{N}(M, \Gamma)$, *Dirichlet* conditions are imposed on ∂M. Let $\mathcal{N}_A(M, \Gamma)$ the jump operator on M_Γ defined in [31] for the trivial bundle with trivial framing and *Alvarez* boundary conditions on ∂M. More precisely, this is defined for a general surface with

boundary as follows. For complex valued functions φ, let φ', φ'' denote the real and imaginary parts, respectively. Also, let $\iota : \partial M \hookrightarrow M$ be the inclusion and $*$ the Hodge operator of the metric induced on the boundary. Then we define

$$\mathsf{DN}_A(M)(f, g) := ((*\iota^*(\bar{\partial}\varphi))', \varphi' \circ \iota) , \tag{2}$$

where (f, g) are real functions on ∂M, and φ is harmonic on M satisfying

$$(\varphi'' \circ \iota, (*\iota^*(\bar{\partial}\varphi))'') = (f, g) .$$

In the case of M_Γ, the jump operator $\mathcal{N}_A(M, \Gamma)$ in $L^2(\Gamma)$ is then the pairwise sum of $\mathsf{DN}_A(M_\Gamma)$ operators as in the case of $\mathcal{N}(M, \Gamma)$. We emphasize that the harmonic extension is such that φ'' satisfies Dirichlet boundary conditions on ∂M, whereas φ' satisfies Neumann conditions. The operator $\mathcal{N}_A(M, \Gamma)$ plays the role of the Neumann jump operator for mixed Dirichlet–Robin type boundary conditions. A key difference, however, is that $\mathcal{N}_A(M, \Gamma)$ is a pseudo-differential operator of order zero. Its determinant is defined as in [9] with the aide of an auxiliary pseudo-differential operator Q of order 1. For simplicity, in this paper we fix a particular Q with $\zeta_Q(0) = 0$ once and for all (cf. (12)). See [31] for more details.

Finally, associated to Γ is a simple graph G_Γ whose vertices $V(G_\Gamma)$ are the connected components $\{M_i\}$ of $M \setminus \Gamma$ and whose edges $E(G_\Gamma)$ are the intersections $\partial M_i \cap \partial M_j$ for distinct i and j, which correspond to a union of possibly more than one component of Γ. The metric g on M gives a *weight function* on G_Γ; namely, a map $\omega_g : E(G_\Gamma) \to \mathbb{R}^+$ obtained by setting $\omega_g(ij)$ to be the sum total of the lengths of the components of Γ in $\partial M_i \cap \partial M_j$. Let $\Delta_{(G_\Gamma, \omega_g)}$ be the associated weighted graph Laplacian.[1] We note in passing that this type of graph Laplacian has previously appeared in the study of small eigenvalues of Laplace operators on hyperbolic surfaces (see [2, 5, 24]). We now can state the main result.

Theorem 1.1 *For any compact connected oriented Riemannian surface M with $\partial M \neq \emptyset$, and $\Gamma \subset M$ as above, the following holds:*

$$\mathscr{I}(M) = \mathscr{I}(M_\Gamma) \left(\det{}^* \Delta_{(G_\Gamma, \omega_g)}\right) \frac{(2 \cdot \mathrm{Det}_Q^* \mathcal{N}_A(M, \Gamma))}{(\mathrm{Det}\, \mathcal{N}(M, \Gamma))^2} .$$

In Sect. 3, we illustrate Theorem 1.1 with an explicit computation for the disk and annulus, and we use the result to obtain asymptotic formulas in the case of multiply connected planar domains (see Theorem 3.1).

As discussed in [29–31], gluing formulas of the type in Theorem 1.1 above are convenient for computing the asymptotic behavior of determinants. In Sect. 4, we use this to prove bounds on $\mathscr{I}(M)$ for degenerating hyperbolic surfaces. This is contained in Theorems 4.1 and 4.7 below. As a consequence, following [22] we define the *height function* on the moduli space of Riemann surfaces with boundary by

$$\mathscr{H}(M) := -\log \mathscr{I}(M) .$$

[1] Graph Laplacians will appear throughout the paper. Notation and the relevant results can be found in Sect. 6, which is independent of the other sections of this paper.

Then we have the following:

Theorem 1.2 *Fix integers $g \geq 0$ and $n \geq 1$, where $n \geq 3$ if $g = 0$. Let $\mathbf{M}(g; b_1, \ldots, b_n)$ denote the moduli space of hyperbolic surfaces of genus g and geodesic boundaries of lengths $b_1, \ldots, b_n$. Then $\mathscr{H}(M)$ is a proper function on $\mathbf{M}(g; b_1, \ldots, b_n)$ if and only if $g = 0$.*

The above is analogous to the result of Osgood–Phillips–Sarnak and Khuri on flat surfaces for the height function associated to the Laplacian with Dirichlet boundary conditions [17, 22].

It is conjectured that nonequivalent surfaces with the same Steklov spectrum are rare (see [14]). Unfortunately, the height $\mathscr{H}(M)$ does not seem to provide much information on this question in higher genus. In genus zero, however, as in [22] we can draw the following consequence.

Corollary 1.3 *Fix a surface M of genus zero with at least three boundary components. Let $\mathcal{F}$ be a family of hyperbolic metrics on M with geodesic boundary that are mutually Steklov isospectral. Then $\mathcal{F}$ is precompact in the C^∞ topology.*

A compactness theorem for Steklov isospectral simply connected planar domains has been proven in [6] and [15].

We also obtain asymptotic results for determinants of Laplace operators on M. For a hyperbolic surface M with nonempty geodesic boundary, let $[\mathrm{Det}\,\Delta_D]_M$ and $[\mathrm{Det}^*\,\Delta_N]_M$ denote the zeta-regularized determinants of the Laplace operators Δ_D and Δ_N on M with Dirichlet and Neumann boundary conditions on ∂M, respectively. We set $\{\kappa_i\}$ to be the collection of all eigenvalues for *both* Dirichlet and Neumann problems on M that satisfy $0 < \kappa_i < 1/4$. As a consequence of Theorem 4.1 and work of Wolpert on the asymptotic behavior of the Selberg zeta function [32], in Corollary 5.1 below we give bounds on the asymptotic behavior of $[\mathrm{Det}\,\Delta_D]_M$ and $[\mathrm{Det}^*\,\Delta_N]_M$. In particular, we have the following:

Corollary 1.4 *Let $\mathscr{H}_D(M) := -\log[\mathrm{Det}\,\Delta_D]_M$ denote the height function for the determinant of the Laplacian with Dirichlet boundary conditions on ∂M. Then $\mathscr{H}_D(M)$ is a proper function on $\mathbf{M}(g; b_1, \ldots, b_n)$.*

We point out that this result has already been obtained by Young-Heon Kim. The estimate found in this paper,

$$\mathscr{H}_D(M) \geq \sum_{\gamma\in\Gamma}\left(\frac{\pi^2}{3\ell(\gamma)} + \frac{3}{2}\log\ell(\gamma)\right) + \frac{1}{2}\sum_i \log(1/\kappa_i) - \log C\,,$$

is on the one hand sharper than that in [18, Thm. 3.3], and in particular it incorporates the small eigenvalues (as suggested should be possible in [18, Rem. 3.3]). In the case of genus zero, we also obtain a more precise statement, and an upper bound (see Corollary 5.1). On the other hand, in Corollary 1.4 the boundary lengths are fixed, whereas Kim's result does not assume this. It may be possible to obtain the additional terms in [18, Thm. 3.3] that account for varying boundary lengths using the methods here, but we have not pursued this.

Finally, in order to prove Theorem 1.2 we found it necessary to derive a general formula for the determinant of a weighted graph Laplacian with a positive diagonal potential (see Theorem 6.1). The result is an extension of Kirchhoff's weighted matrix tree theorem [19], and thus it may be of independent interest.

2 The Gluing Formula

2.1 Preliminary Remarks on Determinants

Here, we briefly review some facts about determinants of operators. For a strictly positive, self-adjoint, elliptic pseudo-differential operator A of positive order acting on a Hilbert space V of functions (or sections of a bundle) on a compact manifold M, possibly with boundary and elliptic boundary conditions, the complex power A^{-s} is trace class for $\operatorname{Re} s \gg 0$ (see [25, 26]). The trace $\zeta_A(s) = \operatorname{tr} A^{-s}$ has a meromorphic continuation to the plane and is regular at $s = 0$. The *zeta-regularized determinant of* A is defined as

$$\operatorname{Det} A := \exp\left(-\zeta_A'(0)\right) . \tag{3}$$

This is extended to operators with a kernel by defining $\operatorname{Det}^* A$ via the restriction of the trace to $(\ker A)^\perp$.

Let $A(\varepsilon)$ be a differential family of such (invertible) operators with

$$A = A(0) , \ B = \frac{d}{d\varepsilon} A(\varepsilon)\big|_{\varepsilon=0},$$

and suppose $A^{-1}B$ is trace class. Then

$$\frac{d}{d\varepsilon} \log \operatorname{Det} A(\varepsilon)\Big|_{\varepsilon=0} = \operatorname{tr}(A^{-1}B) . \tag{4}$$

Determinants of more general operators are defined in [20]. For example, if A has order zero, then a determinant may be defined as follows (see [9] for details). Define

$$\operatorname{Log} A := \frac{i}{2\pi} \int_C dz (\log z)(z - A)^{-1} ,$$

where log is the principal branch and the contour C contains the spectrum of A. Pick a positive self-adjoint pseudo-differential operator Q on M of order 1. We then define

$$\log \operatorname{Det}_Q A := f.p.\ \operatorname{tr}(Q^{-s} \operatorname{Log} A)\Big|_{s=0} , \tag{5}$$

where "f.p." denotes the finite part of the meromorphic extension. If A is not positive, by convention we set (see [31, p. 479]):

$$\log \mathrm{Det}_Q A := \frac{1}{2} \log \mathrm{Det}_Q(A^2) .$$

The formula for variations (4) continues to hold with this definition of determinant.

Finally, we need the following result whose proof is straightforward. Suppose that A is of positive order and acts on V. Let π be the orthogonal projection operator to a finite dimensional subspace V_0, and $V_1 = \ker(1-\pi)$. With respect to the splitting $V = V_0 \oplus V_1$, write

$$A = \begin{pmatrix} A_0 & B^\dagger \\ B & A_1 \end{pmatrix} .$$

Lemma 2.1 *Assume A_1 is invertible. Then the operator A_1^{-s} on V_1 is trace class for* $\mathrm{Re}\, s \gg 0$. *The zeta function $\zeta_{A_1}(s)$ has a meromorphic continuation that is regular at $s = 0$. If* $\mathrm{Det}\, A_1 := \exp(-\zeta'_{A_1}(0))$, *then*

$$\mathrm{Det}\, A = \det(A_0 - B^\dagger A_1^{-1} B)\, \mathrm{Det}(A_1) . \tag{6}$$

Equation (6) holds for zero-th order operators A as well, replacing Det *by* Det_Q, *where we assume that Q preserves the splitting $V_0 \oplus V_1$.*

2.2 Proof of Theorem 1.1

Let us return to the context of the Introduction, where M is a compact Riemannian surface with nonempty boundary. We begin by giving a different expression for the invariant $\mathscr{I}(M)$.

Proposition 2.2 *Let $A(M)$ denote the area of M and κ_g the geodesic curvature of ∂M. Then*

$$\mathscr{I}(M) = \frac{\mathrm{Det}^*\, \Delta_N}{A(M)\, \mathrm{Det}\, \Delta_D} \exp\left(-\frac{1}{2\pi} \int_{\partial M} \kappa_g ds \right) . \tag{7}$$

Proof By the Polyakov–Alvarez formula [1], the right-hand side of (7) is conformally invariant (see [28, eqs. (4) and (5)]). Hence, it suffices to prove (7) in the case of geodesic boundary. Let $\widehat{M}$ be the double of M along ∂M. Then by decomposing the spectrum with respect to the isometric involution on $\widehat{M}$, we have

$$\left[\mathrm{Det}^*\, \Delta\right]_{\widehat{M}} = \left[\mathrm{Det}\, \Delta_D\, \mathrm{Det}^*\, \Delta_N\right]_M . \tag{8}$$

On the other hand, by [3, Theorem B*], cutting $\widehat{M}$ along ∂M gives

$$[\mathrm{Det}^*\, \Delta]_{\widehat{M}} = [\mathrm{Det}\, \Delta_D]_M^2\, \frac{A(\widehat{M})}{\ell(\partial M)}\, \mathrm{Det}^*(2\,\mathsf{DN}(M)) . \tag{9}$$

Now $A(\widehat{M}) = 2A(M)$, and since $\zeta_{\mathsf{DN}(M)}(0) = -1$ (cf. [12, 30]), we have

$$\mathrm{Det}^*(2\,\mathsf{DN}(M)) = \frac{1}{2}\,\mathrm{Det}^*(\mathsf{DN}(M))\ .$$

The result now follows from (8), (9), and the definition (1).

Let $\Gamma \subset M$ be as in the Introduction, and recall that Γ is assumed to be oriented. Hence, the boundary components of M_Γ are signed, depending upon whether the orientations induced from the outward normals agree with those of Γ. This allows us to define a map δ_Γ from functions on ∂M_Γ to functions on Γ by taking the difference of the values on the two sheets of the double cover $\partial M_\Gamma \to \Gamma$. With this understood, we have the following:

Lemma 2.3 *Let M_i, $i = 1, \ldots, p$, denote the connected components of M_Γ. Let $\{\Psi_j\}_{j=1}^p$ be any basis of locally constant functions on M_Γ. Then*

$$\frac{\det^*(\delta\Psi_i, \delta\Psi_j)_\Gamma}{\det(\Psi_i, \Psi_j)_{M_\Gamma}} = \frac{\det^* \Delta_{(G_\Gamma, \omega_g)}}{\prod_{i=0}^p A(M_i)}\ .$$

Proof The statement is independent of the choice of basis, so choose Ψ_i to be the characteristic function on M_i. Clearly, $\det(\Psi_i, \Psi_j)_{M_\Gamma} = \prod_{i=1}^p A(M_i)$. We also have

$$(\delta\Psi_i, \delta\Psi_j)_\Gamma = \begin{cases} \ell(\partial M_i \cap \Gamma)^* & i = j\ , \\ -\ell(\partial M_i \cap \partial M_j) & i \neq j\ . \end{cases}$$

The $*$ means we omit components of Γ that bound only M_i. Now the result follows by the definition of (G_Γ, ω_g) and the graph Laplacian from the Introduction (see Sect. 6 for more details).

Proof of Theorem 1.1 Use a small modification of [3, Theorem B*] to incorporate the boundary ∂M, and write

$$[\mathrm{Det}\,\Delta_D]_M = \prod_{i=0}^p [\mathrm{Det}\,\Delta_D]_{M_i}\,\mathrm{Det}\,\mathcal{N}(M, \Gamma)\ .$$

On the other hand, by a similar modification of [31, Theorem 3.3], we have

$$\frac{\left[\mathrm{Det}\,\Delta_D\,\mathrm{Det}^*\,\Delta_N\right]_M}{2A(M)} = \prod_{i=0}^p \left[\mathrm{Det}\,\Delta_D\,\mathrm{Det}^*\,\Delta_N\right]_{M_i} \frac{\det^*(\delta\Psi_i, \delta\Psi_j)_\Gamma}{\det(\Psi_i, \Psi_j)_{M_\Gamma}}\,\mathrm{Det}_Q^*\,\mathcal{N}_A(M, \Gamma)\ .$$

The result now follows from Proposition 2.2 and Lemma 2.3. Notice that all the factors involving geodesic curvature along Γ cancel pairwise.

3 Examples

3.1 Disks and Annuli

By [7] and [12], or alternatively [28] and eq. (7), or indeed by a direct calculation, it follows that $\mathscr{I}(M) = 1$ for the disk, and $\mathscr{I}(M) = 2\pi/\log\rho$ for the annulus with modulus $(\log\rho)/2\pi$, $1 < \rho < \infty$. If M is a euclidean disk of radius R and Γ the circle centered at the origin of radius r, $0 < r < R$, then Theorem 1.1 states in this case that

$$1 = \frac{4\pi}{\log\rho}(2\pi r)\frac{\mathrm{Det}^*_Q \mathcal{N}_A}{(\mathrm{Det}\,\mathcal{N})^2}\,, \tag{10}$$

where $\rho = R/r$ and $\mathcal{N}, \mathcal{N}_A$ denote the operators for the pair (M, Γ), and we have used that $\det^* \Delta_{(G_\Gamma,\omega_g)} = 2\pi r$ in this case. Let us verify (10) directly. The operators $\mathcal{N}, \mathcal{N}_A$ have eigenvalues $1/r\log\rho$, and $1/2r\log\rho$, corresponding to the constant functions on Γ. It therefore suffices to show that

$$(\mathrm{Det}'\,\mathcal{N})^2 = (2\pi r)^2\,\mathrm{Det}'_Q\,\mathcal{N}_A\,, \tag{11}$$

where the prime indicates the determinant of the operator restricted to the space $L^2_0(\Gamma) \subset L^2(\Gamma)$ orthogonal to the constants. For the operator Q we may choose

$$Q\left(\sum_{n\in\mathbb{Z}} f_n e^{in\theta}\right) = f_0 + \sum_{n\neq 0}|n|\,f_n e^{in\theta}\,, \tag{12}$$

so that $\zeta_Q(0) = 0$ as in the Introduction. Since $\zeta_{\mathcal{N}}(0) = -1$, we must prove

$$\log\mathrm{Det}'\,\widehat{\mathcal{N}} = \frac{1}{2}\log\mathrm{Det}'_Q\,\mathcal{N}_A\,, \tag{13}$$

where $\widehat{\mathcal{N}} = (2\pi r)\mathcal{N}$. Set $\tau = \rho^{-1}$.

Now by a direct calculation one finds (recall (2))

$$\mathcal{N}\left(\sum_{n\neq 0} f_n e^{in\theta}\right) = \sum_{n\neq 0}\frac{|n|a_n}{r} f_n e^{in\theta}\,,$$

$$\mathcal{N}_A\left(\sum_{n\neq 0}\begin{pmatrix} f_n \\ g_n\end{pmatrix} e^{in\theta}\right) = \sum_{n\neq 0} a_n \begin{pmatrix} 0 & -i\sigma(n) \\ i\sigma(n) & -\frac{2r}{|n|}\end{pmatrix}\begin{pmatrix} f_n \\ g_n\end{pmatrix} e^{in\theta}\,,$$

where $a_n = \frac{2}{1-\tau^{2|n|}}$ and $\sigma(n)$ is the sign of n. For fixed R, regard $\widehat{\mathcal{N}}$ and $\mathcal{N}_A$ as operators depending upon τ. Then as $\tau \to 0$,

$$\zeta_{\widehat{\mathcal{N}}(0)}(s) = 2(4\pi)^{-s}\zeta(s)\,,\ (\mathcal{N}_A(0))^2 = 4\cdot\mathrm{id}\,,$$

where $\zeta(s)$ is the Riemann zeta function. Below we use that $\zeta(0) = -1/2$ and $\zeta'(0) = -\frac{1}{2}\log(2\pi)$. Hence, on the one hand,

$$\begin{aligned}\log \mathrm{Det}'_Q \mathcal{N}_A(0) &= \frac{1}{2}\log \mathrm{Det}'_Q(\mathcal{N}_A(0))^2 = \frac{1}{2}\, f.p.\,\mathrm{tr}\left(Q^{-s}\log(4)\cdot \mathrm{id}\right)\big|_{s=0} \\ &= 4\log(2)\, f.p.(\zeta(s))\big|_{s=0} = -2\log(2)\,.\end{aligned}$$

On the other hand,

$$-\zeta'_{\widehat{\mathcal{N}}(0)}(0) = 2(\log(4\pi)\zeta(0) - 2\zeta'(0)) = -\log(2).$$

Thus, (13) is satisfied in this case.

Next, by direct calculation,

$$\frac{d}{d\tau}\log \mathrm{Det}'_Q \mathcal{N}_A = f.p.\,\mathrm{tr}\left(Q^{-s}\widehat{\mathcal{N}}^{-1}\frac{d}{d\tau}\widehat{\mathcal{N}}\right)\Big|_{s=0}.$$

But since the derivative of a_n with respect to τ vanishes rapidly with n, the operator $\widehat{\mathcal{N}}^{-1}(d\widehat{\mathcal{N}}/d\tau)$ is trace class, and so

$$\begin{aligned}\frac{d}{d\tau}\log \mathrm{Det}'\,\widehat{\mathcal{N}} &= \mathrm{tr}\left(\widehat{\mathcal{N}}^{-1}\frac{d}{d\tau}\widehat{\mathcal{N}}\right) = f.p.\,\mathrm{tr}\left(Q^{-s}\widehat{\mathcal{N}}^{-1}\frac{d}{d\tau}\widehat{\mathcal{N}}\right)\Big|_{s=0} \\ &= \frac{d}{d\tau}\log \mathrm{Det}'_Q \mathcal{N}_A\,.\end{aligned}$$

Thus, (13) is proven.

3.2 Multiply Connected Planar Domains

Next, consider a planar domain $M(\boldsymbol{\varepsilon})$, $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_n)$, obtained by removing disks of radius $\varepsilon_i > 0$ (with fixed centers at $a_1, \ldots, a_n$) from a disk of fixed radius, which without loss of generality we may choose to be the unit disk D. The following generalizes the case of the annulus in the previous section.

Theorem 3.1 $\lim_{\boldsymbol{\varepsilon}\to 0} \mathscr{I}(M(\boldsymbol{\varepsilon}))\prod_{i=1}^{n}\log(1/\varepsilon_i) = (2\pi)^n.$

Proof Fix an $\varepsilon_0 > 0$, $\varepsilon_0 < \frac{1}{2}\min\{|a_i - a_j| \mid i \neq j\}$, and such that $D_{\varepsilon_0}(a_i) \Subset D$. Then set $\boldsymbol{\varepsilon}_0 = (\varepsilon_0, \ldots, \varepsilon_0)$. We suppose $\boldsymbol{\varepsilon} < \boldsymbol{\varepsilon}_0$, and let A_{ε_i} denote the annulus $\varepsilon_i \le |z - a_i| \le \varepsilon_0$. Recall that $\mathscr{I}(A_{\varepsilon_i}) = 2\pi/\log(\varepsilon_0/\varepsilon_i)$. Then taking Γ in Theorem 1.1 to be the collection of curves $\partial D_{\varepsilon_0}(a_i)$, we have

$$\mathscr{I}(M(\boldsymbol{\varepsilon})) = \mathscr{I}(M(\boldsymbol{\varepsilon}_0))(\det{}^*\Delta_{(G_\Gamma,\omega_g)})\frac{(2\cdot \mathrm{Det}_Q^*\,\mathcal{N}_A(M(\boldsymbol{\varepsilon}),\Gamma))}{(\mathrm{Det}\,\mathcal{N}(M(\boldsymbol{\varepsilon}),\Gamma))^2}$$
$$\times\prod_{i=1}^{n}\frac{2\pi}{\log(\varepsilon_0/\varepsilon_i)}\,. \tag{14}$$

Since the first two factors on the right-hand side of (14) are fixed independent of the ε_i, it suffices to analyze the determinants of the Neumann jump operators. But by direct calculation, the Dirichlet-to-Neumann operator on an annulus of modulus ρ (with Dirichlet conditions on the inner boundary) is equal to the Dirichlet-to-Neumann operator on the disk of radius ε_0 up to a trace-class operator whose norm $\to 0$ as $\varepsilon_i \to 0$. To be precise, on the n-th Fourier mode, $\mathsf{DN}(A_\varepsilon)$ acts as

$$\mathsf{DN}(A_\varepsilon)(g_n e^{in\theta}) = \frac{|n|}{\varepsilon_0}\left(1+\frac{2\rho^{-2|n|}}{1-\rho^{-2|n|}}\right)g_n e^{in\theta}\,.$$

It follows that as $\varepsilon_i \to 0$, $\mathrm{Det}\,\mathcal{N}(M(\boldsymbol{\varepsilon}),\Gamma)$ tends to the determinant of the corresponding operator on the domain with A_{ε_i} replaced by the disk $D_{\varepsilon_0}(a_i)$. The same holds for $\mathcal{N}_A(M(\boldsymbol{\varepsilon}),\Gamma)$. Applying the gluing formula once again to the limit, we obtain the result.

4 Asymptotics of $\mathscr{I}(M)$

4.1 Genus Zero

As we have seen in Sect. 3.2, general bounds on the invariant $\mathscr{I}(M)$ are obtained from Theorem 1.1 by judicious choices of Γ. This can be done on a case-by-case basis; to get overall uniform estimates is combinatorially complicated. For simplicity, here we treat only the genus zero case in complete generality. In higher genus, we find rough estimates that suffice for the application to the Laplacian with Dirichlet conditions. Recall that by a "short geodesic" on a hyperbolic surface, we mean a simple closed geodesic of length less than the absolute constant c_0 appearing in the collar lemma (cf. [16]).

Theorem 4.1 *Fix positive numbers $b_1,\cdots,b_n$, $n\geq 3$, and $\delta>0$. Then there is a constant $C\geq 1$ depending only on c_0, $(b_1,\cdots,b_n)$, and δ, such that the following holds. For any hyperbolic surface M of genus zero with geodesic boundary components of lengths $b_1,\cdots,b_n$, let Γ be the collection of short geodesics on M and $\Delta_{(G_\Gamma,\omega_g)}$ the graph Laplacian from the Introduction. Then*

$$C^{-1}\frac{\prod_{\gamma\in\Gamma}\ell(\gamma)}{\det(\Delta_{(G_\Gamma,\omega_g)}+D)} \leq \mathscr{I}(M) \leq C\frac{\prod_{\gamma\in\Gamma}\ell(\gamma)}{\det(\Delta_{(G_\Gamma,\omega_g)}+D)}\,, \tag{15}$$

where D is the diagonal matrix with entries δ for the vertices corresponding to the components of M_Γ that intersect ∂M, and zeros elsewhere.

Proof The proof proceeds in several steps. Recall that we denote the connected components of $M \setminus \Gamma$ by M_i.

Step 1 First, about each $\gamma \in \Gamma$ we choose a conformal annulus A_γ of modulus $(\log \rho(\gamma))/2\pi$, $1 < \rho(\gamma) < +\infty$. We choose a local coordinate $1/\rho(\gamma) \leq |z| \leq 1$ for A_γ. By a conformal change we adjust the hyperbolic metric to be euclidean in a neighborhood of the boundary ∂A_γ. Introduce the notation

$$\lambda(\gamma) := 1/\log \rho(\gamma) \, . \tag{16}$$

The significance of $\lambda(\gamma)$ is that it is the value of the DN-operator for A_γ at the boundary $|z| = 1$ for the characteristic function of this boundary. The value the DN-operator at the other boundary is $-\lambda(\gamma)$. By the collar lemma and Wolpert's estimate (cf. [33]), we may choose A_γ such that the length $\ell(\gamma)$ (in the hyperbolic metric) of the geodesic in (M, g) homotopic to γ satisfies $\ell(\gamma) \sim 2\pi^2/\log \rho(\gamma)$. Thus, we have a comparison between $\lambda(\gamma)$ and $\ell(\gamma)$ that is uniform as $\ell(\gamma) \to 0$.

Let

$$\widetilde{\Gamma} := \bigcup_{\gamma \in \Gamma} \partial A_\gamma \, ,$$

and set $\widetilde{M}_i$ to be the connected component of

$$\mathring{M}_{\widetilde{\Gamma}} := M \setminus \bigcup_{\gamma \in \Gamma} A_\gamma$$

that intersects M_i. We let $\overline{M}_{\widetilde{\Gamma}}$ denote the (possibly disconnected) Riemann surface obtained by "capping off" $\mathring{M}_{\widetilde{\Gamma}}$, i.e., replacing each annulus A_γ by a pair of disks. An important point is the bounded geometry of $\widetilde{M}_i$ as M varies in the moduli space of hyperbolic metrics with fixed set Γ of short geodesics. See [32, 33]. In the following, we shall call $\widetilde{M}_i$ an *isolated component* if $\partial \widetilde{M}_i \cap \partial M = \emptyset$.

Step 2 Let $\mathcal{N} = \mathcal{N}(M, \widetilde{\Gamma})$ and $\mathcal{N}_A = \mathcal{N}_A(M, \widetilde{\Gamma})$ be the Neumann jump operators acting on $L^2(\widetilde{\Gamma})$. For each i, let $\chi_i \in L^2(\widetilde{\Gamma})$ denote the function defined by boundary values of the characteristic function of the component $\widetilde{M}_i$, i.e., χ_i is locally constant on $\widetilde{\Gamma}$, and is equal to 1 on $\partial \widetilde{M}_i$ and 0 otherwise. Let us write an orthogonal decomposition

$$L^2(\widetilde{\Gamma}) = V_0 \oplus V_1 \, ,$$

where V_0 is the span of all χ_i for isolated components $\widetilde{M}_i$. By a similar analysis to the one carried out in [29–31], for example, on the orthogonal complement V_1, as the $\ell(\gamma) \to 0$ the operators $\mathcal{N}$ and $\mathcal{N}_A$ converge in trace class to the corresponding operators $\overline{\mathcal{N}}$ and $\overline{\mathcal{N}}_A$ on the capped off components of $\overline{M}_{\widetilde{\Gamma}}$ (see also the proof of Lemma 4.6 below).

Step 3 We must analyze the small eigenvalues of $\mathcal{N}$ and $\mathcal{N}_A$, which occur from the restriction to V_0. For $\mathcal{N}_A$, this refers to the φ'' component. The analysis is therefore identical for both $\mathcal{N}$ and $\mathcal{N}_A$, and so henceforth we deal only with $\mathcal{N}$. For the isolated

components $\widetilde{M}_i$, the harmonic function on $\widetilde{M}_i$ with the boundary values of χ_i is χ_i itself, and so the Dirichlet-to-Neumann operator for $\widetilde{M}_i$ annihilates this. On the other hand, if $\partial\widetilde{M}_i \cap \partial M \neq \emptyset$, then since Dirichlet conditions are imposed on ∂M, the DN operator is nonzero on the boundary values of χ_i. For each collar A_γ, DN is rotationally symmetric, and so preserves constants. In terms of the splitting $V_0 \oplus V_1$, we may write

$$\mathcal{N} = \begin{pmatrix} A & B^\dagger \\ B & \mathcal{N}_0 \end{pmatrix} . \tag{17}$$

Now $\mathcal{N}_0$ is uniformly invertible as the lengths $\ell(\gamma) \to 0$. By Lemma 2.1, we have

$$\operatorname{Det}\mathcal{N} = \det(A - B^\dagger \mathcal{N}_0^{-1} B)\operatorname{Det}(\mathcal{N}_0) . \tag{18}$$

It follows that $\operatorname{Det}\mathcal{N}$ is estimated by $\det(A - B^\dagger\mathcal{N}_0^{-1}B)$. Recall the weighted graph (G_Γ, ω_g) from the Introduction. The key result is the following:

Proposition 4.2 *Fix $\delta > 0$. There is a constant $C \geq 1$ depending only on c_0, $(b_1, \ldots, b_n)$, and δ, such that*

$$C^{-1}\det(\Delta_{(G_\Gamma,\omega_g)} + D) \leq \det(A - B^\dagger\mathcal{N}_0^{-1}B) \leq C\det(\Delta_{(G_\Gamma,\omega_g)} + D) .$$

We postpone the proof of this proposition to the next section.

Step 5 Assuming Proposition 4.2, we complete the proof of Theorem 4.1. By definition

$$\mathscr{I}(M_{\widetilde{\Gamma}}) = \mathscr{I}(\mathring{M}_{\widetilde{\Gamma}}) \prod_{\gamma\in\Gamma} \mathscr{I}(A_\gamma) .$$

Now $\mathscr{I}(A_\gamma) = 2\pi/\log\rho(\gamma) \simeq \ell(\gamma)/\pi$, and because $\mathring{M}_{\widetilde{\Gamma}}$ has bounded geometry over the moduli space, $\mathscr{I}(\mathring{M}_{\widetilde{\Gamma}})$ is bounded from above and below away from zero. Hence, there is an estimate (above and below)

$$\mathscr{I}(M_{\widetilde{\Gamma}}) \simeq C\prod_{\gamma\in\Gamma}\mathscr{I}(A_\gamma) \simeq C\prod_{\gamma\in\Gamma}\ell(\gamma) .$$

Apply Theorem 1.1 to $\widetilde{\Gamma}$. The lengths of the elements of $\widetilde{\Gamma}$ are bounded away from zero, so the factor $\det^*\Delta_{(G_{\widetilde{\Gamma}},\omega_g)}$ in Theorem 1.1 remains bounded above and below away from zero. From the discussion in Step 2 above, eq. (18), and Proposition 4.2, we have

$$\frac{\operatorname{Det}^*_Q \mathcal{N}_A(M,\widetilde{\Gamma})}{(\operatorname{Det}\mathcal{N}(M,\widetilde{\Gamma}))^2} \sim \frac{1}{\operatorname{Det}\mathcal{N}(M,\widetilde{\Gamma})} ,$$

and

$$C^{-1}\det(\Delta_{(G_\Gamma,\omega_M)} + D) \leq \operatorname{Det}\mathcal{N}(M,\widetilde{\Gamma}) \leq C\det(\Delta_{(G_\Gamma,\omega_M)} + D) ,$$

for a constant C. Putting this all together completes the proof.

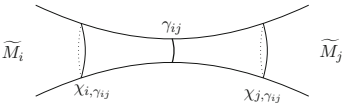


Fig. 1 A collar

4.2 Proof of Proposition 4.2

Here, we relate the action of $\mathcal{N}$ on the locally constant functions in $L^2(\widetilde{\Gamma})$ associated to the isolated components of $\mathring{M}_{\widetilde{\Gamma}}$ to the graph Laplacian $\Delta_{(G_\Gamma,\omega_M)}$. This relationship is contained in the matrix A appearing in (17). Now the key point is that in terms of the graph Laplacian, the form of the modification to the matrix A in Proposition 4.2, $B^\dagger \mathcal{N}_0^{-1} B$, corresponds to adding edges with weights that are at least quadratic in the weights of the graph G_Γ. Using the results in Sect. 6, we then argue that for small weights such a modification gives only a small perturbation of determinants.

To spell this out precisely, let us introduce some convenient notation. Enumerate the components $\widetilde{M}_i$ of $\mathring{M}_{\widetilde{\Gamma}}$, and let L_{ij} denote the adjacency matrix of G_Γ. By the assumption of genus zero, if $L_{ij} \neq 0$, then there is a unique element $\gamma_{ij} \in \Gamma$ bounding M_i and M_j (by definition $\gamma_{ij} = \gamma_{ji}$). Associated to γ_{ij} are two elements of $\widetilde{\Gamma}$, one bounding $\widetilde{M}_i$ and the other $\widetilde{M}_j$. Let $\chi_{i,\gamma_{ij}}$ and $\chi_{j,\gamma_{ij}}$ denote the characteristic functions of these two components of $\widetilde{\Gamma}$ (Fig. 1).

We define a weight function on G_Γ by

$$\omega_M(ij) := \begin{cases} \lambda(\gamma_{ij}) & L_{ij} \neq 0 \\ 0 & L_{ij} = 0 \end{cases} , \tag{19}$$

where $\lambda(\gamma)$ is defined in (16) above. By the discussion in Step 1 of the previous section, there is a constant $\kappa \geq 1$, uniform as the $\ell(\gamma) \to 0$, such that for all i, j,

$$\kappa^{-1}\omega_g(ij) \leq \omega_M(ij) \leq \kappa\omega_g(ij) . \tag{20}$$

Let $G_0 \subset G_\Gamma$ be the (possibly disconnected) subgraph obtained by deleting nonisolated vertices and their edges. We identify the vertices of G_0 with the basis elements χ_i of V_0. Then we

Claim 4.3 *The matrix A in (17) is the restriction to G_0 of the graph Laplacian $\Delta_{(G_\Gamma,\omega_M)}$.*

Proof Let $\widetilde{M}_i$ and $\widetilde{M}_j$ be isolated components. The claim amounts to the statement that

$$\begin{aligned} \langle \mathcal{N}(\chi_i), \chi_j \rangle &= -L_{ij}\lambda(\gamma_{ij}) , \\ \langle \mathcal{N}(\chi_i), \chi_i \rangle &= \sum_{k \text{ isolated}} L_{ik}\lambda(\gamma_{ik}) + \sum_{k \text{ not isolated}} L_{ik}\lambda(\gamma_{ik}) . \end{aligned} \tag{21}$$

This follows by direct calculation of the harmonic extensions of the locally constant functions χ_i (cf. Step 1 of the previous section).

We next consider the other entries of the decomposition (17). As in (21), we have

$$B\chi_i = \sum_j L_{ij}\lambda(\gamma_{ij})\left\{(\chi_{i,\gamma_{ij}})^{\perp} - (\chi_{j,\gamma_{ij}})^{\perp}\right\} . \tag{22}$$

Here, $\perp$ indicates the orthogonal projection to V_1 in the decomposition $V_0 \oplus V_1$. Note that the sum is over *all* components $\widetilde{M}_j$, not just isolated ones. Also, by definition $\mathcal{N}_0\chi_i = 0$. Let us define

$$P_{ij} := \sum_{k,k'} L_{ik}L_{jk'}\lambda(\gamma_{ik})\lambda(\gamma_{jk'})\left\langle \mathcal{N}_0^{-1}\chi_{i,\gamma_{ik}} - \mathcal{N}_0^{-1}\chi_{k,\gamma_{ik}},\, \chi_{j,\gamma_{jk'}} - \chi_{k',\gamma_{jk'}}\right\rangle \tag{23}$$

Since the summand in (23) is skew-symmetric in j and k', we have $\sum_j P_{ij} = 0$, and therefore

$$P_{ii} = -\sum_{j\neq i} P_{ij} . \tag{24}$$

Let $\widehat{G}_\Gamma$ denote the complete graph on the vertices of G_Γ; similarly $\widehat{G}_0 \subset \widehat{G}_\Gamma$ denotes the complete graph on G_0. Define weights for $\widehat{G}_\Gamma$ (possibly zero or nonpositive) by

$$\widehat{\omega}_M(ij) = \omega_M(ij) + P_{ij} , \ i \neq j . \tag{25}$$

If we set (see (32))

$$\widehat{\mu}_M(i) = \mu_M(i) - P_{ii} ,$$

then it follows from (24) that

$$\widehat{\mu}_M(i) = \sum_{j\neq i} \widehat{\omega}_M(ij) .$$

Moreover, if $\widetilde{M}_i$ and $\widetilde{M}_j$ are both isolated, then from (22) we have $P_{ij} = \langle \mathcal{N}_0^{-1}B\chi_i, B\chi_j\rangle$. Combining this with Claim 4.3, we therefore have the following:

Claim 4.4 *The matrix $A - B^\dagger\mathcal{N}_0^{-1}B$ is the restriction to $\widehat{G}_0$ of the graph Laplacian $\Delta_{(\widehat{G}_\Gamma,\widehat{\omega}_M)}$.*

By the discussion in Sect. 6 below (cf. (41)), we conclude

Lemma 4.5 *Fix $\delta > 0$, and let D denote the diagonal matrix with entry δ for all non-isolated components, and zeros elsewhere. Then* $\det(A - B^\dagger\mathcal{N}_0^{-1}B)$ *is the coefficient of δ^k in* $\det(\Delta_{(\widehat{G}_\Gamma,\widehat{\omega}_M)} + D)$*, where k is the number of nonisolated components.*

Given distinct vertices $v_i, v_j \in V(G_\Gamma)$, then since G_Γ is a tree there is a unique geodesic g_{ij} in G_Γ from v_i to v_j. Moreover, there is a 1-1 correspondence $\Gamma \leftrightarrow E(G_\Gamma)$. For $\gamma \in \Gamma$, we shall say $\gamma \in g_{ij}$ if the edge associated to γ lies on g_{ij}.

Lemma 4.6 *Fix ε_0, $0 < \varepsilon_0 < 1$. There are constants $\kappa \geq 1$ and $C > 0$ such that if $\lambda(\gamma)$ is sufficiently small with respect to ε_0 for all $\gamma \in \Gamma$, then*

$$\kappa^{-1}\omega_M(ij) \leq \widehat{\omega}_M(ij) \leq \kappa\omega(ij)\ ,\ L_{ij} \neq 0\ ; \tag{26}$$

$$|\widehat{\omega}_M(ij)| \leq C \prod_{\gamma \in g_{ij}} \lambda(\gamma)\ ,\ L_{ij} = 0\ . \tag{27}$$

Proof By explicit computation, we may write $\mathcal{N}_0 = \overline{\mathcal{N}}_0 + R$, where R is diagonal with respect to the orthogonal decomposition $V_1 \cap \oplus_{\gamma\in\Gamma} L^2(\partial A_\gamma)$, and the component pieces $R_\gamma \to 0$ in trace class as $\lambda(\gamma) \to 0$. If f_n denotes the n-th Fourier mode of a function f on one boundary component of ∂A_γ, $n \neq 0$, then the n-th Fourier mode of $R_\gamma(f)$ on this component is

$$\frac{|n|\rho^{-2|n|}}{1-\rho^{-2|n|}} f_n\ ,$$

and on the other component of ∂A_γ it is

$$-\frac{2|n|\rho^{-|n|}}{1-\rho^{-2|n|}} f_{-n}\ .$$

In particular, given ε_0 then for sufficiently small $\lambda(\gamma)$ the norm of R is bounded by $\varepsilon_0\lambda(\gamma)$. Since $\overline{\mathcal{N}}_0^{-1}$ is uniformly bounded, we have $\mathcal{N}_0 = \overline{\mathcal{N}}_0(1 + \overline{\mathcal{N}}_0^{-1}R)$, where $\overline{\mathcal{N}}_0^{-1}R$ has small norm in trace class bounded on each component by $\ell(\gamma)$. Let f be supported on $\partial\widetilde{M}_i$ and g on $\partial\widetilde{M}_j$, and fix $p \geq 1$. We claim there is a constant $C' > 0$, independent of i, j, p and the $\lambda(\gamma)$, such that

$$|\langle(\overline{\mathcal{N}}_0^{-1}R)^p f, g\rangle| \leq C'\varepsilon_0^p \|f\|\|g\| \prod_{\gamma\in g_{ij}} \lambda(\gamma)\ .$$

This follows easily by induction on p. We now apply this estimate, and use the expression

$$\mathcal{N}_0^{-1} = \overline{\mathcal{N}}_0^{-1} + \sum_{p=1}^{\infty} (-1)^p (\overline{\mathcal{N}}_0^{-1}R)^p \overline{\mathcal{N}}_0^{-1}$$

in the definition (23) of P_{ij}. For example, for $i \neq j$, one of the terms is

$$\begin{aligned}
&\sum_{k,k'} L_{ik}L_{jk'}\lambda(\gamma_{ik})\lambda(\gamma_{jk'})\langle\mathcal{N}_0^{-1}\chi_{k,\gamma_{ik}}, \chi_{k',\gamma_{jk}}\rangle \\
&\quad = \sum_{k} L_{ik}L_{jk}\lambda(\gamma_{ik})\lambda(\gamma_{jk})\langle\overline{\mathcal{N}}_0^{-1}\chi_{k,\gamma_{ik}}, \chi_{k,\gamma_{jk}}\rangle \\
&\qquad + \sum_{k,k'} L_{ik}L_{jk'}\lambda(\gamma_{ik})\lambda(\gamma_{jk'}) O\left(\prod_{\gamma\in g_{kk'}} \lambda(\gamma)\right)\ ,
\end{aligned} \tag{28}$$

where we have used the fact that $\overline{\mathcal{N}}_0^{-1}(\chi_{k,\gamma_{ik}})$ is supported on $\partial\widetilde{M}_k$ for any i. Notice that the second term on the right-hand side of (28) contains $\lambda(\gamma)$ for every $\gamma \in g_{ij}$, and therefore satisfies both (i) and (ii). The first term vanishes if $L_{ij} \neq 0$, and so automatically satisfies (i). If $L_{ij} = 0$, it contributes $\lambda(\gamma_{ik})\lambda(\gamma_{jk})$ for each vertex k "subadjacent" to i and j (see Fig. 2). In particular, this term satisfies (ii). The other terms in (23) are treated similarly.

Finally, we complete the proof of Proposition 4.2. Fix $\delta > 0$, and suppose M_Γ has k nonisolated components. By the result of Lemma 4.5 and the expansion (33), if the weights $\lambda(\gamma)$ are sufficiently small compared to δ for all $\gamma \in \Gamma$, then the determinants are dominated by the δ^k coefficients. It therefore suffices to relate $\det(\Delta_{(\widehat{G}_\Gamma,\widehat{\omega}_M)} + D)$ to $\det(\Delta_{(G_\Gamma,\omega_g)} + D)$. From (27) and Corollary 6.4, there is a constant $C \geq 1$ such that

$$C^{-1}\det(\Delta_{(G_\Gamma,\widetilde{\omega}_M)} + D) \leq \det(\Delta_{(\widehat{G}_\Gamma,\widehat{\omega}_M)} + D) \leq C\det(\Delta_{(G_\Gamma,\widetilde{\omega}_M)} + D)\,,$$

where $\widetilde{\omega}_M$ is the restriction of the weight function $\widehat{\omega}_M$ on $\widehat{G}_\Gamma$ to G_Γ. By (26) and Corollary 6.2, there is constant $C_1 \geq 1$ such that

$$C_1^{-1}\det(\Delta_{(G_\Gamma,\omega_M)} + D) \leq \det(\Delta_{(G_\Gamma,\widetilde{\omega}_M)} + D) \leq C_1\det(\Delta_{(G_\Gamma,\omega_M)} + D)\,.$$

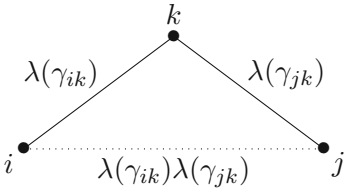
Finally, using (20) and the same comparison between ω_M and ω_g gives upper and lower bounds on $\det(\Delta_{(G_\Gamma,\omega_g)} + D)$. Combining these statements completes the proof of Proposition 4.2.

4.3 Higher Genus

In higher genus, the graph G_Γ will not be a tree in general. This leads to a more complicated perturbation of the graph Laplacian. Nevertheless, it is clear from the proof of Theorem 4.1 that there is a uniform upper bound on $\mathrm{Det}\,\mathcal{N}(M,\widetilde{\Gamma})$. Indeed, the discussion concerned the low eigenvalues, whereas as the operator on the orthogonal complement in the previous section converges in trace class. As a consequence, directly from Theorem 1.1, we have the following:

Theorem 4.7 *Fix positive numbers $g \geq 1$ and $b_1, \cdots, b_n$, $n \geq 1$. Then there is a positive constant C, depending only on c_0 and $(b_1, \cdots, b_n)$, such that the following holds. For any hyperbolic surface M of genus zero with geodesic boundary components of lengths $b_1, \cdots, b_n$ and short geodesics Γ,*

Fig. 2 Added edge



$$C^{-1} \prod_{\gamma \in \Gamma} \ell(\gamma) \leq \mathscr{I}(M) \, .$$

5 Further Results

5.1 Properness and Steklov Isospectral Surfaces

Now we provide proofs of the other consequences of Theorems 1.1 and 4.1.

Proof of Theorem 1.2 Let $\{M_j\}$ be a sequence of genus zero hyperbolic surfaces with geodesic boundaries of fixed lengths $b_1, \ldots, b_n$. After passing to a subsequence, we may assume there is a nonempty collection Γ_j of geodesics all of whose lengths $\ell(\gamma) \to 0$ as $j \to \infty$, and all other geodesics have lengths bounded away from zero. Since the M_j have genus zero, In this case, $(\#\Gamma + 1) \prod \omega_{M_j}(\gamma) = \det{}^* \Delta_{(G_\Gamma, \omega_{M_j})}$ (see (39)), and each $\omega_{M_j}(\gamma)$ is comparable to the length $\ell_{M_j}(\gamma)$. We can then use (15) and Corollary 6.3 below to conclude that

$$\mathscr{I}(M_j) \leq C \, \max \left\{ \ell(\gamma_1) \cdots \ell(\gamma_{k-1}) \mid \gamma_1, \ldots, \gamma_{k-1} \in \Gamma_j \text{ distinct} \right\} \, , \tag{29}$$

where k is the number of nonisolated components of $(M_j)_{\Gamma_j}$ that intersect ∂M_j. We may assume k is constant and C is independent of j. Then (29) implies that

$$\mathcal{H}(M_j) \geq (k-1) \min_{\gamma \in \Gamma_j} \log(1/\ell_j(\gamma)) - \log C \, . \tag{30}$$

For a connected tree with more than one vertex, there are at least two vertices having only a single edge. Since the components of $(M_j)_{\Gamma_j}$ must have at least 3 boundary components, this implies $k \geq 2$. Since $\ell_j(\gamma) \to 0$, (30) implies that $\mathcal{H}(M_j)$ is unbounded along $\{M_j\}$.

If $g \neq 0$, then we may find a family of surfaces M_ε with a geodesic γ_ε of length $\ell(\varepsilon) \sim 1/\log(1/\varepsilon)$, such that $M_\varepsilon \setminus \gamma_\varepsilon$ consists of two components: one component M'_ε containing all components of the boundary $\partial M(\varepsilon)$, and an isolated component N_ε obtained by removing a disk of radius ε from a genus one Riemann surface N. As in the proof above, we may choose an annulus A_ε about γ_ε whose boundary lengths are bounded above and below. Now apply Theorem 1.1 to the case where $\Gamma = \partial A_\varepsilon$. Then the Neumann jump operators $\mathcal{N}$ and $\mathcal{N}_A$ both have a small eigenvalue $\sim 1/\log(1/\varepsilon)$ corresponding to the constant function 1 on the component of ∂A_ε meeting N_ε, and 0 on the other component of ∂A_ε. As in the proof above, orthogonal to this space, the operators converge up to trace class to the corresponding operators on N and the surface M'_ε union a disk. The small eigenvalue cancels the vanishing of $\mathscr{I}(A_\varepsilon)$ in the gluing formula, with the remaining factors bounded. Hence, $\mathcal{H}(M_\varepsilon)$ remains bounded as $\varepsilon \to 0$.

Proof of Corollary 1.3 By [10, Theorem 1.7], the lengths of the boundary components of all the members of $\mathcal{F}$ are equal to some fixed lengths $(b_1, \ldots, b_n)$. From Theorem

1.2, $\mathcal{F}$ is contained in a compact subset of the moduli space $\mathbf{M}(0; b_1, \ldots, b_n)$. The result then follows as in [22].

5.2 Dirichlet and Neumann Laplacians

Corollary 5.1 *Consider the situation in Theorem 4.1. Let $\{\kappa_i\}$ be the collection of small eigenvalues, as in the Introduction. Then the constant C may be chosen such that for any hyperbolic surface M of genus zero with geodesic boundary components of lengths $b_1, \cdots, b_n$,*

$$\begin{aligned} C^{-1} \prod_{\gamma\in\Gamma} \exp(-\pi^2/3\ell(\gamma))\ell^{-3/2}(\gamma) \left(\det(\Delta_{(G_\Gamma,\omega_M)} + D) \prod \kappa_i\right)^{1/2} \\ \leq [\mathrm{Det}\, \Delta_D]_M \\ \leq C \prod_{\gamma\in\Gamma} \exp(-\pi^2/3\ell(\gamma))\ell^{-3/2}(\gamma) \\ \times \left(\det(\Delta_{(G_\Gamma,\omega_M)} + D) \prod \kappa_i\right)^{1/2}, \end{aligned}$$

and

$$\begin{aligned} C^{-1} \prod_{\gamma\in\Gamma} \exp(-\pi^2/3\ell(\gamma))\ell^{-1/2}(\gamma) \left(\frac{\prod \kappa_i}{\det(\Delta_{(G_\Gamma,\omega_M)} + D)}\right)^{1/2} \\ \leq [\mathrm{Det}^*\, \Delta_N]_M \\ \leq C \prod_{\gamma\in\Gamma} \exp(-\pi^2/3\ell(\gamma))\ell^{-1/2}(\gamma) \\ \times \left(\frac{\prod \kappa_i}{\det(\Delta_{(G_\Gamma,\omega_M)} + D)}\right)^{1/2}. \end{aligned}$$

For $g \geq 1$ and $n \geq 1$, we have

$$[\mathrm{Det}\, \Delta_D]_M \leq C \prod_{\gamma\in\Gamma} \exp(-\pi^2/3\ell(\gamma))\ell^{-3/2}(\gamma) \left(\prod \kappa_i\right)^{1/2}.$$

Proof Let $\widehat{M}$ be the double of M. Then decomposing the spectrum with respect to the isometric involution, the small eigenvalues for the Laplacian on the closed surface $\widehat{M}$ are exactly the collection $\{\kappa_i\}$. Moreover, since the boundary lengths of M are fixed, we may ignore them in the asymptotics. Hence, the short geodesics of $\widehat{M}$ correspond to the short geodesics in M and their mirrors in the double. By [32, Theorem 5.3] there is a constant $B > 1$ such that

$$B^{-1} \leq \frac{[\mathrm{Det}^*\, \Delta]_{\widehat{M}}}{\prod_{\gamma\in\Gamma} \exp(-2\pi^2/3\ell(\gamma))\ell^{-2}(\gamma) \prod_i \kappa_i} \leq B\,. \tag{31}$$

On the other hand, from (7) and (8), we have

$$[\mathrm{Det}\,\Delta_D]_M^2 = \frac{[\mathrm{Det}^*\,\Delta]_{\widehat{M}}}{A(M)\,\mathscr{I}(M)} \quad , \quad [\mathrm{Det}\,\Delta_N]_M^2 = [\mathrm{Det}^*\,\Delta]_{\widehat{M}}\,A(M)\,\mathscr{I}(M)\ .$$

The result now follows from (31) and Theorems 4.1 and 4.7.

6 Graph Laplacians

6.1 Matrix Tree Theorem with Potential

For the proof of Theorem 1.2, we require the results in this section, perhaps well known, but for which we have been unable to locate precise statements in the vast literature on this subject. For the sake of completeness, we therefore provide proofs here. This will also allow us to review the construction and basic facts of graph Laplacians. The main result, Theorem 6.1, is an extension of the weighted matrix tree theorem of Kirchhoff for the graph Laplacian with an added diagonal potential. Corollary 6.3 then gives a comparison of the determinants of the graph Laplacians with and without the potential.

Let G be an undirected graph with vertex and edge sets $V(G)$ and $E(G)$, respectively. Label the elements of $V(G)$ by $v_i \in V$, $i = 1, \ldots, n$. For $i \neq j$ we say $(ij) \in E(G)$ if there is an edge between v_i and v_j. We always assume G is *simple*, by which we mean there is at most one edge between distinct vertices, and no edge from a vertex to itself. A weight function on G is a map $\omega : E(G) \to \mathbb{R}$. The weight defines (and is determined by) an associated $n \times n$ symmetric matrix:

$$\omega_{ij} := \begin{cases} \omega(ij) & \text{if } (ij) \in E\ , \\ 0 & \text{otherwise.} \end{cases}$$

If we set $\mu_i = \sum_{(ij)\in E} \omega_{ij}$, then the (weighted) graph Laplacian is the $n \times n$ symmetric matrix:

$$(\Delta_{(G,\omega)})_{ij} = \begin{cases} -\omega_{ij} & (ij) \in E\ , \\ \mu_i & i = j\ , \\ 0 & \text{otherwise.} \end{cases} \tag{32}$$

The weight ω is positive if $\omega(ij) > 0$ for all $(ij) \in E(G)$. When the weights are positive, the matrix $\Delta_{(G,\omega)}$ is positive semidefinite with a zero eigenvalue of multiplicity 1 if G is connected. We let $\det{}^*\Delta_{(G,\omega)}$ denote the product of the nonzero eigenvalues. By a *potential* we mean a function $\delta : V(G) \to \mathbb{R}$. If $\delta_i = \delta(v_i)$, then δ is represented by a diagonal matrix with entries δ_i, which we will typically denote by D. Given a potential, we shall say a vertex v is *marked* if $\delta(v) \neq 0$. The potential is positive if $\delta(v)$ is either zero or positive for every v.

For a connected graph G, let $\mathsf{Sp}(G)$ be the set of spanning trees of G, i.e., connected trees $T \subset G$ such that $V(T) = V(G)$. For a tree T with ℓ marked vertices $v_1, \dots, v_\ell$, let $\mathscr{E}(T; v_1, \dots, v_\ell)$ denote the set of collections of $(\ell-1)$ edges $e_1, \dots, e_{\ell-1} \in E(T)$ such that each of the ℓ connected components of $T \setminus e_1 \cup \cdots \cup e_{\ell-1}$ contains exactly one marked vertex v_j. Finally, for $T \in \mathsf{Sp}(G)$ and $S \in \mathscr{E}(T; v_1, \dots, v_\ell)$, we define a *multiplicity*:

$$m(T, S) = \#\left\{(T', S') \mid T' \in \mathsf{Sp}(G)\ ,\ S' \in \mathscr{E}(T'; v_1, \dots, v_\ell)\ ,\ T' \setminus S' = T \setminus S\right\}$$

For elements of the set above, we shall say that (T', S') *is equivalent to* (T, S). With this understood, we are ready to state the main result.

Theorem 6.1 *Let (G, ω) be a connected weighted graph with n vertices $v_1, \dots, v_n$, and let $\Delta_{(G,\omega)}$ be the graph Laplacian. Fix a potential $\delta : V(G) \to \mathbb{R}$, $\delta_i = \delta(v_i)$ with associated diagonal matrix D. Then*

$$\det(\Delta_{(G,\omega)} + D) = \sum_{T \in \mathsf{Sp}(G)} \sum_{\ell=1}^{n} \sum_{1 \le i_1 < \cdots < i_\ell \le n} \sum_{S \in \mathscr{E}(T; v_{i_1}, \dots, v_{i_\ell})} \frac{\delta_{i_1} \cdots \delta_{i_\ell}}{m(T, S)} \prod_{e \in E(T) \setminus S} \omega(e)\ . \tag{33}$$

Theorem 6.1 will be proved in the next section. First, let us draw some conclusions. An immediate consequence of (33) is the following important

Corollary 6.2 *Suppose (G, ω) is a connected graph with positive weights and a positive potential δ. For $\kappa \ge 1$, there is $C \ge 1$ depending only on κ, G, and δ, such that the following holds. For any weight function $\widetilde{\omega}$ on G with*

$$\kappa^{-1} \omega(e) \le \widetilde{\omega}(e) \le \kappa\, \omega(e) \tag{34}$$

for all $e \in E(G)$, we have

$$C^{-1} \det(\Delta_{(G,\omega)} + D) \le \det(\Delta_{(G,\widetilde{\omega})} + D) \le C \det(\Delta_{(G,\omega)} + D)\ .$$

In the following, we suppose δ has exactly k nonzero entries $\delta_1, \dots, \delta_k$ at $v_1, \dots, v_k$, $1 \le k \le n$. Suppose first that $k = 1$. Notice that in this case, no edges are removed: in the expression (33) the sum over S (and therefore also the multiplicities) is absent. For $\varepsilon > 0$,

$$\frac{d}{d\varepsilon} \log\det(\Delta_{(G,\omega)} + \varepsilon D) = \operatorname{tr}\left((\Delta_{(G,\omega)} + \varepsilon D)^{-1} D\right) = \delta_1 (\Delta_{(G,\omega)} + \varepsilon D)^{-1}_{11}\ ;$$
$$\frac{d}{d\varepsilon} \det(\Delta_{(G,\omega)} + \varepsilon D) = \delta_1 \det((\Delta_{(G,\omega)} + D)^{[1]}) = \delta_1 \det(\Delta^{[1]}_{(G,\omega)})\ .$$

Here, we have introduced the following notation: if $A = (A_{ij})$ is an $n \times n$ matrix, we denote by $A^{[k]}$ the $(n-1) \times (n-1)$ matrix obtained by deleting the k-th row and the k-th column. Since $\det \Delta_{(G,\omega)} = 0$, by integration we get

$$\det(\Delta_{(G,\omega)} + D) = \delta_1 \det(\Delta^{[1]}_{(G,\omega)}) \,. \tag{35}$$

Now by the weighted matrix tree theorem (cf. [27, Thm. VI.27]),

$$\det(\Delta^{[1]}_{(G,\omega)}) = \sum_{T \in \mathsf{Sp}(G)} \prod_{e \in E(T)} \omega(e) \,, \tag{36}$$

and so we obtain (33). In the proof of Theorem 6.1 below, we do not reprove (36) but rather use it as a starting point for an inductive argument.

For $k \geq 2$, the appearance of the multiplicity $m(T, E)$ is a new feature in the generalized matrix tree expression (33). Its necessity is immediate from the δ^n term in case $k = n$, $\delta_1 = \cdots = \delta_n = \delta$. More illuminating is the simple example in Fig. 3.

Here, the weights are $\delta(v_i) = \delta_i$, $i = 1, 2$ and zero otherwise. Then one calculates the $\delta_1\delta_2$ term directly

$$\det(\Delta_{(G,\omega)} + D) = \delta_1\delta_2(\omega_{13} + \omega_{23})(\omega_{14} + \omega_{24}) + \cdots \tag{37}$$

There are 4 spanning trees for G, obtained by removing a single edge. For each tree T, there are two possible edges S that can be removed to separate v_1 from v_2. Thus there are 8 terms in the $\delta_1\delta_2$ sum in (33). But the multiplicity of each pair (T, S) is clearly 2, corresponding to switching the edge removed to define the tree T with the edge S removed from T. The 8 terms thus reduce to the 4 terms in (37).

A second special case is where G is a tree. For $\delta_i > 0$, $i = 1, \ldots, k$, and zero otherwise, from (33) we have

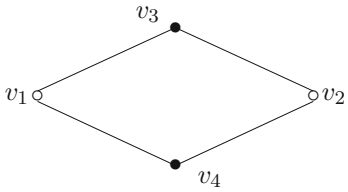
$$\begin{aligned} \det(\Delta_{(G,\omega)} + D) \;\geq\; & \delta_1 \cdots \delta_k \min\{\omega(e_1) \cdots \omega(e_{n-k}) \mid e_1, \ldots, e_{n-k} \in \\ & E(G) \text{ distinct}\} \end{aligned} \tag{38}$$

We are mostly interested in the case where the edge weights are much smaller than the δ_i's. The estimate above can probably be improved. However, notice that in the example (37), $\omega_{13}\omega_{23}$ (or $\omega_{14}\omega_{24}$) do not appear in the $\delta_1\delta_2$ term. If ω_{14} and ω_{24} are big compared to the other two weights, we cannot replace min by max in (38).

For a connected graph, the weighted matrix tree theorem (the equality (36), which holds for any principal minor) implies

$$\det{}^* \Delta_{(G,\omega)} = n \sum_{T \in \mathsf{Sp}(G)} \prod_{e \in E(T)} \omega(e) \,. \tag{39}$$

Fig. 3 Example of multiplicity



In case G is a tree, there is only one term in the sum. Hence, from (35), (36), and (38) we obtain

Corollary 6.3 *Let (G, ω) be a weighted tree with n vertices, and suppose D has exactly $k \geq 1$ nonzero entries $\delta_1, \ldots, \delta_k > 0$. Then if $k = 1$,*

$$\det{}^*\Delta_{(G,\omega)} = \frac{n}{\delta_1} \det(\Delta_{(G,\omega)} + D) \ ,$$

and if $k \geq 2$,

$$\frac{\det{}^*(\Delta_{(G,\omega)})}{\det(\Delta_{(G,\omega)} + D)} \leq \frac{n}{\delta_1 \cdots \delta_k} \max \{\omega(e_1) \cdots \omega(e_{k-1}) \mid e_1, \ldots, e_{k-1} \in E(G) \text{ distinct}\} \ .$$

Finally, an important technical result for this paper is the following, which is used in Sect. 4. Let (G, ω) be a connected, weighted tree with n vertices and positive weights. Let $\widehat{G}$ be the complete graph on the vertices of G. Fix $\kappa_0 > 0$. Suppose $\widehat{\omega}$ is a system of weights (not necessarily positive) for $\widehat{G}$ satisfying

(i) $\widehat{\omega}(e) = \omega(e)$ for all $e \in E(G)$;
(ii) For $\widehat{e} \in E(\widehat{G}) \setminus E(G)$ between vertices v_1 and v_2,

$$|\widehat{\omega}(\widehat{e})| \leq \kappa_0 \omega(e)$$

for any $e \in E(G)$ along the geodesic in G from v_1 to v_2.

Corollary 6.4 *Fix (G, ω) as above, and let $\delta : V(G) \to \mathbb{R}_{\geq 0}$ be a positive potential. Then for $\kappa_0 > 0$ sufficiently small (depending upon (G, ω) and δ), there is a constant $C \geq 1$ depending only on G, δ, and κ_0, such that if $(\widehat{G}, \widehat{\omega})$ satisfies (i) and (ii) above,*

$$C^{-1} \det(\Delta_{(G,\omega)} + D) \leq \det(\Delta_{(\widehat{G},\widehat{\omega})} + D) \leq C \det(\Delta_{(G,\omega)} + D) \ .$$

Proof We wish to compare the terms appearing in (33) for (G, ω) and $(\widehat{G}, \widehat{\omega})$. Let $\widehat{T}$ be a spanning tree for $\widehat{G}$. Consider a component subtree $\widehat{T}'$ of $\widehat{T} \setminus \widehat{S}$, for a separating set of edges $\widehat{S}$. Suppose that $\widehat{T}'$ contains an edge $\widehat{e}$ not in $E(G)$. Let v_1 and v_2 be the two vertices of $\widehat{e}$. Since $\widehat{T}'$ contains only one marked vertex, we may assume v_1 is not marked. Let g be the geodesic in G from v_1 to v_2. Because $\widehat{T}'$ is a tree, we cannot have $g \subset \widehat{T}'$, since then $g \cup \widehat{e}$ would be a cycle. Hence, let e be the first edge in g (going from v_1 to v_2) that is not contained in $\widehat{T}'$. Then if we replace $\widehat{e}$ by e we obtain a new spanning tree $\widehat{T}_1$ (with the same separating set $\widehat{S}$) with fewer edges that are not in $E(G)$. Moreover, by (ii) the product of the edges in $\widehat{T} \setminus \widehat{S}$ is strictly less (in absolute value) than that of $\widehat{T}_1 \setminus \widehat{S}$. Continuing in this way, we find a new spanning tree $\widehat{T}_\bullet$ of $\widehat{G}$ such that $\widehat{T}_\bullet \setminus \widehat{S} \subset G$. Now there is a unique separating set $S \subset E(G)$ such that $\widehat{T}_\bullet \setminus \widehat{S} = G \setminus S$. Thus, the term in the expansion (33) for $\widehat{G}$ corresponding to $(\widehat{T}, \widehat{S})$ is dominated by the term (G, S) in the expansion for G. This completes the proof.

6.2 Proof of Theorem 6.1

The proof is by induction on n and k = the number of nonzero entries of D. Thus, we assume (33) holds for graphs with fewer than n vertices and any D. We have seen in (36) that by the usual weighted matrix tree theorem, the result holds for all n and $k = 1$. Suppose now that $k \geq 2$, and that (33) holds for n vertices and potentials with fewer than k nonzero entries. We must show that for D with exactly k nonzero entries,

$$\begin{aligned}&\det(\Delta_{(G,\omega)} + D)\\ &= \sum_{T\in \mathsf{Sp}(G)} \sum_{\ell=1}^{k} \sum_{1\leq i_1<\cdots<i_\ell\leq k} \sum_{S\in\mathscr{E}(T;v_{i_1},\ldots,v_{i_\ell})} \frac{\delta_{i_1}\cdots\delta_{i_\ell}}{m(T,S)} \prod_{e\in E(T)\setminus S} \omega(e)\,. \end{aligned} \tag{40}$$

Set $\delta_k = \delta(v_k)$. By the same argument used above to derive (36), we have

$$\det(\Delta_{(G,\omega)} + D) = \det(\Delta_{(G,\omega)} + D)\big|_{\delta_k=0} + \delta_k \det((\Delta_{(G,\omega)} + D)^{[k]})\,. \tag{41}$$

We view the second term on the right-hand side as the determinant of a new weighted graph $\widetilde{G}$ with potential $\widetilde{D}$, obtained by deleting v_k and all edges at v_k. The weight function $\widetilde{\omega}$ is the restriction of ω to $\widetilde{G}$. Importantly, since the edges of v_k have been deleted, the μ_j differ from $\widetilde{\mu}_j$, and $\widetilde{D}$ is determined by the rule

$$\widetilde{\delta}_j = \begin{cases} \delta_j & (jk)\notin E(G)\,,\\ \delta_j + \omega_{jk} & (jk)\in E(G)\,, \end{cases} \tag{42}$$

for all $v_j \in \widetilde{G}$. With this interpretation we have

$$\det((\Delta_{(G,\omega)} + D)^{[k]}) = \det(\Delta_{(\widetilde{G},\widetilde{\omega})} + \widetilde{D})\,. \tag{43}$$

By induction on k, we may assume the first term on the right-hand side of (41) satisfies (33), with δ_k set to zero. This accounts for all the terms on the right-hand side of (40) where $i_\ell \leq k-1$. The remaining terms all have $i_\ell = k$, and therefore a factor of δ_k. Given (43), in order to complete the proof we must show that

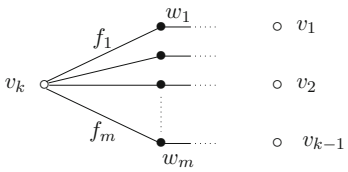
$$\begin{aligned}\det(\Delta_{(\widetilde{G},\widetilde{\omega})} + \widetilde{D}) &= \sum_{\ell=1}^{k} \sum_{1\leq i_1<\cdots<i_{\ell-1}\leq k-1} \delta_{i_1}\cdots\delta_{i_{\ell-1}}\\ &\quad\times \sum_{T\in\mathsf{Sp}(G)} \sum_{S\in\mathscr{E}(T;v_{i_1},\ldots,v_{i_{\ell-1}},v_k)} \frac{1}{m(T,S)} \prod_{e\in E(T)\setminus S} \omega(e)\,. \end{aligned} \tag{44}$$

In the sum above, $\ell = 1$ is taken to mean that no δ_i's appear. Let $w_1,\ldots,w_m$ be the vertices adjacent to v_k, with edges $f_1,\ldots,f_m$. See Fig. 4.

Let us first assume that

(i) $G\setminus\{v_k\}$ *is connected*;

Fig. 4 Marked vertex



(ii) *None of the w_j's are marked in G.*

Thus, by (42), $\widetilde{G}$ has $(k+m-1)$ marked points at $v_1, \ldots, v_{k-1}$ and $w_1, \ldots, w_m$. Since $\widetilde{G}$ has $(n-1)$ vertices and we have assumed the result holds in this case for all k, by induction we have

$$\det(\Delta_{(\widetilde{G},\widetilde{\omega})} + \widetilde{D}) = \sum_{\substack{\ell=1,\ldots,k \\ \ell'=0,\ldots,m}} \sum_{\substack{1\le i_1<\cdots<i_{\ell-1}\le k-1 \\ 1\le j_1<\cdots<j_{\ell'}\le m}} \delta_{i_1}\cdots\delta_{i_{\ell-1}} \times \sum_{\widetilde{T}\in \mathsf{Sp}(\widetilde{G})} \sum_{\widetilde{S}\in\mathscr{E}(\widetilde{T};v_{i_1},\ldots,v_{i_{\ell-1}},w_{j_1},\ldots,w_{j_{\ell'}})} \frac{\widetilde{\delta}_{j_1}\cdots\widetilde{\delta}_{j_{\ell'}}}{m(\widetilde{T},\widetilde{S})} \prod_{e\in E(\widetilde{T})\setminus\widetilde{S}} \omega(e)\,, \tag{45}$$

where by $\ell' = 0$ we mean no $\widetilde{\delta}_j$'s appear, and in the sum we do not allow both $\ell = 1$ and $\ell' = 0$. In order to prove the equality of the right-hand sides of (44) and (45), for a fixed choice of $i_1, \ldots, i_{\ell-1}$, we must find a correspondence between trees and edge sets in $\widetilde{G}$ and G, modulo equivalences.

Case 1. Suppose first that $\ell' \ge 1$. Let $\widetilde{T} \in \mathsf{Sp}(\widetilde{G})$, $\widetilde{S} \in \mathscr{E}(\widetilde{T}; i_1, \ldots, i_{\ell-1}, j_1, \ldots, j_{\ell'})$, so that $\#\widetilde{S} = \ell + \ell' - 2$. Let $\widetilde{e}_1, \ldots, \widetilde{e}_{\ell'-1} \in \widetilde{S}$ be the edges that separate $w_{j_1}, \ldots, w_{j_{\ell'}}$. To be precise, there is a unique geodesic g_{12} in $\widetilde{T}$ from w_{j_1} to w_{j_2}, and by definition of the set $\mathscr{E}$ there is an edge in $\widetilde{S}$ that is a segment of g_{12}. Choose $\widetilde{e}_1$ to be the nearest such edge to w_{j_1} in the simplicial metric. Now consider the geodesic g_{23} from w_{j_2} to w_{j_3}. This may or may not be separated by $\widetilde{e}_1$. If it is, then g_{23} intersects the geodesic g_{13} from w_{j_1} to w_{j_3}. We then choose $\widetilde{e}_2$ to be the nearest edge to w_{j_1} along this geodesic. If g_{23} is not separated, choose $\widetilde{e}_2$ to be the nearest edge to w_{j_2}. Continuing in this way, we determine the collection $\widetilde{e}_1, \ldots, \widetilde{e}_{\ell'-1} \in \widetilde{S}$. Now define $T \subset G$ by

$$T = \left(\widetilde{T} \setminus \widetilde{e}_1 \cup \cdots \cup \widetilde{e}_{\ell'-1}\right) \cup f_{j_1} \cup \cdots \cup f_{j_{\ell'}} \cup \{v_k\}\,.$$

We claim that T is a connected tree. Being a subset of $\widetilde{T}$, $T \cap \widetilde{G}$ is a tree. By construction, $w_{j_1}, \ldots, w_{j_{\ell'-1}}$ are in distinct components of $T \cap \widetilde{G}$. It follows that T is a tree as well. That T is connected follows from the fact that the connected components of $\widetilde{T}\setminus\widetilde{e}_1\cup\cdots\cup\widetilde{e}_{\ell'-1}$ are in 1-1 correspondence with the $\{w_{j_i}\}$. Now the remaining $\ell-1$ edges in $\widetilde{S}$ – let us denote them $e_1, \ldots, e_{\ell-1}$ – provide an element $S \in \mathscr{E}(T; i_1, \ldots, i_{\ell-1}, v_k)$. Indeed, removing $e_1, \ldots, e_{\ell-1}$ separates the $v_{i_1}, \ldots, v_{i_{\ell-1}}$ from all the $w_{j_1}, \ldots, w_{j_{\ell'-1}}$, and further removing $\widetilde{e}_1 \cup \cdots \cup \widetilde{e}_{\ell'-1}$ separates the $w_{j_1}, \ldots, w_{j_{\ell'-1}}$ from themselves in $\widetilde{T}$. It follows that in T, v_k is separated from the $v_{i_1}, \ldots, v_{i_{\ell-1}}$ as well. It is clear that equivalent pairs $(\widetilde{T}, \widetilde{S})$ give equivalent counterparts (T, S). Indeed,

$$T \setminus S = (\widetilde{T} \setminus \widetilde{S}) \cup f_{j_1} \cup \cdots \cup f_{j_{\ell'}} \cup \{v_k\} \ .$$

This construction may be reversed. Starting from the pair (T, S), we construct $(\widetilde{T}, \widetilde{S})$ as follows. Let $f_{j_1}, \ldots, f_{j_{\ell'}}$ be all the edges in $T \setminus S$ from v_k. The first step is to replace (T, S) with an equivalent pair (T', S') so that $f_{j_1}, \ldots, f_{j_{\ell'}}$ are the only edges from v_k in T'. Let $f_p \in S$ be another such edge, to w_p. Let $e \in E(\widetilde{G})$ be the edge realizing the minimal distance from the component of $T \cap \widetilde{G}$ containing w_p to the other components. Then if we let $T' = (T \setminus f_p) \cup e$, $S' = (S \setminus \{f_p\}) \cup \{e\}$, then clearly T' is a tree. Hence, we may assume, up to equivalence, that the edges in T from v_k are not in S. Now the components of $T \cap \widetilde{G}$ are in 1-1 correspondence with the w_{i_j}. Let $\widetilde{e}_1, \ldots, \widetilde{e}_{\ell'-1}$ be edges in $\widetilde{G}$ minimizing the distances between these components. We set

$$\begin{aligned} \widetilde{T} &= (T \cap \widetilde{G}) \cup \widetilde{e}_1 \cup \cdots \cup \widetilde{e}_{\ell'-1} \ , \\ \widetilde{S} &= \left\{\widetilde{e}_1, \ldots, \widetilde{e}_{\ell'-1}, e_1, \ldots, e_{\ell-1}\right\} \ . \end{aligned}$$

Then the pair $(\widetilde{T}, \widetilde{S})$ is the desired inverse, modulo equivalence of the previous construction. Finally, notice from (42) that in this construction, $\widetilde{\delta}_{j_i} = \omega(f_{j_i})$. Hence,

$$\prod_{e \in T \setminus S} \omega(e) = \widetilde{\delta}_{j_1} \cdots \widetilde{\delta}_{j_{\ell'}} \prod_{e \in \widetilde{T} \setminus \widetilde{S}} \omega(e) \ .$$

We have therefore found a correspondence of terms in (44) with some $f_j \in T \setminus S$, and terms in (45) with $\ell' \geq 1$. As seen above, equivalent pairs $(\widetilde{T}, \widetilde{S})$ give equivalent counterparts (T, S).

Case 2. Now suppose $\ell' = 0$, i.e., none of the points $w_1, \ldots, w_m$ are marked. Note that by our rule this forces $\ell \geq 2$. Let $\widetilde{T} \in \mathsf{Sp}(\widetilde{G})$, $\widetilde{S} \in \mathscr{E}(\widetilde{T}; i_1, \ldots, i_{\ell-1})$, so that now $\#\widetilde{S} = \ell - 2$. If nonempty, enumerate the elements of $\widetilde{S}$ by $\widetilde{e}_1, \ldots, \widetilde{e}_{\ell-2}$. Write a disjoint union

$$\{w_1, \ldots, w_m\} = \mathcal{C}_1 \sqcup \cdots \sqcup \mathcal{C}_q \ ,$$

so that each $\mathcal{C}_i$ lies in a distinct connected component of $\widetilde{T} \setminus \widetilde{e}_1 \cup \cdots \cup \widetilde{e}_{\ell-2}$. As in the previous case, we may choose a subset of $\widetilde{S}$, which after renumbering we assume to be $\widetilde{e}_1, \ldots, \widetilde{e}_{q-1}$, so that each $\mathcal{C}_i$ lies in a distinct connected component of $\widetilde{T} \setminus \widetilde{e}_1 \cup \cdots \cup \widetilde{e}_{q-1}$. For each $\mathcal{C}_i$, choose an edge f_{j_i} from one of the elements of $\mathcal{C}_i$ to v_k. We obtain a tree $T \in \mathsf{Sp}(G)$ by adding the edges $f_{j_1}, \ldots, f_{j_q}$ to $\widetilde{T}$, and deleting $\widetilde{e}_1, \ldots, \widetilde{e}_{q-1}$. The new edge set is

$$S = \{f_{j_1}, \ldots, f_{j_q}, \widetilde{e}_q, \ldots, \widetilde{e}_{\ell-2}\} \in \mathscr{E}(T; v_{i_1}, \ldots, v_{i_{\ell-1}}, v_k) \ .$$

Clearly, the choice of f_{j_i}'s give equivalent pairs (T, S). Similarly, equivalent pairs $(\widetilde{T}, \widetilde{S})$ give equivalent pairs (T, S). In this case, $T \setminus S = (\widetilde{T} \setminus \widetilde{S}) \cup \{v_k\}$. Going the other way, suppose the edges from v_k of a spanning tree T are $f_{j_1}, \ldots, f_{j_q}$, and that these are all contained in an edge set S. Let $\widetilde{e}_q, \ldots, \widetilde{e}_{\ell-2}$ denote the remaining edges

in S. Then $T \cap \widetilde{G}$ has exactly q connected components in 1-1 correspondence with the w_{j_i}. Find $\widetilde{e}_1, \ldots, \widetilde{e}_{q-1}$ in $\widetilde{G}$ connecting these components of w_{i_j}'s, in a manner exactly the same as in Case 1. We then set

$$\widetilde{T} = (T \cap \widetilde{G}) \cup \widetilde{e}_1 \cup \cdots \cup \widetilde{e}_{q-1} \ ,$$
$$\widetilde{S} = \{\widetilde{e}_1, \ldots, \widetilde{e}_{\ell-2}\} \ .$$

This is inverse to the previous construction. Thus, we have a correspondence between terms in (44) with $f_j \notin T \setminus S$, $j = 1, \ldots, m$, and terms in (45) with no $\widetilde{\delta}_j$'s.

We now address assumptions (i) and (ii). Suppose that $G \setminus \{v_k\}$ is not connected. Notice that the right-hand side of (45) is multiplicative and that extending spanning trees of each component of $G \setminus \{v_k\}$ to include v_k uniquely determines a spanning tree of G. Hence, since the analysis above applies to each component assumption (i) may be dropped. For (ii), if one of the points, e.g., w_1, is marked in G, then after relabeling we may assume $w_1 = v_{k-1}$. Then

$$\widetilde{\delta}(w_1) = \delta(v_{k-1}) + \omega(f_1) \ ,$$

and $\widetilde{G}$ has $k + m - 2$ marked points. In the expression (45), terms involving $\widetilde{\delta}(w_1)$ split into terms with $\delta(v_{k-1})$ and those with $\omega(f_1)$. The latter correspond to terms in (44) with v_{k-1} unmarked, just as in the cases considered above. The terms involving $\delta(v_{k-1})$ correspond to terms in (44) with v_{k-1} marked. In this case, in the definition of T, we include f_1 in the set S, but otherwise proceed as above. This completes the proof.

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