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Maximum clique deleted from ramsey graphs of a graph and paths

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Abstract
For graphs F, G and H, let $F \to (G, H)$ signify that any red-blue edge coloring of F contains either a red G or a blue H, hence the Ramsey number $R(G, H)$ is the smallest r such that $K_r \to (G, H)$. Define K_t as the surplus clique of (G, H) if $K_r \setminus K_t \to (G, H)$, where $r = R(G, H)$. For any graph G with $s(G) = 1$, we shall show that the maximum order of surplus clique of (G, P_n) is exactly $\lceil \frac{n}{2} \rceil$ for large n.

Keywords Ramsey number · Ramsey surplus clique · Path · Book

1 Introduction

For a graph $G = (V, E)$, we write $v(G) = |V|$ and $e(G) = |E|$ as usual. Let $\alpha(G)$ and $\chi(G)$ be the independence number and the chromatic number of G, respectively. Furthermore, denote by $s(G)$ the chromatic surplus of G that is the minimum size of a color class in a proper $\chi(G)$-coloring of vertices of G.

For vertex disjoint graphs G and H, denote by $G + H$ the join of G and H obtained by additional edges connecting $V(G)$ and $V(H)$ completely. Let $G \cup H$ be the disjoint union of G and H, and nG the vertex disjoint copies of G. Call $B_m^{(k)} = K_k + mK_1$ **book** consisting of m complete graphs K_{k+1} that share a common K_k, in which the common K_k is called the base of the book. We shall write $B_m^{(2)}$ as B_m that is the first non-trivial book for $k = 2$. As usual, denote by T_n the tree of n vertices, in which P_n is a path and $K_{1,n-1}$ is a star of n vertices particularly. Let C_n be a cycle of n vertices.

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If $V(G) \subseteq V(K_r)$, let $K_r \setminus G$ be the graph obtained from K_r by deleting the edges of G from K_r, hence $K_r \setminus K_1 = K_r$. If H is a subgraph of G and $V(H) \subseteq V(G)$, we denote by $G - H$ the subgraph of G induced by $V(G) \setminus V(H)$.

For graphs F, G and H, let $F \to (G, H)$ signify that any red-blue edge coloring of F contains either a red G or a blue H. The Ramsey number is defined as

$$R(G, H) = \min\{r \mid K_r \to (G, H)\}.$$

It is easy to see that if $v(F) < R(G, H)$, then $F \not\to (G, H)$. Call F a Ramsey graph of (G, H) if $F \to (G, H)$ and $v(F) = R(G, H)$. So K_r is a Ramsey graph of (G, H) with $r = R(G, H)$. However, some subgraph L of K_r may be "surplus" in sense that $K_r \setminus L \to (G, H)$.

Since $r = R(G, H)$ is the minimum order of K_r such that $K_r \to (G, H)$, one can ask how add a new vertex v and edges to connect v with some vertices of K_{r-1} such that the resultant graph is a Ramsey graph of (G, H). Let $K_{r-1} \sqcup K_{1,t}$ be the graph obtained from K_{r-1} and an additional vertex v that is adjacent to t vertices of K_{r-1}. Hook and Isaak (2011) defined the star-critical Ramsey number as follows.

Definition 1 Hook and Isaak (2011) Let G and H be graphs. The star-critical Ramsey number $R_*(G, H)$ is defined as

$$R_*(G, H) = \min\{t \mid K_{r-1} \sqcup K_{1,t} \to (G, H)\},$$

where $r = R(G, H)$.

The idea of surplus subgraphs initiated from (Wang and Li 2020; Wang et al. 2021), in which the star-critical Ramsey number $R^*(G, H)$ is defined as

$$R^*(G, H) = \max\{t \mid K_r \setminus K_{1,t} \to (G, H)\}, \tag{1.1}$$

in which $r = R(G, H)$ and $R^*(G, H)$ is the maximum size of surplus star in K_r.

Clearly, $R_*(G, H) + R^*(G, H) = R(G, H) - 1$. Many star-critical Ramsey numbers $R_*(G, H)$ and $R^*(G, H)$ for various pairs G and H have been determined, see, e.g, (Haghi et al. 2017; Hao and Lin 2018; Hook 2015; Hook and Isaak 2011; Jayawardene et al. 2021; Li and Li 2015; Liu and Chen 2021; Wang and Li 2020; Wang et al. 2021; Wu et al. 2015; Zhang et al. 2016).

However, unlike $R_*(G, H)$ that is defined for star only, the definition of $R^*(G, H)$ can be extended by replacing the surplus star $K_{1,t}$ in (1.1) with another graph. For example, we can define path-critical Ramsey number as $\max\{t \mid K_r \setminus P_t \to (G, H)\}$.

After publications of (Wang and Li 2020; Wang et al. 2021), the authors are suggested to rename the critical Ramsey number defined in (Wang and Li 2020; Wang et al. 2021) to avoid confusion with the star-critical Ramsey number defined earlier in (Hook and Isaak 2011). We now give the following definition.

Definition 2 Let G and H be graphs. Define Ramsey surplus clique number $R^\omega(G, H)$ as the maximum size of surplus clique of (G, H), namely,

$$R^\omega(G, H) = \max\left\{t \mid K_r \setminus K_t \to (G, H)\right\}, \tag{1.2}$$

where $r = R(G, H)$.

It is easy to define other Ramsey surplus numbers by replacing the surplus K_t in (1.2). For example, in addition to (1.1) and (1.2), we can define Ramsey surplus path number $R^{\pi}(G, H)$ and Ramsey surplus matching number $R^{\mu}(G, H)$ by replacing K_t in (1.2) with path P_t and matching M_t, respectively. In this note, we consider the Ramsey surplus clique numbers in form $R^{\omega}(G, P_n)$.

Burr (1981) called a connected graph H to be G-good if $v(H) \geq s(G)$ and

$$R(G, H) = (\chi(G) - 1)(v(H) - 1) + s(G).$$

The following result of Chvátal (1977) says that T_n is K_m-good for any positive m and n as

$$R(K_m, T_n) = (m - 1)(n - 1) + 1.$$

Among trees T_n, paths are of particular interest. Erdős et al. (1985) showed that P_n is G-good if n is large. Furthermore, Pokrovskiy and Sudakov (2017) proved that P_n is G-good if $n \geq 4v(G)$, which improved the lower bound of n in Pei and Li (2016) as a linear form on $v(G)$.

Lemma 1 Pokrovskiy and Sudakov (2017) *Let G be a graph. If $n \geq 4v(G)$, then P_n is G-good.*

The Ramsey surplus clique number $R^{\omega}(C_4, P_n)$ was determined in Wang and Li (2020) to be $\lceil \frac{n}{2} \rceil$ for $n \geq 4$. In this note, we obtain $R^{\omega}(G, P_n)$ as $\lceil \frac{n}{2} \rceil$ for graph in form of $K_1 + G$. We first give a general upper bound $R^{\omega}(G, P_n) \leq \lceil \frac{n}{2} \rceil$ for sufficiently large n, and then we determine some reasonable magnitudes for n such that the upper bound becomes an equality.

Our main results are as follows.

Theorem 1 *Let G be a connected graph with $v(G) \geq 2$. Then*

$$R^{\omega}(G, P_n) \leq \left\lceil \frac{n}{2} \right\rceil + r - (\chi(G) - 1)(n - 1) - s(G),$$

where $r = R(G, P_n)$. Particularly, if P_n is G-good, then

$$R^{\omega}(G, P_n) \leq \left\lceil \frac{n}{2} \right\rceil. \tag{1.3}$$

Let $M(G)$ be the maximum number of vertices in a color class among all proper vertex coloring of G with $\chi(G)$ colors. We shall prove the following result for $n \geq 9\chi(G)M(G) + 1$, but we only write it in a weaker form as $n \geq 9v^2(G)$ as we believe that n can have a lower bound of linear on $v(G)$ similar to that in Lemma 1.

Theorem 2 *Let G be a graph with $s(G) = 1$. If $n \geq 9v^2(G)$, then*

$$R^{\omega}(G, P_n) = \left\lceil \frac{n}{2} \right\rceil.$$

The lower bounds of n in following results are reasonable for cases $G \in \{K_m, B_m^{(k)}\}$, in which $G \neq K_{1,m}$ as $k \geq 2$ in Theorem 4.

Theorem 3 *If integers $m, n \geq 2$, then $R^{\omega}(K_m, P_n) = \lceil \frac{n}{2} \rceil$.*

Theorem 4 *Let $m \geq 1,\ k \geq 2$ and $n \geq 4(m+k)$ be integers. Then*

$$R^{\omega}(B_m^{(k)}, P_n) = \left\lceil \frac{n}{2} \right\rceil.$$

It is natural to propose the following problem.

Problem 1 Determine general graph G such that (1.3) becomes an equality for large n.

2 Proofs of the main results

For a red-blue edge colored G, denote by G^R and G^B the subgraphs of G induced by the red edges and blue edges, respectively. For subset $S \subseteq V(G)$, denote by $G[S]$ the subgraph of G induced by S. For subgraph $H \subseteq G$ and $v \in V(G)$, denote by $N_H(v)$ the set of neighbors of v in H. Let $N_H^R(v)$ and $N_H^B(v)$ be the sets of all red and blue neighbors of v in H in a red-blue edge colored G, respectively. Hence $d_H^R(v) = |N_H^R(v)|$ and $d_H^B(v) = |N_H^B(v)|$. Therefore $d_H(v) = d_H^R(v) + d_H^B(v)$.

For the figures in this paper, red edges are solid line and blue edges are long dashed line. Short dashed lines represent sets. If all the edges between two sets are red (or blue), a solid (or long dashed) line is drawn between the sets.

Proof of Theorem 1. Let $\ell = \lceil n/2 \rceil + r - (\chi(G) - 1)(n-1) - s(G)$, where $r = R(G, P_n)$. For convenience, let $\chi = \chi(G)$ and $s = s(G)$.

Consider the graph $F = K_r \setminus K_{\ell+1}$. Note that

$$F = K_{r-\ell-1} + (\ell+1)K_1.$$

Color the edges of F by red and blue. Denote by R and B the subgraphs induced by red edges and blue edges, respectively.

$$B = (\chi - 2)K_{n-1} \cup K_{s-1} \cup (K_{\lfloor n/2 \rfloor - 1} + (\ell+1)K_1).$$

Then color the rest edges of F by red. We can see that B contains no P_n, since the order of the longest path of each component of B is at most $n-1$. Furthermore, since $\chi(R) = \chi$ and $s(R) = s - 1$, R contains no G. Then we can deduce that

$$R^{\omega}(G, P_n) \leq \left\lceil \frac{n}{2} \right\rceil + r - (\chi(G) - 1)(n-1) - s(G),$$

completing the proof. □

Before proceeding to the proof of Theorem 2, we need the following result, in which G is a star $K_{1,m}$.

Lemma 2 Parsons (1974) *Let* $m \geq 2$. *If* $n \geq 2m-1$, *then*

$$R(K_{1,m}, P_n) = n.$$

Lemma 3 *Let integers* $m \geq 2$ *and* $n \geq 6m+1$. *Then* $R^{\omega}(K_{1,m}, P_n) = \lceil n/2 \rceil$.

Proof By Theorem 1, we only need to prove the lower bound. Let $r = R(K_{1,m}, P_n) = n$. Denote by $G' = K_r \backslash K_{\lceil n/2 \rceil} = K_{\lfloor n/2 \rfloor} + \lceil n/2 \rceil K_1$. Set $X = K_{\lfloor n/2 \rfloor}$ and $Y = \lceil n/2 \rceil K_1$.

Assume that G' contains neither a red $K_{1,m}$ nor a blue P_n, and we shall find a contradiction. Let $P_\ell = v_1 v_2 \ldots v_\ell$ be a longest blue path in G'.

Case 1. If $(V(G') \backslash V(P_\ell)) \cap V(X) \neq \emptyset$, we may assume that $u \in (V(G') \backslash V(P_\ell)) \cap V(X)$. Note that u can not be adjacent to two consecutive vertices in P by blue, otherwise we get a longer path. So $d_P^R(u) \geq \lceil \ell/2 \rceil$. Note that $\ell \geq \lfloor n/2 \rfloor + 1$ as $R(K_{1,m}, P_{\lfloor n/2 \rfloor + 1}) = \lfloor n/2 \rfloor + 1$. Thus

$$d_P^R(u) \geq \frac{\lfloor n/2 \rfloor + 1}{2} \geq m,$$

yielding a red $K_{1,m}$ with central vertex u.

Case 2. If $(V(G') \setminus V(P_\ell)) \cap V(X) = \emptyset$, then $V(G') \setminus V(P) \subseteq V(Y)$. We may assume $V(G') \setminus V(P) = \{y_1, \ldots, y_t\}$. Then either at least one of the $\{v_1, v_\ell\}$ belongs to $V(X)$, or there exist two consecutive vertices v_{i-1}, v_i such that $v_{i-1}, v_i \in V(X)$ since $|V(P) \cap V(X)| \geq |V(P) \cap V(Y)|$.

Suppose $v_1 \in V(X)$. So $v_1 y_j$ is red for any $j = 1, \ldots, t$ and $d_P^R(v_1) \leq m - t - 1$, implying that $t \leq m-1$ and

$$d_P^B(v_1) \geq \ell - 1 - (m - t - 1) = \ell - m + t.$$

Thus we get at least $\ell - m + t - 1$ new paths with new end vertices. So at least $\ell - m + t - 1 - (\lceil n/2 \rceil - t)$ new end vertices belong to $V(X)$, which are adjacent to $\{y_1, \ldots, y_t\}$ by red completely. Since $\ell = n - t$ and

$$\ell - m + t - (\lceil n/2 \rceil - t) \geq n - m - \lceil n/2 \rceil \geq m,$$

we get a red $K_{1,m}$ with some central vertex y_i.

Suppose $v_{i-1}, v_i \in V(X)$, then $v_1 v_i$ and $v_{i-1} v_\ell$ are red, otherwise we get a new P_s with end vertex $v_{i-1} \in V(X)$ or $v_i \in V(X)$, which is proved above. So we may assume $v_1, v_\ell \in V(Y)$.

Note that $d_P^R(v_1) \leq m-1$. So $d_P^B(v_1) = d_X^B(v_1) \geq \lfloor n/2 \rfloor - (m-1)$. Thus we get at least $\lfloor n/2 \rfloor - m$ new paths with v_{i-1}, v_i still being two consecutive vertices in these new paths since $v_1 v_i$ is red. Therefore at least $\frac{\lfloor n/2 \rfloor - m}{2}$ new end vertices are adjacent to v_{i-1} or v_i by red as the index of new end vertex can be larger than i or smaller than $i-1$. Since $\frac{\lfloor n/2 \rfloor - m}{2} \geq m$, we get a red $K_{1,m}$ with central vertex v_{i-1} or v_i, completing the proof. □

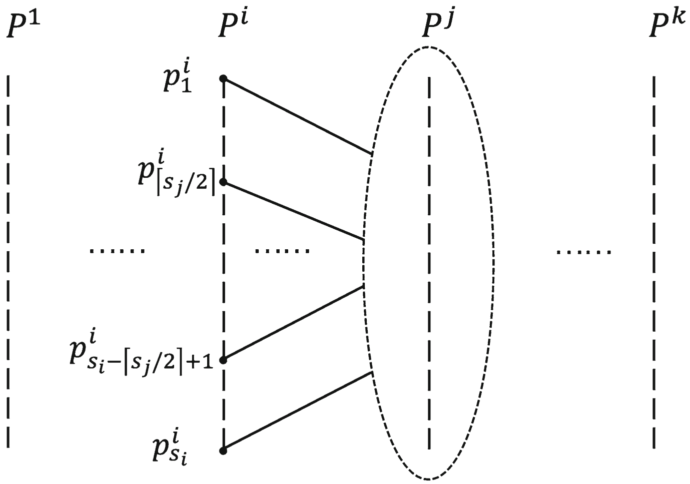

Fig. 1 The edge coloring between part vertices of $V(P^i)$ and $V(P^j)$

Lemma 4 *Let G be a graph. If $n \geq 9\chi(G)M(G)+1$, then*

$$R^{\omega}(K_1+G, P_n) = \left\lceil \frac{n}{2} \right\rceil.$$

Proof By Theorem 1, we only need to prove the lower bound. Let $k = \chi(G)$. Let m be the largest size of k color classes of G. By Lemma 1, $r = R(K_1+G, P_n) = k(n-1)+1$ for $n \geq 4(v(G)+1)$. Denote by $H = K_r \backslash K_{\lceil n/2\rceil} = K_{r-\lceil n/2\rceil} + \lceil n/2\rceil K_1$. Set $X = K_{r-\lceil n/2\rceil}$ and $Y = \lceil n/2\rceil K_1$. We may assume that H contains neither a blue P_n nor a red K_1+G. We shall find a contradiction.

Choose the blue paths $P^1, P^2, \cdots, P^t$ in H as follows: $\cup_{j=1}^{t} V(P^j) = V(H)$, P^1 is a longest blue path in H, and, if $t > 1$, P^{i+1} is a longest blue path in the graph induced by $V(H) \setminus \cup_{j=1}^{i} V(P^j)$ for $1 \leq i \leq t-1$. Denote by s_j the numbers of vertices on the path P^j. So $s_1 \geq s_2 \geq \ldots \geq s_t$. Note the fact that $t \geq k+1$. Otherwise if $t \leq k$, then $s_1 \geq r/k \geq n$, yielding a blue P_n. Furthermore, the graph induced by $V(H) \setminus \cup_{j=1}^{k-1} V(P^j)$ contains no red K_1+G, and then we have

$$s_k \geq \frac{\Big(r - \left\lceil \frac{n}{2} \right\rceil - (k-1)(n-1)\Big) - 1}{k} + 1 \geq \frac{n-1}{2k}$$

since $R(K_1+G, P_n) = k(n-1)+1$ and $s_j \leq n-1$ for $j = 1, \ldots, k-1$.

Fix an orientation of each P^j and denote by $P^j = p_1^j p_2^j \ldots p_{s_j}^j$ for $j = 1, \ldots, k$. Note that each p_ℓ^i with $1 \leq \ell \leq \lceil s_j/2\rceil$ and $s_i - \lceil s_j/2\rceil + 1 \leq \ell \leq s_i$ for $1 \leq i \leq j-1$ is adjacent to $V(P^j)$ by red edges only, as shown in Fig. 1. Otherwise we shall get a longer blue path than P^i, implying that most edges among these blue paths are red. Then we shall show that actually, all of the existing edges among these blue paths can only be red.

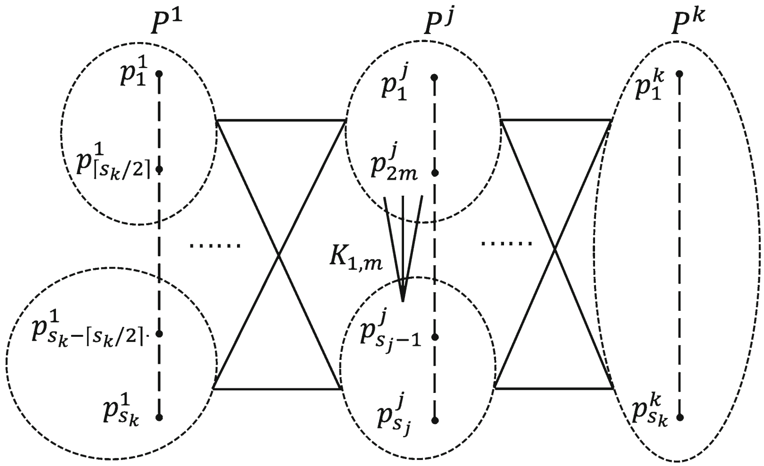

Fig. 2 The case that $V(P^j)$ induces a red $K_{1,m}$

For any path $P^j = p_1^j p_2^j \dots p_{s_j}^j$ with $j = 1, \dots, k$, there must be a blue edge between sets $\{p_1^j, \dots, p_{2m}^j\}$ and $\{p_{s_j-1}^j, p_{s_j}^j\}$. Otherwise if all edges between sets $\{p_1^j, \dots, p_{2m}^j\}$ and $\{p_{s_j-1}^j, p_{s_j}^j\}$ are red, then $V(P^j)$ induces a red $K_{1,m}$ since at least half of the vertices in $\{p_1^j, \dots, p_{2m}^j\}$ and $\{p_{s_j-1}^j, p_{s_j}^j\}$ belong to $V(X)$. Note that all existing edges among sets

$$\begin{aligned}
&\{p_1^1, \dots, p_{\lceil s_k/2 \rceil}^1, p_{s_1-\lceil s_k/2 \rceil+1}^1, \dots, p_{s_1}^1\},\\
&\{p_1^2, \dots, p_{\lceil s_k/2 \rceil}^2, p_{s_1-\lceil s_k/2 \rceil+1}^2, \dots, p_{s_2}^2\},\\
&\dots\dots,\\
&\{p_1^k, \dots, p_{s_k}^k\}
\end{aligned}$$

are red, along with the red $K_{1,m}$ in the graph induced by $V(P^j)$, we get a red $K_1 + G$, as shown in Fig. 2. Thus each path P^j contains a blue cycle of length at least $s_j - 2m$.

Denote by C^j the blue cycle in P^j. Note that each vertex in $V(C^i)$ is adjacent to $V(P^j)$ by red edges only, for any $1 \le i < j \le k$. Otherwise we shall get a longer path than P^i, contradicting to the fact that P^i is a longest path in the graph induced by $V(H) \setminus \cup_{\ell=1}^{i-1} V(P^\ell)$. Thus for any $1 \le i < j \le k$, vertices of $V(P^i)$ are adjacent to $V(P^j)$ by red edges only, as shown in Fig. 3.

If there is a vertex $u \in V(H) \setminus \cup_{\ell=1}^{k} V(P^\ell)$ that is adjacent to each P^j by red edges only for $1 \le j \le k$, then we get a red $K_1 + G$ with central vertex u. So we may assume that for any vertex $v \in V(H) \setminus \cup_{\ell=1}^{k} V(P^\ell)$, there exists some i, $1 \le i \le k$ such that $d_{P^i}^B(u) \ge 1$. Consider any vertex $v \in V(H) \setminus \cup_{j=1}^{k} V(P^j)$, then v must be adjacent to $k-1$ blue paths by red edges only. Otherwise suppose vu_i and vu_j are blue with $u_i \in V(P^i)$ and $u_j \in V(P^j)$ for $i < j$. Then similarly, we can get a longer blue path than P^i, contradicting to the fact that P^i is a longest path in the graph induced by $V(H) \setminus \cup_{\ell=1}^{i-1} V(P^\ell)$.

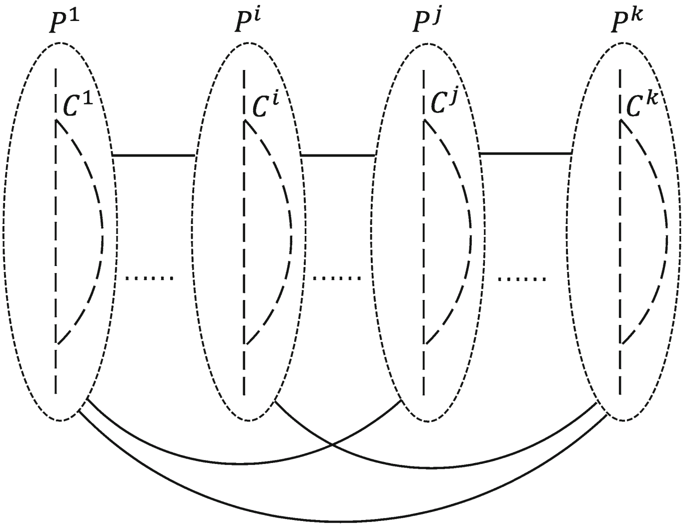

Fig. 3 The edge coloring between $V(P^i)$ and $V(P^j)$

Renew each set $V(P^j)$ to Q^j by adding blue neighbors of $V(P^j)$ in $V(H) \setminus \cup_{\ell=1}^{k} V(P^\ell)$. Note that each vertex in Q^j is adjacent to Q^i by red edges only for $1 \le i, j \le k$ and $i \ne j$. By pigeonhole principle, there must be a Q^j with $|Q^j| \ge n$. If the graph induced by Q^j contains a red $K_{1,m}$, then we get a red $K_1 + G$. So if there is no red $K_{1,m}$ in the graph induced by Q^j, then by Lemma 3, we get a blue P_n, completing the proof. □

Proof of Theorem 2. Since $s(G) = 1$, we know that there is a vertex v in G such that $G_v = G - \{v\}$ has $\chi(G_v) = \chi(G) - 1$. By Lemma 4, we have $n \ge 9\chi(G)M(G) \ge 9\chi(G_v)M(G_v)$, and thus $R^\omega(K_1 + G_v, P_n) = \lceil \frac{n}{2} \rceil$. Then, from $K_r \setminus K_{\lceil n/2 \rceil} \to K_1 + G_v$, we know $K_r \setminus K_{\lceil n/2 \rceil} \to G$ as $K_1 + G_v$ contains G, and the assertion follows. □

We shall separate the main proof of Theorem 3 into several lemmas.

Definition 3 For any integers $n, m \ge 2$, let $r = R(K_m, P_n) = (n-1)(m-1)+1$. For $i = 1, 2, \ldots, m-1$, define the graph $H_i = K_{s_i} + q_i K_1$, where s_i and q_i are integers with $0 \le q_i \le \lceil n/2 \rceil - 1$, $s_i + q_i = n - 1$ and $\sum_{i=1}^{m-1} q_i = \lceil n/2 \rceil - 1$. Define the graph G to be $K_{r-1} \setminus K_{\lceil n/2 \rceil - 1}$ with a red-blue edge coloring such that

$$G^B = \bigcup_{i=1}^{m-1} H_i$$

and G^R is the complement graph of G^B in G.

Lemma 5 *For integers $n \ge 2$ and $t \ge \lfloor n/2 \rfloor + 1$, let G be a red-blue edge colored graph with $G = X + Y$, where $X = K_t$ and $Y = (\lceil n/2 \rceil - 1)K_1$. If each vertex*

$x \in V(X)$ has $d_G^B(x) \geq n-1$, and each vertex $y \in V(Y)$ has $d_G^B(y) \geq \lfloor n/2 \rfloor$, then G must contain a blue P_n.

Proof Denote by $P_n = p_1 p_2 \dots p_n$. We shall embed P_n in G^B. Embed p_1 in $V(Y)$, and then embed p_2 in $V(X)$ since $d_G^B(p_1) \geq \lfloor n/2 \rfloor$. Then we shall always conduct the following principle in each step: If $d_Y^B(p_i) \geq 1$, then we shall embed p_{i+1} in $V(Y)$ for $i = 2, \dots, n-1$. Namely, we embed $V(P_n)$ in Y as much as possible. Let $P_\ell = p_1 p_2 \dots p_\ell$, which is the longest path that we can embed in G^B. If $\ell \geq n$, then we are done. Suppose $\ell \leq n-1$. If $p_\ell \in V(X)$, then we can find a blue neighbor of p_ℓ outside P_ℓ since $d_G^B(p_\ell) \geq n-1$, yielding a contradiction. So we may assume $p_\ell \in V(Y)$. As p_ℓ can only be adjacent to vertices in $V(X)$, we can divide $V(P_\ell) \cap V(X)$ into sets A and B such that

$$A = \{p_i | p_i \in V(X), p_{i+1} \in V(X)\}, B = \{p_i | p_i \in V(X), p_{i+1} \in V(Y)\}.$$

Note that any vertex in A can not be adjacent to p_ℓ in G^B since it will violate the principle. Therefore p_ℓ can only be adjacent to vertices in B. Since $|B| \leq |Y| - 1 \leq \lceil n/2 \rceil - 2$, we have

$$d_{X-P_\ell}^B(p_\ell) \geq \lfloor n/2 \rfloor - (\lceil n/2 \rceil - 2) \geq 1.$$

Then we can find a longer path $P_{\ell+1}$, completing the proof. □

Lemma 6 *For a red-blue edge colored $G = K_{r-\lceil n/2 \rceil} + (\lceil n/2 \rceil - 1)K_1$, where $r = R(K_m, P_n)$, if G is (K_m, P_n)-free, then G must be the graph described in Definition 3.*

Proof It is trivial for $m = 2$ as $s_1 = n - \lceil n/2 \rceil$ and $q_1 = \lceil n/2 \rceil - 1$, yielding $G^B = K_{s_1} + q_1 K_1$ and G^R is empty. So we may assume $m \geq 3$. Let $X = K_{r-\lceil n/2 \rceil}$ and $Y = (\lceil n/2 \rceil - 1)K_1$.

There must be a vertex $x_1 \in V(X)$ such that $d_G^R(x_1) \geq (n-1)(m-2)$. Suppose to the contrary that for each vertex $x \in V(X)$, $d_G^R(x) \leq (n-1)(m-2) - 1$. Then we have

$$d_G^B(x) \geq (n-1)(m-1) - 1 - [(n-1)(m-2) - 1] \geq n - 1.$$

Note that for any vertex $y \in V(Y)$, $d_X^R(y) \leq (n-1)(m-2)$. Otherwise there is a red K_{m-1} or a blue P_n in $G[N_X^R(y)]$. Along with vertex y, we get a red K_m. Thus

$$d_G^B(y) = d_X^B(y) \geq (n-1)(m-1) + 1 - \left\lceil \frac{n}{2} \right\rceil - (n-1)(m-2) \geq \left\lfloor \frac{n}{2} \right\rfloor.$$

So by Lemma 5, we can find a blue P_n.

Define subgraph $G_1 \subseteq G$ such that $V(G_1) \subseteq N_G^R(x_1)$ and $v(G_1) = (n-1)(m-2)$. Note that G_1 contains neither a red K_{m-1} nor a blue P_n. Denote by $H_1 = G - G_1$ and $V(H_1) = S_1 \cup Q_1$ with $S_1 \subseteq V(X)$ and $Q_1 \subseteq V(Y)$. Let $s_1 = |S_1|$ and $q_1 = |Q_1|$.

Note that $s_1 + q_1 = n - 1$. Similarly, we can show that there must be a vertex $x_2 \in V(G_1) \cap V(X)$ such that $d^R_{G_1}(x_2) \geq (n-1)(m-3)$.

We shall continue the following procedure. In Step j with $2 \leq j \leq m-3$, define subgraph $G_j \subseteq G_{j-1}$ such that $V(G_j) \subseteq N^R_{G_{j-1}}(x_j)$ and $v(G_j) = (n-1)(m-j-1)$. Note that G_j contains neither a red K_{m-j} nor a blue P_n. Let $G_j = X_j + Y_j$ with $X_j = K_{r-\lceil n/2\rceil - s_1 - \ldots - s_{j-1}}$ and $Y_j = (\lceil n/2\rceil - 1 - q_1 - \ldots - q_{j-1})K_1$. We may assume that there must be a vertex $x_{j+1} \in V(X_j)$ such that $d^R_{G_j}(x_{j+1}) \geq (n-1)(m-j-2)$. Otherwise if for each vertex $v \in V(X_j)$, $d^R_{G_j}(v) \leq (n-1)(m-j-2) - 1$, then we have

$$d^B_{G_j}(v) \geq (n-1)(m-j-1) - 1 - [(n-1)(m-j-2) - 1] \geq n-1.$$

Note that for any vertex $y \in V(Y_j)$, we have $d^R_{G_j}(y) = d^R_{X_j}(y) \leq (n-1)(m-j-2)$. Otherwise we can find a red K_{m-j-1} or a blue P_n in the graph induced by $N^R_{X_j}(y)$. Along with vertices $x_1, \ldots, x_j$ and y, we get a red K_m. Thus $d^B_{X_j}(y)$ is at least

$$(n-1)(m-j-1) - \left(\left\lceil \frac{n}{2} \right\rceil - 1 - q_1 - \ldots - q_{j-1}\right) - (n-1)(m-j-2),$$

which implies

$$d^B_{G_j}(y) = d^B_{X_j}(y) \geq \left\lfloor \frac{n}{2} \right\rfloor.$$

Since $v(G_j) \geq (n-1)(m-j-1) \geq n$, by Lemma 5, we can find a blue P_n. So there must be a vertex $x_{j+1} \in V(X_j)$ such that $d^B_{G_j}(x_{j+1}) \geq (n-1)(m-j-2)$.

Define subgraph $G_{j+1} \subseteq G_j$ such that $V(G_{j+1}) \subseteq N^R_{G_j}(x_{j+1})$ and $v(G_{j+1}) = (n-1)(m-j-2)$. Note that G_{j+1} contains neither a red K_{m-j-1} nor a blue P_n. Denote by $H_j = G_j - G_{j+1}$ and $V(H_j) = S_j \cup Q_j$ with $S_j \subseteq V(X)$ and $Q_j \subseteq V(Y)$. Let $s_j = |S_j|$ and $q_j = |Q_j|$. Note that $s_j + q_j = n-1$.

In the last step, namely Step $m-2$. Similarly, we have graph G_{m-2} with $v(G_{m-2}) = n-1$, sets S_{m-2} and Q_{m-2} with $s_{m-2} = |S_{m-2}|$ and $q_{m-2} = |Q_{m-2}|$. Graph G_{m-2} contains no red edge, otherwise there is a red K_m along with $x_1, \ldots, x_{m-2}$. Set $S_{m-1} = V(G_{m-2}) \cap V(X)$ and $Q_{m-1} = V(G_{m-2}) \cap V(Y)$. Select vertex $x_{m-1} \in S_{m-1}$.

Now consider the coloring of edges among S_{m-2}, Q_{m-2}, S_{m-1} and Q_{m-1}. All the edges between $V(H_{m-2})$ and S_{m-1} are red. Otherwise if there is a blue edge uw with $u \in V(H_{m-2})$ and $v \in S_{m-1}$, then we get a blue P_n. All the edges between S_{m-2} and Q_{m-2} are blue. Otherwise there is a red edge uw, and we get a red K_m with $V(K_m) = \{x_1, x_2, \ldots, x_{m-3}, u, w, x_{m-1}\}$. Similarly, all the edges between Q_{m-1} and S_{m-2} are red, see Fig. 4

Continue this backwards procedure. In Step k, $2 \leq k \leq m-3$, all edges between $V(H_{m-k-2})$ and S_{m-k-1} are red. Otherwise if there is a blue edge uw with $u \in V(H_{m-k-2})$ and $v \in S_{m-k-1}$, then we get a blue P_n. All the edges between S_{m-k-2} and Q_{m-k-2} are blue. Otherwise if there is a red edge uw, then we get a red K_m

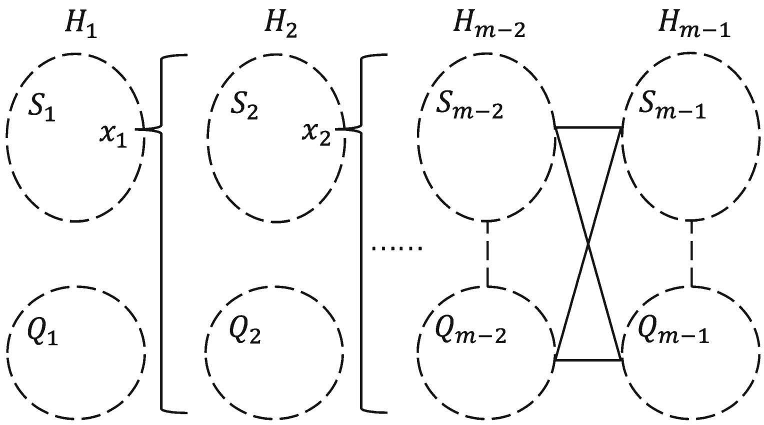

Fig. 4 The edge coloring of G

with $V(K_m) = \{x_1, x_2, \dots, x_{m-k-2}, u, w, x_{m-k}, \dots, x_{m-1}\}$. Similarly, all the edges between Q_{m-k-1} and S_{m-k-2} are red, completing the proof. □

Proof of Theorem 3. We only need to prove the lower bound. Consider the red-blue edge coloring of $G = K_{r-1}\backslash K_{\lceil n/2\rceil-1} = K_{r-\lceil n/2\rceil} + (\lceil n/2\rceil - 1)K_1$, where $r = R(K_m, P_n)$. Assume that G is (K_m, P_n)-free, otherwise we are done. By Lemma 6, G must be the graph described in Definition 3. Now we consider an extra vertex v_0. As there is no blue P_n, v_0 is adjacent to $V(X)$ by red edges completely, yielding a red K_m. □

Rousseau and Sheehan (1978) gave the following result.

Lemma 7 Rousseau and Sheehan (1978) *Let m and n be positive integers. If $n > (6m+7)/4$, then*

$$R(B_m, P_n) = 2n - 1.$$

We shall prove the $R^{\omega}(B_m^{(k)}, P_n)$ by induction on $k \geq 2$, for which the following result is needed as the initial step that gives a better bound for n than that from Theorem 1 for the case $k = 2$.

Theorem 5 *Let integers $m \geq 1$ and $n \geq (8m+4)/3$. Then $R^{\omega}(B_m, P_n) = \lceil n/2\rceil$.*

Proof We only need to prove the lower bound. Let $r = R(B_m, P_n) = 2n - 1$. Denote by $G = K_r\backslash K_{\lceil n/2\rceil} = K_{r-\lceil n/2\rceil} + \lceil n/2\rceil K_1$. Set $X = K_{r-\lceil n/2\rceil}$ and $Y = \lceil n/2\rceil K_1$.

Assume that G contains neither a red B_m nor a blue P_n, and we shall find a contradiction. Let $P_\ell = v_1 v_2 \dots v_\ell$ be a longest blue path in G. Then there is no blue edge between $\{v_1, v_\ell\}$ and $V(G - P_\ell)$, otherwise we will get a longer blue path than P_ℓ. As X contains no red B_m and

$$v(X) = 2n - 1 - \lceil n/2\rceil = n + \left\lfloor \frac{n}{2} \right\rfloor - 1 \geq 2\left(\left\lfloor \frac{3n-1}{4} \right\rfloor\right) - 1,$$

we have

$$\ell \geq \left\lfloor \frac{3n-1}{4} \right\rfloor \geq 2m$$

for sufficiently large n. Thus $2m \leq \ell \leq n-1$.

Case 1. If $v_1, v_\ell \in V(X)$, we have $v(G-P_\ell) \geq 2n-1-(n-1) \geq n$, implying that v_1 and v_ℓ share n common red neighbors outside P_ℓ. If $v_1 v_\ell$ is red, then we can get a red B_m. Thus $v_1 v_\ell$ is blue, yielding a blue cycle C_ℓ. For $1 \leq i \leq \ell$, v_i can be seen as the end vertex of P_ℓ, and v_i is adjacent to $V(G-P_\ell)$ by red edges only. Since

$$|V(X) \cap V(G-P_\ell)| \geq 2n-1-\left\lceil \frac{n}{2} \right\rceil-(n-1) \geq m,$$

$G-P_\ell$ has at least m vertices in $V(X)$. So the graph induced by $V(P_\ell)$ contains no red edge, otherwise we get a red B_m. Since $|V(X) \cap V(P_\ell)| \geq \lceil \ell/2 \rceil \geq m$, the graph induced by $V(G-P_\ell)$ contains no red edge. Note that $v(G-P_\ell) \geq n$ and $|V(G-P_\ell) \cap V(Y)| \leq \lceil n/2 \rceil$, yielding a blue P_n.

Case 2. If $v_1 \in V(X)$ and $v_\ell \in V(Y)$, since $|V(X) \cap V(G-P_\ell)| \geq m$ as mentioned, v_1 and v_ℓ share at least m common red neighbors outside P_ℓ in X. Thus $v_1 v_\ell$ is blue, yielding a blue cycle C_ℓ. For $1 \leq i \leq \ell$, v_i is adjacent to $V(G-P_\ell)$ by red edges only. Similarly, the graph induced by $V(P_\ell)$ contains no red edge, and the graph induced by $V(G-P_\ell)$ contains no red edge either. Note that $v(G-P_\ell) \geq n$ and $|V(G-P_\ell) \cap V(Y)| \leq \lceil n/2 \rceil$, yielding a blue P_n.

Case 3. If $v_1, v_\ell \in V(Y)$, then $v_2, v_{\ell-1} \in V(X)$. There is no blue edge between $\{v_2, v_{\ell-1}\}$ and $V(X) \cap V(G-P_\ell)$, otherwise we will get a new P_s as in Case 1 or Case 2. Set $U = \{v_1, v_2, v_{\ell-1}, v_\ell\}$. Note that there is no blue edge between U and $V(X) \cap V(G-P_\ell)$.

Since $|V(X) \cap V(G-P_\ell)| \geq m$ as mentioned, $v_2 v_{\ell-1}$ is blue. Now we get a new blue $P_{\ell-1} = v_1 v_2 v_{\ell-1} v_{\ell-2} \ldots v_3$. There is no blue edge between v_3 and $V(X) \cap V(G-P_\ell)$, otherwise we will get a new P_ℓ as in Case 2. Similarly, we can get a new $P_{\ell-1} = v_\ell v_{\ell-1} v_2 v_3 \ldots v_{\ell-2}$, and there is no blue edge between $v_{\ell-2}$ and $V(X) \cap V(G-P_\ell)$. Renew set $U = \{v_1, v_2, v_3, v_{\ell-2}, v_{\ell-1}, v_\ell\}$. Note that there is no blue edge between U and $V(X) \cap V(G-P_\ell)$.

So $v_2 v_{\ell-2}$ and $v_3 v_{\ell-1}$ are blue, otherwise we get a red B_m. Then we get two new paths $P_{\ell-1} = v_1 v_2 v_3 v_{\ell-1} v_{\ell-2} \ldots v_4$ and $P_{\ell-1} = v_\ell v_{\ell-1} v_{\ell-2} v_2 v_3 \ldots v_{\ell-3}$. Renew set $U = \{v_1, v_2, v_3, v_4, v_{\ell-3}, v_{\ell-2}, v_{\ell-1}, v_\ell\}$. Note that there is no blue edge between U and $V(X) \cap V(G-P_\ell)$.

Continue this procedure and renew set U until $U = V(P_\ell)$. We can deduce that there is no blue edge between $V(P_\ell)$ and $V(X) \cap V(G-P_\ell)$, and the graph induced by $V(P_\ell)$ contains no red edge. So the graph induced by $V(X) \cap V(G-P_\ell)$ contains no red edge.

Set $U_1 = \{v_i | v_i \in V(X) \cap V(P_\ell)\}$, $V_1 = \{v_i | v_i \in V(Y) \cap V(P_\ell)\}$, $U_2 = \{v_i | v_i \in V(X) \cap V(G-P_\ell)\}$, and $V_2 = \{v_i | v_i \in V(Y) \cap V(G-P_\ell)\}$, where $U_1 \cup U_2 = V(X)$ and $V_1 \cup V_2 = V(Y)$. Note that $|U_1| \geq |V_1| - 1$, since $U_1 \cup V_1$ induces a blue path P_ℓ. If $|U_1| \geq |V_1|$, then we can find a new blue path P_ℓ with one of the end vertices belonging to $V(X)$ as in Case 2. If $|U_1| = |V_1| - 1$, then

$$|U_1| = \left\lfloor \frac{\ell}{2} \right\rfloor \leq \left\lfloor \frac{n-1}{2} \right\rfloor$$

and

$$|U_1| + |U_2| = 2n - 1 - \left\lceil \frac{n}{2} \right\rceil,$$

which implies

$$|U_2| \geq 2n - 1 - \left\lceil \frac{n}{2} \right\rceil - \left\lfloor \frac{n-1}{2} \right\rfloor \geq n - 1.$$

So U_2 induces a new blue P_ℓ as in Case 1, completing the proof. □

Proof of Theorem 4. By Lemma 1, for any integers $k \geq 1$, $m \geq 1$ and $n \geq 4(m+k)$, $R(B_m^{(k)}, P_n) = k(n-1)+1$. We only need to proof the lower bound. Let $r = R(B_m^{(k)}, P_n) = k(n-1)+1$. Denote by $G = K_r \backslash K_{\lceil n/2 \rceil} = K_{r-\lceil n/2 \rceil} + \lceil n/2 \rceil K_1$. Set $X = K_{r-\lceil n/2 \rceil}$ and $Y = \lceil n/2 \rceil K_1$.

Now we shall prove by induction on k. It holds for $k = 2$ by Theorem 5. So we may assume it holds for $k-1$ with $k \geq 3$, that is,

$$K_{(k-1)(n-1)+1} \setminus K_{\lceil n/2 \rceil} \to (B_m^{(k-1)}, P_n).$$

For any vertex $x \in V(X)$, . we have

$$d_G^R(x) \leq (k-1)(n-1).$$

Otherwise if there is a vertex $v \in V(X)$ such that $d_G^R(v) \geq (k-1)(n-1)+1$, then by induction, $N_G^R(v)$ induces a red $B_m^{(k-1)}$ or a blue P_n, hence a red $B_m^{(k)}$ or a blue P_n along with v. So for any vertex $x \in V(X)$, we have

$$d_G^B(x) \geq k(n-1) - (k-1)(n-1) = n-1.$$

For any vertex $y \in V(Y)$, we have

$$d_G^R(y) = d_X^R(y) \leq (k-1)(n-1),$$

otherwise we get a red $B_m^{(k-1)}$ or a blue P_n, hence a red $B_m^{(k)}$ or a blue P_n along with y. Thus for any vertex $y \in V(Y)$, we have

$$d_G^B(y) = d_X^B(y) \geq k(n-1) + 1 - \left\lceil \frac{n}{2} \right\rceil - (k-1)(n-1) = \left\lfloor \frac{n}{2} \right\rfloor.$$

So by Lemma 5, we can find a blue P_n. □

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Data availability Enquiries about data availability should be directed to the authors.

Declarations

Conflict of interest The authors have not disclosed any conflict of interest.

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