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On maximizing monotone or non-monotone k-submodular functions with the intersection of knapsack and matroid constraints

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Abstract

A k-submodular function is a generalization of a submodular function. The definition domain of a k-submodular function is a collection of k-disjoint subsets instead of simple subsets of ground set. In this paper, we consider the maximization of a k-submodular function with the intersection of a knapsack and m matroid constraints. When the k-submodular function is monotone, we use a special analytical method to get an approximation ratio $\frac{1}{m+2}(1-e^{-(m+2)})$ for a nested greedy and local search algorithm. For non-monotone case, we can obtain an approximate ratio $\frac{1}{m+3}(1-e^{-(m+3)})$.

Keywords k-Submodularity · Knapsack constraint · Matroid constraint · Approximation algorithm

Mathematics Subject Classification 90C27 · 68W40 · 68W25

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1 Introduction

Given a ground set G containing n elements and $k \in N_+$, refer $(X_1, \ldots, X_k)$ as k-disjoint subsets, with $X_i \subseteq G$, $\forall i \in [k]$ and $X_i \cap X_j = \emptyset$, $\forall i \neq j \in [k]$; write $(k+1)^G$ as the family of k disjoint subsets. Define join and meet operations for any $\mathbf{x} = (X_1, \ldots, X_k)$ and $\mathbf{y} = (Y_1, \ldots, Y_k)$ in $(k+1)^G$, that is,

$$\mathbf{x} \sqcup \mathbf{y} := (X_1 \cup Y_1 \setminus (\bigcup_{i \neq 1} X_i \cup Y_i), \ldots, X_k \cup Y_k \setminus (\bigcup_{i \neq k} X_i \cup Y_i)),$$
$$\mathbf{x} \sqcap \mathbf{y} := (X_1 \cap Y_1, \ldots, X_k \cap Y_k).$$

The join operation removes some points with different positions in $\mathbf{x}$ and $\mathbf{y}$, that is, points v with $v \in X_i$, $v \in Y_j$, $\forall i \neq j \in [k]$. And the meet operation is just an intersection operation of sets.

A function $f : (k+1)^G \to R$ is said to be k-submodular (Huber and Kolmogorov 2012) if

$$f(\mathbf{x}) + f(\mathbf{y}) \geq f(\mathbf{x} \sqcup \mathbf{y}) + f(\mathbf{x} \sqcap \mathbf{y}),$$

for any $\mathbf{x}$ and $\mathbf{y}$ in $(k+1)^G$. The k-submodular function is a generalization of a submodular function. Note that the definition domain of k-submodular function is a collection of k disjoint subsets instead of simple subsets. When $k = 1$, a k-submodular function becomes a submodular function.

1.1 Related work

There have been many research results on monotone submodular maximization problem. Nemhauser et al. (1978) firstly achieved a greedy $(1 - 1/e)$-approximation algorithm under a cardinality constraint, which was known as a tight bound. Later, Sviridenko (2004) designed a combinatorial $(1 - 1/e)$ approximate algorithm under a knapsack constraint. For this problem, Ene and Nguyen (2019) also offered an approximate ratio of $(1 - 1/e - \varepsilon)$ by using multilinear extention function, which only needed approximate linear running time. With a matroid constraint, Calinescu et al. (2011) got an approximate ratio of $(1 - 1/e)$, by using the continuous greedy method and pipage rounding technique. Filmus and Ward (2014) designed a combination algorithm using local search technique, which also achieved an approximate ratio of $(1 - 1/e)$. More recently, Sarpatwar et al. (2019) contributed an algorithm with an approximate ratio of $\frac{1-e^{-(m+1)}}{m+1}$ combining the greedy algorithm and local search techniques for maximization problem of submodular function subject to the intersection of a knapsack and m matroid constraints. For maximizing non-monotone submodular functions, Lee et al. (2010) presented a $(\frac{1}{m+2+\frac{1}{m}+\varepsilon})$ approximation algorithm under m matroid constraints, and a $(\frac{1}{5} - \varepsilon)$ approximation algorithm under m knapsack constraints. Feldman et al. (2011) and Chekuri et al. (2014) studied constant factor approximation algorithms to maximize a multilinear extension of the submodular function over a down-closed polytope, respectively. The fractional solution could be rounded with contention resolution schemes. For more references on submodular maximization, see Bian et al. (2017); Calinescu et al. (2011); Ene and Nguyen (2019); Feldman and Naor (2013); Filmus and Ward (2014); Huang et al. (2022); Liu et al. (2022b); Sviridenko (2004); Yoshida (2019).

As a generalization of submodular function, the k-submodular function still has diminishing marginal benefits, where the definition domain is extended from the collection of simple subsets to the collection of k disjoint subsets. Many practical applications can be attributed to the k-submodular maximization problem. Ohsaka and Yoshida (2015) studied influence maximization with k topics and sensor placement with k sensors both based on k-submodular maximization with a size constraint. Rafiey and Yoshida (2020) applied k-submodular maximization to facility location.

In recent years, many researches on k-submodular maximization has sprung up. For k-submodular maximization without monotonicity assumption, Ward and Zivny (2014) studied the unconstrained problem and gave a deterministic greedy algorithm and a randomized greedy algorithm achieving the approximate ratio of $1/3$ and $\frac{1}{1+a}$ with $a = \max\{1, \sqrt{\frac{k-1}{4}}\}$, respectively. Later, the approximation ratio was improved to $1/2$ by Iwata et al. (2016). And Oshima (2021) also contributed a $\frac{k^2+1}{2k^2+1}$-approximate algorithm. For monotone k-submodular maximization, Ward and Zivny (2014) showed a $1/2$-approximate algorithm without constraint, and then it was improved to $k/(2k-1)$ by Iwata et al. (2016), which is asymptotically tight. Ohsaka and Yoshida (2015) introduced a construction method between current solution and optimal solution to obtain a $1/2$-approximate ratio, for a total size constraint. Using the similar construction method, a $1/2$-approximate ratio could be also achieved by Sakaue (2017) for a matroid constraint. Tang et al. (2022) contributed a $\frac{1}{2}(1-e^{-1})$-approximate algorithm with a knapsack constraint. Xiao et al. found that this result could be improved to $\frac{1}{2}(1-e^{-2})$. Recently, Liu et al. (2022a) designed a nested greedy and local search $\frac{1}{2(m+1)}(1-e^{-(m+1)})$-approximation algorithm for monotone k-submodular maximization subject to the intersection of a knapsack and m matroid constraints.

1.2 Our contributions

In this paper, we consider the k-submodular maximization subject to the intersection of a knapsack and m matroid constraints, and discuss the results in monotone and non monotone cases respectively. The main contributions of this paper are as follows:

– We improve the approximate ratio from $\frac{1}{2(m+1)}(1-e^{-(m+1)})$ in Liu et al. (2022a) to $\frac{1}{m+2}(1-e^{-(m+2)})$ for monotone k-submodular maximization problem with the intersection of a knapsack and m matroid constraints. In the theoretical analysis of the algorithm, we no longer rely on the conclusion of the greedy algorithm for unconstrained k-submodular maximization problem, and use the properties of k-submodular function to get the new result. Note that our result will be $\frac{1}{3}(1-e^{-3})$ when $m = 1$, it improves the result $\frac{1}{4}(1-e^{-2})$ in Liu et al. (2022a) with the intersection of a knapsack and a matroid constraint.

– We extend the approximation algorithm to non-monotone case. By increasing the number of enumeration points in the algorithm and using the pairwise monotone property, we achieve a $\frac{1}{m+3}(1-e^{-(m+3)})$ approximate ratio. It is easy to know that we have a $\frac{1}{4}(1-e^{-4})$ approximate ratio for the non-monotone k-submodular maximization problem with the intersection of a knapsack and a matroid constraint.

1.3 Organization

Organize our paper as follows: In Sect. 2, we introduce notations, properties and some basic results about k-submodular function. In Sect. 3, we give and explain the nested greedy and local search algorithm. In Sects. 4 and 5, we present our theoretical analysis and show the main results for monotone case and non-monotone case, respectively.

2 Preliminaries

2.1 *k*-Submodular function

In this paper, we set $k \geq 2$ and $k \in N_+$, because k-submodular function is submodular function when $k = 1$. For any two k disjoint subsets $\mathbf{x}, \ \mathbf{y} \in (k+1)^G$, we need to introduce a remove operation and a partial order, i.e.

$$\mathbf{x} \setminus \mathbf{y} := (X_1 \setminus Y_1, \dots, X_k \setminus Y_k),$$

$$\mathbf{x} \preceq \mathbf{y}, \text{ if } X_i \subseteq Y_i, \forall i \in [k].$$

Define one-item $\mathbf{1}_{v,i} := (X_1, \dots, X_k)$, where $X_i = \{v\}$ and $X_{j \neq i} = \emptyset$, and empty-item $\mathbf{0} := (\emptyset, \dots, \emptyset)$. Denote the support set $U(\mathbf{x}) := \bigcup_{i=1}^{k} X_i$.

Given a function $f : (k+1)^G \to R$, for any $\mathbf{x} \in (k+1)^G$, $v \in G \setminus U(\mathbf{x})$ and $i \in [k]$, it is said to be monotone if its marginal gain satisfies:

$$f_{\mathbf{x}}(\mathbf{1}_{v,i}) = f(\mathbf{x} \sqcup \mathbf{1}_{v,i}) - f(\mathbf{x}) \geq 0.$$

From Ohsaka and Yoshida (2015), f is *pairwise monotone* if

$$f_{\mathbf{x}}(\mathbf{1}_{v,i}) + f_{\mathbf{x}}(\mathbf{1}_{v,j}) \geq 0,$$

for any $\mathbf{x} \in (k+1)^G$, $v \in G \setminus U(\mathbf{x})$ and $i \neq j \in [k]$. And f is *orthant submodular*, if

$$f_{\mathbf{x}}(\mathbf{1}_{v,i}) \geq f_{\mathbf{y}}(\mathbf{1}_{v,i}),$$

for $\mathbf{x} \preceq \mathbf{y} \in (k+1)^G$, $v \in G \setminus U(\mathbf{y})$ and $i \neq j \in [k]$. As below, a k-submodular function has a well-known equivalent definition (Ward and Zivny 2014).

Definition 1 A function $f : (k+1)^G \to R$ is k-submodular iff it is pairwise monotone and orthant submodular.

Obviously, the monotonicity of f implies pairwise monotonicity. For a monotone function $f : (k+1)^G \to R$, the k-submodularity is equivalent to the orthant submodularity. In addition, a k-submodular function also has the following useful property (Ohsaka and Yoshida 2015).

Lemma 1 *Given a k-submodular function f, we have*

$$f(\mathbf{y}) - f(\mathbf{x}) \leq \sum_{\mathbf{1}_{v,i} \preceq \mathbf{y} \setminus \mathbf{x}} f_{\mathbf{x}}(\mathbf{1}_{v,i}),$$

for any $\mathbf{x}, \mathbf{y} \in (k+1)^G$ *and* $\mathbf{x} \preceq \mathbf{y}$.

Given a fixed k disjoint subsets $\mathbf{y} \in (k+1)^G$, define a family of k disjoint subsets $D(\mathbf{y}) := \{\mathbf{x} \in (k+1)^G \mid \mathbf{y} \preceq \mathbf{x}\}$. In the later analysis, we need to construct a function $g(\mathbf{x}) : D(\mathbf{y}) \to R$ by temporarily hiding $\mathbf{y}$. In order to maintain the regularity, we can set a k-submodular function $g(\mathbf{x}) = f(\mathbf{x}) - f(\mathbf{y})$, which is still a k-submodular function.

Lemma 2 *Given a k-submodular function* $f : (k+1)^G \to R$ *and* $\mathbf{y} \in (k+1)^G$, *then* $g(\mathbf{x}) = f(\mathbf{x}) - f(\mathbf{y}) : D(\mathbf{y}) \to R$ *is a k-submodular function and* $g(\mathbf{y}) = 0$.

2.2 Knapsack and matroid constraints

Given $\mathcal{L} \subseteq 2^G$, a pair $(G, \mathcal{L})$ is an independence system if $(\mathcal{M}1)$ and $(\mathcal{M}2)$ hold, and a set A is an independence set if $A \in \mathcal{L}$. Further, the independence system $(G, \mathcal{L})$ is said to be a matroid if $(\mathcal{M}3)$ holds.

Definition 2 Given $\mathcal{L} \subseteq 2^G$ and a pair $\mathcal{M} = (G, \mathcal{L})$ is a matroid if
$(\mathcal{M}1)$: $\emptyset \in \mathcal{L}$.
$(\mathcal{M}2)$: $A \subseteq B$ and $B \in \mathcal{L} \Longrightarrow A \in \mathcal{L}$.
$(\mathcal{M}3)$: $A, B \in \mathcal{L}$ and $|A| > |B| \Longrightarrow \exists\, v \in A \setminus B$, s.t. $B \cup \{v\} \in \mathcal{L}$.

For $m \in N_+$ and each $j \in [m]$, $\mathcal{L}_j$ is a collection of independent sets, and $\mathcal{M}_j = (G, \mathcal{L}_j)$ is a matroid. Given a nonnegative bound B, and for each element $v \in G$, there is a nonnegative weight w_v. Without losing generality, we assume that w_v and B are integers. Otherwise, we can always enlarge them to integers in the same proportion. Let $w_{\mathbf{x}} = \sum_{v \in U(\mathbf{x})} w_v$. The k-submodular maximization problem with the intersection of a knapsack and m matroid constraints is

$$\max_{\mathbf{x} \in (k+1)^G} \{f(\mathbf{x}) \mid w_{\mathbf{x}} \leq B \text{ and } U(\mathbf{x}) \in \bigcap_{j=1}^{m} \mathcal{L}_j\}. \tag{1}$$

For any $A \in G$, we use $[A]^m$ to express a collection of subsets of A, whose size does not exceed m. Given an independence set $A \in \bigcap_{j=1}^m \mathcal{L}_j$ and a pair $(\bar{a}, b)$ with $\bar{a} \in [A]^m$ and $b \in G \setminus A$, we refer the pair $(\bar{a}, b)$ as a m-swap $(\bar{a}, b)$ if $(A \setminus \bar{a}) \cup \{b\} \in \bigcap_{j=1}^m \mathcal{L}_j$. The next lemma ensures that there exists some m-swap $(\bar{a}, b)$ between two independence sets. The detailed proof of Lemma 3 is given by Sarpatwar et al. (2019).

Lemma 3 *Assume two independence sets $A, B \in \bigcap_{j=1}^m \mathcal{L}_j$, then we can construct a mapping $y : B \backslash A \to [A \backslash B]^m$, such that $(A \backslash \bar{a}) \cup \{b\} \in \bigcap_{j=1}^m \mathcal{L}_j$ with $b \in B \backslash A$, $\bar{a} \in [A \backslash B]^m$, and each element $a \in A \backslash B$ appears in mapping y no more than m times.*

In the later theoretical proof, the following Lemma 4 (Nemhauser et al. 1978) needs to be used.

Lemma 4 *Given two fixed $P, D \in N_+$ and a sequence of nonnegative real numbers $\{\gamma_i\}_{i \in [P]}$, then we have*

$$\frac{\sum_{i=1}^{P} \gamma_i}{\min_{t \in [P]} (\sum_{i=1}^{t-1} \gamma_i + D\gamma_t)} \geq 1 - (1 - \frac{1}{D})^P \geq 1 - e^{-P/D}. \tag{2}$$

3 Algorithm overview

3.1 Greedy algorithm

Firstly, we introduce a Greedy Algorithm (f, G) from Ward and Zivny (2014). By Definition 1, k-submodularity of f implies pairwise monotonicity, that is, $f_{\mathbf{x}}(\mathbf{1}_{v,i}) + f_{\mathbf{x}}(\mathbf{1}_{v,j}) \geq 0$ for any $\mathbf{x} \in (k+1)^G$, $v \notin U(\mathbf{x})$ and $i \neq j \in [k]$. It means that there are no two positions $i \neq j \in [k]$ such that $f_{\mathbf{x}}(\mathbf{1}_{v,i}) < 0$ and $f_{\mathbf{x}}(\mathbf{1}_{v,j}) < 0$ both hold. For k-submodular maximization problem without constraint, there is always an optimal solution $\mathbf{x}^*$ satisfying $U(\mathbf{x}^*) = G$. In Greedy Algorithm (f, G), we enter a set G and give a fixed order to the points in G, that is $G = \{v_1, \ldots, v_{|G|}\}$. Each current solution $\mathbf{x}_l$ is obtained by $\mathbf{x}_{l-1}$ adding $v_l \in G \backslash U(\mathbf{x}_{l-1})$ with a greedy position $i_l \in [k]$ for $l = 1, \ldots, |G|$.

Algorithm 1 Greedy Algorithm (f, G)

Require: A k-submodular $f : (k+1)^G \to R_+$ and a set $G = \{v_1, \ldots, v_{|G|}\}$
Ensure: A k-disloint set $\mathbf{x}_{|G|} \in (k+1)^G$
1: $\mathbf{x}_0 \leftarrow \mathbf{0}$
2: **for** $l = 1$ to $|G|$ **do**
3: $\quad i_l \leftarrow \arg\max_{i \in [k]} f_{\mathbf{x}_{l-1}}(\mathbf{1}_{v_l,i})$
4: $\quad \mathbf{x}_l \leftarrow \mathbf{x}_{l-1} \sqcup \mathbf{1}_{v_l,i_l}$
5: **end for**
6: **return** $\mathbf{x}_{|G|}$

3.2 Nested greedy and local search algorithm KM-KM

Next, we present a nested greedy and local search algorithm for problem (1), which is inspired by Liu et al. (2022a). For simplicity, we call it KM-KM. If the objective function f is monotone, we choose $\lambda = 2$ in KM-KM. Otherwise, we need to choose $\lambda \geq \frac{(m+1)(m+3)}{m+2+e^{-(m+3)}}$, because of the proof of the approximate ratio.

KM-KM starts with $\mathbf{x}^\lambda \preceq \mathbf{x}^*$ obtained by enumerating with the largest marginal profits, where $\mathbf{x}^*$ is an optimal solution of problem (1). If $|U(\mathbf{x}^*)| \leq \lambda$, we can find $\mathbf{x}^*$ by enumerating $\mathbf{x} \in (k+1)^G$ with $|U(\mathbf{x})| \leq |U(\mathbf{x}^*)|$. Therefore, we only consider the case when $|U(\mathbf{x}^*)|$ is greater than λ. For a positive integer $t \geq \lambda$, we define t-th iteration as the process when KM-KM finds a suitable m-swap $(\bar{a}^t, b^t)$ to update $\mathbf{x}^t$. Clearly $|(U(\mathbf{x}^t \backslash \mathbf{x}^\lambda) \backslash \bar{a}^t) \cup \{b^t\}| = |U(\mathbf{x}^{t+1} \backslash \mathbf{x}^\lambda)|$. If the current m-swap $(\bar{a}^t, b^t)$ satisfies all the conditions in line 11, KM-KM performs line 12-18 and breaks loop 9-19 to update S^m in line 8. In line 12 of KM-KM, we consider the elements in $(U(\mathbf{x}^t \backslash \mathbf{x}^\lambda) \backslash \bar{a}^t) \cup \{b^t\}$, and add them to Greedy Algorithm in the same order as in KM-KM. For $l \in \{1, \ldots, |(U(\mathbf{x}^t \backslash \mathbf{x}^\lambda) \backslash \bar{a}^t) \cup \{b^t\}|\}$, Greedy Algorithm $(f(\tilde{\mathbf{x}}^{t+1} \sqcup \mathbf{x}^\lambda), (U(\mathbf{x}^t \backslash \mathbf{x}^\lambda) \backslash \bar{a}^t) \cup \{b^t\})$ reorders the positions i of points $v_l \in (U(\mathbf{x}^t \backslash \mathbf{x}^\lambda) \backslash \bar{a}^t) \cup \{b^t\}$. Define $\tilde{\mathbf{x}}_l^{t+1}$ as the current solution, such that $\tilde{\mathbf{x}}_l^{t+1} = \tilde{\mathbf{x}}_{l-1}^{t+1} \sqcup \mathbf{1}_{v_l, i_l}$. If current m-swap $(\bar{a}^t, b^t)$ violates any conditions in line 11, KM-KM will remove it and continue to pick the next m-swap. Finally, KM-KM breaks all loops when $S^m = \emptyset$ in line 9 and return $\mathbf{x}^t$. We define the time when KM-KM outputs $\mathbf{x}^t$ as T and $T \geq \lambda + 1$.

Algorithm 2 KM-KM (G, B, M, λ)

Require: A k-submodular function $f : (k+1)^G \to R_+$, a bound $B \in N_+$, m matroids $(G, \mathcal{L}_j)$ for $j \in [m]$ and $\lambda \in N_+$

Ensure: A k-disloint set $\mathbf{x}^t \in (k+1)^G$ satisfying $w_{\mathbf{x}^t} \leq B$ and $U(\mathbf{x}^t) \in \bigcap_{j=1}^m \mathcal{L}_j$

1: $\mathbf{x}^0 \leftarrow \mathbf{0}$
2: **for** $t = 0$ to $\lambda - 1$ **do**
3: $\quad \mathbf{x}^{t+1} \leftarrow \arg \max_{|U(\mathbf{x})| = t+1, \mathbf{x}^t \preceq \mathbf{x} \preceq \mathbf{x}^*} f(\mathbf{x})$
4: **end for**
5: Let $t = \lambda$ and $switch = false$
6: **while** $switch = false$ **do**
7: $\quad switch \leftarrow true$
8: $\quad$ Generate a collection of all m-swaps $S^m = S^m(U(\mathbf{x}^t)) \backslash \{m\text{-swap } (\bar{a}, b) \mid \bar{a} \cap U(\mathbf{x}^\lambda) \neq \emptyset\}$
9: $\quad$ **while** $switch = true$ and $S^m \neq \emptyset$ **do**
10: $\quad\quad$ Pick a m-swap $(\bar{a}, b)$ from S^m with a maximum value $\rho(\bar{a}, b) = \max_{i \in [k], \mathbf{1}_{a,j} \preceq \mathbf{x}^t} \frac{f((\mathbf{x}^t \backslash \bigsqcup_{a \in \bar{a}} \mathbf{1}_{a,j}) \sqcup \mathbf{1}_{b,i}) - f(\mathbf{x}^t)}{w_b}$ and call it the m-swap $(\bar{a}^t, b^t)$
11: $\quad\quad$ **if** $\rho(\bar{a}^t, b^t) > 0$ and $w_{\mathbf{x}^t} - w_{\bar{a}^t} + w_{b^t} \leq B$ **then**
12: $\quad\quad\quad \tilde{\mathbf{x}}^{t+1} \leftarrow$ **Greedy Algorithm** $(f(\tilde{\mathbf{x}}^{t+1} \sqcup \mathbf{x}^\lambda), \ (U(\mathbf{x}^t \setminus \mathbf{x}^\lambda) \setminus \bar{a}^t) \cup \{b^t\})$
13: $\quad\quad\quad \mathbf{x}^{t+1} \leftarrow \tilde{\mathbf{x}}^{t+1} \sqcup \mathbf{x}^\lambda$
14: $\quad\quad\quad w_{\mathbf{x}^{t+1}} \leftarrow w_{\mathbf{x}^t} - w_{\bar{a}^t} + w_{b^t}$
15: $\quad\quad\quad switch \leftarrow false$
16: $\quad\quad\quad t \leftarrow t + 1$
17: $\quad\quad$ **end if**
18: $\quad\quad S^m \leftarrow S^m \backslash \{m\text{-swap } (\bar{a}^t, b^t)\}$
19: $\quad$ **end while**
20: **end while**
21: **return** $\mathbf{x}^t$

3.3 A construction method for analysis

In order to give an approximate ratio analysis, we introduce a construction method based on Algorithm 2. Mark $\mathbf{x}^*$ as an optimal solution of problem (1).

Given a fixed iteration step $t \geq \lambda + 1$ in KM-KM and $l \in \{1, \ldots, |U(\mathbf{x}^t \backslash \mathbf{x}^\lambda)|\}$. Define $\mathbf{x}_l^t = \tilde{\mathbf{x}}_l^t \sqcup \mathbf{x}^\lambda$, then $\mathbf{x}_{|U(\mathbf{x}^t \backslash \mathbf{x}^\lambda)|}^t = \mathbf{x}^t$. We further construct two sequences $\{\mathbf{o}_{l-1/2}^t\}$ and $\{\mathbf{o}_l^t\}$ such that $\mathbf{o}_{l-1/2}^t = (\mathbf{x}^* \sqcup \mathbf{x}_l^t) \sqcup \mathbf{x}_{l-1}^t$, $\mathbf{o}_l^t = (\mathbf{x}^* \sqcup \mathbf{x}_l^t) \sqcup \mathbf{x}_l^t$ and $\mathbf{o}_{l=0}^t = \mathbf{x}^*$. Note that $\mathbf{x}_{l-1}^t \preceq \mathbf{x}_l^t \preceq \mathbf{o}_l^t$, $\mathbf{o}_{l-1/2}^t \preceq \mathbf{o}_l^t$ and $U(\mathbf{o}_{|U(\mathbf{x}^t \backslash \mathbf{x}^\lambda)|}^t) \backslash U(\mathbf{x}^t) = U(\mathbf{x}^*) \backslash U(\mathbf{x}^t)$.

By Lemma 2, define a k-submodular function $g(\mathbf{x}) = f(\mathbf{x}) - f(\mathbf{x}^\lambda) : D(\mathbf{x}^\lambda) \to R_+$. The construction method has the following conclusions. The detailed proofs of them are shown in the Appendix.

Lemma 5 *Given a fixed iteration step $t \geq \lambda + 1$ in KM-KM and an optimal solution $\mathbf{x}^*$ for problem (1), we have*:

(i) *when the objective function f is monotone,*

$$g(\mathbf{o}_{l-1}^t) - g(\mathbf{o}_l^t) \leq g(\mathbf{x}_l^t) - g(\mathbf{x}_{l-1}^t), \tag{3}$$

$$g(\mathbf{x}^*) \leq g(\mathbf{o}_{|U(\mathbf{x}^t \backslash \mathbf{x}^\lambda)|}^t) + g(\mathbf{x}^t). \tag{4}$$

(ii) *when the objective function f is non-monotone,*

$$g(\mathbf{o}_{l-1}^t) - g(\mathbf{o}_l^t) \leq 2[g(\mathbf{x}_l^t) - g(\mathbf{x}_{l-1}^t)], \tag{5}$$

$$g(\mathbf{x}^*) \leq g(\mathbf{o}_{|U(\mathbf{x}^t \backslash \mathbf{x}^\lambda)|}^t) + 2g(\mathbf{x}^t). \tag{6}$$

4 Analysis for monotone *k*-submodular maximization with a knapsack and *m* matroid constraints

In this section, we will explain in detail how to obtain the approximate ratio for problem (1). Our framework of proof is inspired by Sviridenko (2004); Sarpatwar et al. (2019); Liu et al. (2022a). To simplify the process of analyzing approximate ratio, we give several lemmas. The detailed proofs of them are shown in the Appendix.

Lemma 6 *Given a fixed iteration step $t \geq \lambda + 1$ in KM-KM and an optimal solution $\mathbf{x}^*$ for problem (1), there exists a mapping y :*

$$U(\mathbf{o}_{|U(\mathbf{x}^t \backslash \mathbf{x}^\lambda)|}^t) \backslash U(\mathbf{x}^t) \to [U(\mathbf{x}^t) \backslash U(\mathbf{x}^*)]^m$$

such that $(U(\mathbf{x}^t) \backslash \bar{y}(b)) \cup \{b\} \in \bigcap_{j=1}^m \mathcal{L}_j$, for $b \in U(\mathbf{o}_{|U(\mathbf{x}^t \backslash \mathbf{x}^\lambda)|}^t) \backslash U(\mathbf{x}^t)$, $\bar{y}(b) \in [U(\mathbf{x}^t) \backslash U(\mathbf{x}^*)]^m$, and each element $a \in U(\mathbf{x}^t) \backslash U(\mathbf{x}^*)$ appears in mapping y no more than m times. Then we have

$$g(\mathbf{o}^t_{|U(\mathbf{x}^t\setminus\mathbf{x}^\lambda)|}) \leq \sum_{\mathbf{1}_{b,i}\preceq(\mathbf{o}^t_{|U(\mathbf{x}^t\setminus\mathbf{x}^\lambda)|}\setminus\mathbf{x}^t)} [g((\mathbf{x}^t\setminus \bigsqcup_{y(b)\in\bar{y}(b)} \mathbf{1}_{y(b),j}) \sqcup \mathbf{1}_{b,i}) - g(\mathbf{x}^t\setminus \bigsqcup_{y(b)\in\bar{y}(b)} \mathbf{1}_{y(b),j})] + g(\mathbf{x}^t) \tag{7}$$

and

$$\sum_{\mathbf{1}_{b,i}\preceq(\mathbf{o}^t_{|U(\mathbf{x}^t\setminus\mathbf{x}^\lambda)|}\setminus\mathbf{x}^t)} [g(\mathbf{x}^t) - g(\mathbf{x}^t\setminus \bigsqcup_{y(b)\in\bar{y}(b)} \mathbf{1}_{y(b),j})] \leq mg(\mathbf{x}^t). \tag{8}$$

for $\mathbf{1}_{y(b),j} \preceq \mathbf{x}^t\setminus\mathbf{x}^\lambda$.

Let us assume that there exists a m-swap $(\bar{y}(b), b)$ with respect to $\mathbf{x}^T$ and $\mathbf{x}^*$ satisfying $w_{\mathbf{x}^T} - w_{\bar{y}(b)} + w_b > B$, when KM-KM runs. Let $t^* + 1$ be the iteration which appears a m-swap $(\bar{y}(b^{t^*}), b^{t^*})$ in $S^m(U(\mathbf{x}^{t^*})) \setminus \{m\text{−swap } (\bar{a}, b) \mid \bar{a} \cap U(\mathbf{x}^\lambda) \neq \emptyset\}$ violating $w_{\mathbf{x}^{t^*}} - w_{\bar{y}(b^{t^*})} + w_{b^{t^*}} \leq B$, with $b^{t^*} \in U(\mathbf{x}^*)\setminus U(\mathbf{x}^{t^*})$ and $\bar{y}(b^{t^*}) \in [(U(\mathbf{x}^t)\setminus U(\mathbf{x}^*))]^m$, for the first time.

Lemma 7 *Considering the current solution* $\mathbf{x}^{t^*}$ *and the* m*-swap* $(\bar{y}(b^{t^*}), b^{t^*})$ *mentioned above, we have*

$$f((\mathbf{x}^{t^*} \setminus \bigsqcup_{y(b^{t^*})\in\bar{y}(b^{t^*})} \mathbf{1}_{y(b^{t^*}),j^{t^*}}) \sqcup \mathbf{1}_{b^{t^*},i^{t^*}}) - f(\mathbf{x}^{t^*}) \leq \frac{1}{2} \cdot f(\mathbf{x}^\lambda), \tag{9}$$

where $\mathbf{1}_{y(b^{t^*}),j^{t^*}} \preceq \mathbf{x}^{t^*}\setminus\mathbf{x}^\lambda$*, if* f *is monotone.*

Lemma 8 *Given* $t \in \{\lambda+1, \ldots, t^*\}$ *in KM-KM for problem (1), we have*

$$\sum_{\mathbf{1}_{b,i}\preceq(\mathbf{o}^t_{|U(\mathbf{x}^t\setminus\mathbf{x}^\lambda)|}\setminus\mathbf{x}^t)} [g((\mathbf{x}^t\setminus \bigsqcup_{y(b)\in\bar{y}(b)} \mathbf{1}_{y(b),j}) \sqcup \mathbf{1}_{b,i}) - g(\mathbf{x}^t)] \leq (B - w_{\mathbf{x}^\lambda})\rho_t, \tag{10}$$

for $\mathbf{1}_{y(b),j} \preceq \mathbf{x}^t\setminus\mathbf{x}^\lambda$.

Lemma 9 *Given* $t \in \{\lambda+1, \ldots, t^*\}$ *in KM-KM,* α, β, r *are positive constants satisfying* $1 - \frac{1}{\alpha}(1 - e^{-\beta}) - r \geq 0$ *and* $\mathbf{x}^*$ *be an optimal solution of problem (1). If*

$$g(\mathbf{x}^*) \leq \alpha[g(\mathbf{x}^t) + \frac{(B - w_{\mathbf{x}^\lambda})}{\beta}\rho_t]$$

and

$$f((\mathbf{x}^{t^*} \setminus \bigsqcup_{y(b^{t^*})\in\bar{y}(b^{t^*})} \mathbf{1}_{y(b^{t^*}),j^{t^*}}) \sqcup \mathbf{1}_{b^{t^*},i^{t^*}}) - f(\mathbf{x}^{t^*}) \leq r \cdot f(\mathbf{x}^\lambda)$$

hold, we have

$$f(\mathbf{x}^{t^*}) \geq \frac{1}{\alpha}(1 - e^{-\beta}) f(\mathbf{x}^*). \tag{11}$$

Theorem 1 *If the objective function f is monotone for problem (1), we can obtain a $\frac{1}{m+2}(1 - e^{-(m+2)})$-approximate solution in KM-KM by setting $\lambda = 2$.*

Proof When there is no qualified m-swap$(\bar{a}, b) \in S^m$, KM-KM will break all loops and output $\mathbf{x}^T$. Using Lemma 3 between $U(\mathbf{x}^t)$ and $U(\mathbf{x}^*)$, for a fixed $t \geq \lambda$, there exists a mapping $y : U(\mathbf{x}^*)\backslash U(\mathbf{x}^t) \to [U(\mathbf{x}^t)\backslash U(\mathbf{x}^*)]^m$ such that $(U(\mathbf{x}^t)\backslash \bar{y}(b)) \cup \{b\} \in \bigcap_{j=1}^m \mathcal{L}_j$, for $b \in U(\mathbf{x}^*)\backslash U(\mathbf{x}^t)$ and $\bar{y}(b) \in [U(\mathbf{x}^t)\backslash U(\mathbf{x}^*)]^m$. Thus, there are some m-swaps $(\bar{y}(b), b)$ with respect to $\mathbf{x}^t$ and $\mathbf{x}^*$.

When $t = T$, according to whether the conditions in line 11 of KM-KM are violated, consider dividing m-swaps $(y(b), b)$ with respect to $\mathbf{x}^T$ and $\mathbf{x}^*$ into two cases.

Case 1: Considering the m-swaps $(\bar{y}(b), b)$ with respect to $\mathbf{x}^T$ and $\mathbf{x}^*$, they were all rejected just due to $\rho(\bar{y}(b), b) \leq 0$ instead of knapsack constraint.

Due to our assumption about the m-swaps, we get

$$g((\mathbf{x}^T \backslash \bigsqcup_{y(b) \in \bar{y}(b)} \mathbf{1}_{y(b),j}) \sqcup \mathbf{1}_{b,i}) \leq g(\mathbf{x}^T). \tag{12}$$

Since f is monotone, we combine formula (4) in Lemma 5 and formula (7) in Lemma 6, then use formula (12) and formula (8) in Lemma 6 to get $g(\mathbf{x}^*) \leq (m+2)g(\mathbf{x}^T)$. Finally, we have $f(\mathbf{x}^*) \leq (m+2)f(\mathbf{x}^T) - (m+1)f(\mathbf{x}^\lambda) \leq (m+2)f(\mathbf{x}^T)$ due to nonnegativity of f. Therefore, we find a $\frac{1}{m+2}$-approximate solution in case 1, if f is monotone.

Case 2: Considering the m-swaps $(\bar{y}(b), b)$ with respect to $\mathbf{x}^T$ and $\mathbf{x}^*$, there exists at least one satisfying $w_{\mathbf{x}^t} - w_{\bar{y}(b)} + w_b > B$.

For a fixed $t \geq \lambda$, KM-KM selects a qualified m-swap$(\bar{a}^t, b^t)$ to update $\mathbf{x}^t$ in each t-th iteration. In $t^* + 1$ iteration, KM-KM checks m-swap $(\bar{y}(b^{t^*}), b^{t^*})$, where $b^{t^*} \in U(\mathbf{x}^*)\backslash U(\mathbf{x}^{t^*})$ and $\bar{y}(b^{t^*}) \in [(U(\mathbf{x}^t)\backslash U(\mathbf{x}^*))]^m$, in line 11 and removed it due to $w_{\mathbf{x}^{t^*}} - w_{\bar{y}(b^{t^*})} + w_{b^{t^*}} > B$, for the first time. Define $\rho_t := \rho(\bar{a}^t, b^t)$ for $t \in \{\lambda, \ldots, t^* - 1\}$ and

$$\rho_{t^*} := \frac{f((\mathbf{x}^{t^*} \setminus \bigsqcup_{y(b^{t^*}) \in \bar{y}(b^{t^*})} \mathbf{1}_{y(b^{t^*}), j^{t^*}}) \sqcup \mathbf{1}_{b^{t^*}, i^{t^*}}) - f(\mathbf{x}^{t^*})}{w_{b^{t^*}}}.$$

When $t \in \{\lambda + 1, \ldots, t^*\}$, we combine formula (4) in Lemma 5 and formula (7) in Lemma 6, then rewrite formula (7) in Lemma 6 as below

$$
\begin{aligned}
& g((\mathbf{x}^t \setminus \bigsqcup_{y(b)\in\bar{y}(b)} \mathbf{1}_{y(b),j}) \sqcup \mathbf{1}_{b,i}) - g((\mathbf{x}^t \setminus \bigsqcup_{y(b)\in\bar{y}(b)} \mathbf{1}_{y(b),j})) \\
&= [g((\mathbf{x}^t \setminus \bigsqcup_{y(b)\in\bar{y}(b)} \mathbf{1}_{y(b),j}) \sqcup \mathbf{1}_{b,i}) - g(\mathbf{x}^t)] \\
&+ [g(\mathbf{x}^t) - g((\mathbf{x}^t \setminus \bigsqcup_{y(b)\in\bar{y}(b)} \mathbf{1}_{y(b),j}))].
\end{aligned} \tag{13}
$$

Using formula (8) in Lemma 6 and Lemma 8, we can get $g(\mathbf{x}^*) \leq (m+2)[g(\mathbf{x}^t) + \frac{(B-w_{\mathbf{x}^\lambda})}{m+2}\rho_t]$. By formula (9) in Lemma 7, we set $r = \frac{1}{2}$ in Lemma 9. Therefore, $f(\mathbf{x}^{t^*}) \geq \frac{1}{m+2}(1-e^{-(m+2)})f(\mathbf{x}^*)$ holds immediately. So we get the approximate ratio of $\frac{1}{m+2}(1-e^{-(m+2)})$ in case 2, if f is monotone. □

As above, we show a $\frac{1}{m+2}(1-e^{-(m+2)})$-approximate ratio for monotone k-submodular maximization with a knapsack and m matroid constraints. Due to our conclusion, we improve the approximate ratio of monotone k-submodular maximization with a knapsack and m matroid constraints (Liu et al. 2022a) from $\frac{1}{2(m+1)}(1-e^{-(m+1)})$ to $\frac{1}{m+2}(1-e^{-(m+2)})$.

When $m = 1$, i.e. monotone k-submodular maximization with a knapsack and a matroid constraints, we have the corresponding conclusion as below. It also improves the result $\frac{1}{4}(1-e^{-2})$ in Liu et al. (2022a).

Corollary 1 *If the objective function f is monotone for problem (1) with $m = 1$, we can obtain a $\frac{1}{3}(1-e^{-3})$-approximate solution in KM-KM by setting $\lambda = 2$.*

5 Analysis for non-monotone *k*-submodular maximization with a knapsack and *m* matroid constraints

In this section, we further study non-monotone k-submodular maximization with a knapsack and m matroids constraints. In fact, the impact of monotonicity of f is not reflected in Lemmas 6, 8, 9. So we only need to give the following Lemma 10. Using Lemmas 6, 8, 9 10, we can get an approximate ratio $\frac{1}{m+3}(1-e^{-(m+3)})$.

Lemma 10 *Considering the current solution $\mathbf{x}^{t^*}$ and the m-swap $(\bar{y}(b^{t^*}), b^{t^*})$ as in Lemma 7, we have*

$$
f((\mathbf{x}^{t^*} \setminus \bigsqcup_{y(b^{t^*})\in\bar{y}(b^{t^*})} \mathbf{1}_{y(b^{t^*}),j^{t^*}}) \sqcup \mathbf{1}_{b^{t^*},i^{t^*}}) - f(\mathbf{x}^{t^*}) \leq \frac{m+1}{\lambda} \cdot f(\mathbf{x}^\lambda), \tag{14}
$$

where $\mathbf{1}_{y(b^{t^*}),j^{t^*}} \preceq \mathbf{x}^{t^*} \setminus \mathbf{x}^\lambda$.

Theorem 2 *If the objective function f is non-monotone for problem (1), we can obtain a $\frac{1}{m+3}(1-e^{-(m+3)})$-approximate solution in KM-KM by setting $\lambda \geq \frac{(m+1)(m+3)}{m+2+e^{-(m+3)}}$.*

Proof When $t = T$, similar to Theorem 1 in Sect. 4, we consider dividing the m-swaps $(y(b), b)$ with respect to $\mathbf{x}^T$ and $\mathbf{x}^*$ into two cases.

Case 1: Considering the m-swaps $(\bar{y}(b), b)$ with respect to $\mathbf{x}^T$ and $\mathbf{x}^*$, they were all rejected just due to $\rho(\bar{y}(b), b) \le 0$ instead of knapsack constraint.

Combine formula (6) in Lemma 5 and formula (7) in Lemma 6, then use formula (12) and formula (8) in Lemma 6 to get $g(\mathbf{x}^*) \le (m+3)g(\mathbf{x}^T)$. Finally, we have $f(\mathbf{x}^*) \le (m+3)f(\mathbf{x}^T) - (m+2)f(\mathbf{x}^\lambda) \le (m+3)f(\mathbf{x}^T)$ due to nonnegativity of f. Therefore, we find a $\frac{1}{m+3}$-approximate solution in case 1, if f is non-monotone in problem (1).

Case 2: Considering the m-swaps $(\bar{y}(b), b)$ with respect to $\mathbf{x}^T$ and $\mathbf{x}^*$, there exists at least one satisfying $w_{\mathbf{x}^t} - w_{\bar{y}(b)} + w_b > B$.

When $t \in \{\lambda + 1, \ldots, t^*\}$, we combine formula (6) in Lemma 5, formula (7) in Lemma 6 and formula(13). Then use formula (8) in Lemma 6 and Lemma 8, we can get $g(\mathbf{x}^*) \le (m+3)[g(\mathbf{x}^t) + \frac{(B - w_{\mathbf{x}^\lambda})}{m+3}\rho_t]$. By formula (14) in Lemma 10, we set $r = \frac{m+1}{\lambda}$ in Lemma 9. Therefore, $f(\mathbf{x}^{t^*}) \ge \frac{1}{m+3}(1 - e^{-(m+3)})f(\mathbf{x}^*)$ holds immediately. So we get the approximate ratio of $\frac{1}{m+3}(1 - e^{-(m+3)})$ in case 2, if f is non-monotone in problem (1). □

As above, we show a $\frac{1}{m+3}(1 - e^{-(m+3)})$-approximate ratio for non-monotone k-submodular maximization with a knapsack and m matroid constraints. Due to our conclusion, we extend monotone k-submodular maximization with a knapsack and m matroid constraints (Liu et al. 2022a) to non-monotone case.

When $m = 1$, i.e. non-monotone k-submodular maximization with a knapsack and a matroid constraints, we have the corresponding conclusion as below.

Corollary 2 *If the objective function f is non-monotone for problem (1) with $m = 1$, we can obtain a $\frac{1}{4}(1 - e^{-4})$-approximate solution in KM-KM by setting $\lambda = 3$.*

6 Conclusions

In our paper, based on a nested greedy and local search algorithm KM-KM (Liu et al. 2022a) and a construction method (Nguyen and Thai 2020), we improve the approximate ratio for problem (1) (Liu et al. 2022a) from $\frac{1}{2(m+1)}(1 - e^{-(m+1)})$ to $\frac{1}{m+2}(1 - e^{-(m+2)})$ by enumerating $\lambda = 2$ items with the largest marginal profits in the optimal solution. The conclusion can get $\frac{1}{3}(1 - e^{-3})$-approximate ratio for problem (1) with $m = 1$. Furthermore, we extend the conclusion to non-monotone case and get the approximate ratio $\frac{1}{m+3}(1 - e^{-(m+3)})$ for problem (1) by enumerating $\lambda \ge \frac{(m+1)(m+3)}{m+2+e^{-(m+3)}}$ items with $\lambda \in N_+$. The conclusion can get $\frac{1}{4}(1 - e^{-4})$ -approximate ratio for problem (1) with $m = 1$. And we need to enumerate $\lambda = 3$ items with the largest marginal profits in the optimal solution.

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Declarations

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Appendix

Proof of Lemma 5:

When k-submodular function f is monotone, the conclusions are as follows. Due to monotonicity of f, $f_{\mathbf{x}^t_{l-1}}(\mathbf{1}_{v_l.i_l}) \geq 0$ holds in Greedy Algorithm. By the definition of g, we have $g_{\mathbf{x}^t_{l-1}}(\mathbf{1}_{v_l.i_l}) \geq 0$. For v_l in l-th iteration of Greedy Algorithm, we compare the position i_l with $\mathbf{1}_{v_l,i_l} \preceq \mathbf{x}^t_l$ and i_* with $\mathbf{1}_{v_l,i_*} \preceq \mathbf{x}^*$.

If $v_l \in U(\mathbf{x}^*)$ with $i_* = i_l$, then $\mathbf{o}^t_{l-1} = \mathbf{o}^t_l$. Therefore, we have

$$g(\mathbf{o}^t_{l-1}) - g(\mathbf{o}^t_l) = 0 \leq g(\mathbf{x}^t_l) - g(\mathbf{x}^t_{l-1}). \tag{15}$$

If $v_l \in U(\mathbf{x}^*)$ with $i_* \neq i_l$, then $\mathbf{o}^t_{l-1} = \mathbf{o}^t_{l-1/2} \sqcup \mathbf{1}_{v_l,i_*}$ and $\mathbf{o}^t_l = \mathbf{o}^t_{l-1/2} \sqcup \mathbf{1}_{v_l,i_l}$. By monotonicity of f, greedy choice of Greedy Algorithm and orthant submodularity, we get

$$\begin{aligned} g(\mathbf{o}^t_{l-1}) - g(\mathbf{o}^t_l) &\leq g(\mathbf{o}^t_{l-1}) - g(\mathbf{o}^t_{l-1/2}) \\ &= g_{\mathbf{o}^t_{l-1/2}}(\mathbf{1}_{v_l,i_*}) \\ &\leq g_{\mathbf{x}^t_{l-1}}(\mathbf{1}_{v_l,i_*}) \\ &\leq g(\mathbf{x}^t_l) - g(\mathbf{x}^t_{l-1}). \end{aligned} \tag{16}$$

If $v_l \notin U(\mathbf{x}^*)$, then $\mathbf{o}^t_{l-1/2} = \mathbf{o}^t_l = \mathbf{o}^t_{l-1} \sqcup \mathbf{1}_{v_l,i_l}$, we have

$$g(\mathbf{o}^t_{l-1}) - g(\mathbf{o}^t_l) \leq 0 \leq g(\mathbf{x}^t_l) - g(\mathbf{x}^t_{l-1}). \tag{17}$$

In summary, we have

$$g(\mathbf{o}^t_{l-1}) - g(\mathbf{o}^t_l) \leq g(\mathbf{x}^t_l) - g(\mathbf{x}^t_{l-1}). \tag{18}$$

Sum it for l from 1 to $|U(\mathbf{x}^t \backslash \mathbf{x}^\lambda)|$ and get

$$\begin{aligned} g(\mathbf{x}^*) - g(\mathbf{o}^t_{|U(\mathbf{x}^t \backslash \mathbf{x}^\lambda)|}) &= \sum_{l=1}^{|U(\mathbf{x}^t \backslash \mathbf{x}^\lambda)|} [g(\mathbf{o}^t_{l-1}) - g(\mathbf{o}^t_l)] \\ &\leq \sum_{l=1}^{|U(\mathbf{x}^t \backslash \mathbf{x}^\lambda)|} g(\mathbf{x}^t_l) - g(\mathbf{x}^t_{l-1}) \\ &= g(\mathbf{x}^t). \end{aligned} \tag{19}$$

When k-submodular function f is non-monotone, the conclusion will change as below. Due to pairwise monotonicity, $f_{\mathbf{x}^t_{l-1}}(\mathbf{1}_{v_l.i_l}) \geq 0$ holds in Greedy Algorithm. By the definition of g, we have $g_{\mathbf{x}^t_{l-1}}(\mathbf{1}_{v_l.i_l}) \geq 0$.

If $v_l \in U(\mathbf{x}^*)$ with $i_* = i_l$, we have

$$g(\mathbf{o}^t_{l-1}) - g(\mathbf{o}^t_l) = 0 \leq 2[g(\mathbf{x}^t_l) - g(\mathbf{x}^t_{l-1})]. \tag{20}$$

If $v_l \in U(\mathbf{x}^*)$ with $i_* \neq i_l$, we get

$$\begin{aligned} g(\mathbf{o}^t_{l-1}) - g(\mathbf{o}^t_l) &= g_{\mathbf{o}^t_{l-1/2}}(\mathbf{1}_{v_l,i_*}) + g_{\mathbf{o}^t_{l-1/2}}(\mathbf{1}_{v_l,i'}) \\ &\quad - [g_{\mathbf{o}^t_{l-1/2}}(\mathbf{1}_{v_l,i_l}) + g_{\mathbf{o}^t_{l-1/2}}(\mathbf{1}_{v_l,i'})] \\ &\leq g_{\mathbf{o}^t_{l-1/2}}(\mathbf{1}_{v_l,i_*}) + g_{\mathbf{o}^t_{l-1/2}}(\mathbf{1}_{v_l,i'}) \\ &\leq 2g_{\mathbf{x}^t_{l-1}}(\mathbf{1}_{v_l,i_l}) \\ &= 2[g(\mathbf{x}^t_l) - g(\mathbf{x}^t_{l-1})] \end{aligned} \tag{21}$$

for any $i' \in [k]$ with $i' \neq i_l$. Due to pairwise monotonicity, we get the first inequality. By greedy choice of Greedy Algorithm and orthant submodularity, the second holds.

If $v_l \notin U(\mathbf{x}^*)$, similar to above, we have

$$\begin{aligned} g(\mathbf{o}^t_{l-1}) - g(\mathbf{o}^t_l) &= g_{\mathbf{o}^t_{l-1}}(\mathbf{1}_{v_l,i'}) - [g_{\mathbf{o}^t_{l-1}}(\mathbf{1}_{v_l,i'}) + g_{\mathbf{o}^t_{l-1}}(\mathbf{1}_{v_l,i_l})] \\ &\leq g_{\mathbf{o}^t_{l-1}}(\mathbf{1}_{v_l,i'}) \\ &\leq g_{\mathbf{o}^t_{l-1}}(\mathbf{1}_{v_l,i_l}) \\ &\leq g_{\mathbf{x}^t_{l-1}}(\mathbf{1}_{v_l,i_l}) \\ &\leq 2[g(\mathbf{x}^t_l) - g(\mathbf{x}^t_{l-1})]. \end{aligned} \tag{22}$$

In summary, we have

$$g(\mathbf{o}^t_{l-1}) - g(\mathbf{o}^t_l) \leq 2[g(\mathbf{x}^t_l) - g(\mathbf{x}^t_{l-1})]. \tag{23}$$

Sum it for l from 1 to $|U(\mathbf{x}^t \backslash \mathbf{x}^\lambda)|$ and get

$$\begin{aligned} g(\mathbf{x}^*) - g(\mathbf{o}^t_{|U(\mathbf{x}^t\backslash\mathbf{x}^\lambda)|}) &= \sum_{l=1}^{|U(\mathbf{x}^t\backslash\mathbf{x}^\lambda)|} [g(\mathbf{o}^t_{l-1}) - g(\mathbf{o}^t_l)] \\ &\leq \sum_{l=1}^{|U(\mathbf{x}^t\backslash\mathbf{x}^\lambda)|} 2[g(\mathbf{x}^t_l) - g(\mathbf{x}^t_{l-1})] \\ &= 2g(\mathbf{x}^t). \end{aligned} \tag{24}$$

Proof of Lemma 6:

Due to $\mathbf{o}^t_{|U(\mathbf{x}^t \backslash \mathbf{x}^\lambda)|} = (\mathbf{x}^* \sqcup \mathbf{x}^t) \sqcup \mathbf{x}^t$, we have $U(\mathbf{o}^t_{|U(\mathbf{x}^t \backslash \mathbf{x}^\lambda)|}) \backslash U(\mathbf{x}^t) = U(\mathbf{x}^*) \backslash U(\mathbf{x}^t)$ and $\mathbf{x}^t \preceq \mathbf{o}^t_{|U(\mathbf{x}^t \backslash \mathbf{x}^\lambda)|}$. By Lemma 3 between $U(\mathbf{x}^*)$ and $U(\mathbf{x}^t)$, there exists a mapping y :

$$U(\mathbf{o}^t_{|U(\mathbf{x}^t \backslash \mathbf{x}^\lambda)|}) \backslash U(\mathbf{x}^t) \rightarrow [U(\mathbf{x}^t) \backslash U(\mathbf{x}^*)]$$

such that $(U(\mathbf{x}^t) \backslash \bar{y}(b)) \cup \{b\} \in \bigcap_{j=1}^m \mathcal{L}_j$, for $b \in U(\mathbf{o}^t_{|U(\mathbf{x}^t \backslash \mathbf{x}^\lambda)|}) \backslash U(\mathbf{x}^t)$, $\bar{y}(b) \in [U(\mathbf{x}^t) \backslash U(\mathbf{x}^*)]^m$, and each element $a \in U(\mathbf{x}^t) \backslash U(\mathbf{x}^*)$ appears in mapping y no more than m times. Using Lemma 1 and the mapping $y : b \rightarrow \bar{y}(b)$, we get

$$\begin{aligned} g(\mathbf{o}^t_{|U(\mathbf{x}^t \backslash \mathbf{x}^\lambda)|}) &\leq \sum_{\mathbf{1}_{b,i} \preceq (\mathbf{o}^t_{|U(\mathbf{x}^t \backslash \mathbf{x}^\lambda)|} \backslash \mathbf{x}^t)} [g(\mathbf{x}^t \sqcup \mathbf{1}_{b,i}) - g(\mathbf{x}^t)] + g(\mathbf{x}^t) \\ &\leq \sum_{\mathbf{1}_{b,i} \preceq (\mathbf{o}^t_{|U(\mathbf{x}^t \backslash \mathbf{x}^\lambda)|} \backslash \mathbf{x}^t)} [g((\mathbf{x}^t \backslash \bigsqcup_{y(b) \in \bar{y}(b)} \mathbf{1}_{y(b),j}) \sqcup \mathbf{1}_{b,i}) \\ &\quad - g(\mathbf{x}^t \backslash \bigsqcup_{y(b) \in \bar{y}(b)} \mathbf{1}_{y(b),j})] + g(\mathbf{x}^t) \end{aligned} \tag{25}$$

for $\mathbf{1}_{y(b),j} \preceq \mathbf{x}^t \backslash \mathbf{x}^\lambda$.

Then we give the proof of second inequality as follows.

For fixed iteration step $t \geq \lambda + 1$ in KM-KM, the ground set of Greedy Algorithm(f, G) is $G = \{v_1, \ldots, v_{|U(\mathbf{x}^t \backslash \mathbf{x}^\lambda)|}\} = U(\mathbf{x}^t \backslash \mathbf{x}^\lambda)$ in a fixed order as we mentioned earlier.

Each $b \in U(\mathbf{o}^t_{|U(\mathbf{x}^t \backslash \mathbf{x}^\lambda)|}) \backslash U(\mathbf{x}^t)$ will be mapped to $\bar{y}(b) \in [U(\mathbf{x}^t) \backslash U(\mathbf{x}^*)]^m$. Let $p = |\bar{y}(b)| \in [m]$ for each such b, then write $\bar{y}(b) = \{v_{q_1}, \ldots, v_{q_p}\}$, where $1 \leq q_1 < \ldots q_p \leq m$.

By our settings, we have

$$\begin{aligned} &\sum_{\mathbf{1}_{b,i} \preceq (\mathbf{o}^t_{|U(\mathbf{x}^t \backslash \mathbf{x}^\lambda)|} \backslash \mathbf{x}^t)} [g(\mathbf{x}^t) - g(\mathbf{x}^t \backslash \bigsqcup_{y(b) \in \bar{y}(b)} \mathbf{1}_{y(b),j})] \\ &= \sum_{\mathbf{1}_{b,i} \preceq (\mathbf{o}^t_{|U(\mathbf{x}^t \backslash \mathbf{x}^\lambda)|} \backslash \mathbf{x}^t)} [g(\mathbf{x}^t) - g(\mathbf{x}^t \backslash \bigsqcup_{l=q_1}^{q_p} \mathbf{1}_{v_l,j_l})] \\ &\leq \sum_{\mathbf{1}_{b,i} \preceq (\mathbf{o}^t_{|U(\mathbf{x}^t \backslash \mathbf{x}^\lambda)|} \backslash \mathbf{x}^t)} \sum_{r=q_1}^{q_p} [g(\mathbf{x}^\lambda \sqcup (\bigsqcup_{l=1}^{r} \mathbf{1}_{v_l,j_l})) - g(\mathbf{x}^\lambda \sqcup (\bigsqcup_{l=1}^{r-1} \mathbf{1}_{v_l,j_l}))] \\ &\leq m \cdot \sum_{r=1}^{|U(\mathbf{x}^t \backslash \mathbf{x}^\lambda)|} [g(\mathbf{x}^\lambda \sqcup (\bigsqcup_{l=1}^{r} \mathbf{1}_{v_l,j_l})) - g(\mathbf{x}^\lambda \sqcup (\bigsqcup_{l=1}^{r-1} \mathbf{1}_{v_l,j_l}))] \\ &= m \cdot [g(\mathbf{x}^t) - g(\mathbf{x}^\lambda)] \\ &\leq m \cdot g(\mathbf{x}^t) \end{aligned} \tag{26}$$

for $\mathbf{1}_{y(b),j} \preceq \mathbf{x}^t \backslash \mathbf{x}^\lambda$. The first inequality is due to orthant submodularity. As we mentioned, each element $a \in U(\mathbf{x}^t)\backslash U(\mathbf{x}^*)$ appears in mapping y no more than m times. In Greedy Algorithm(f, G), all marginal gains $g_{\mathbf{x}}(\mathbf{1}_{v_l, j_l}) \geq 0$ for non-monotone or monotone k-submodular function f input. Therefore, we get the second inequality. The third inequality needs nonnegativity of g.

Proof of Lemma 7:

For problem (1), input a monotone k-submodular function f and $\lambda = 2$ in KM-KM. In the fixed $t^* + 1$ iteration, considering the current solution $\mathbf{x}^{t^*}$ and the m-swap $(\bar{y}(b^{t^*}), b^{t^*})$, we have

$$
\begin{aligned}
& f((\mathbf{x}^{t^*} \setminus \mathbf{1}_{y(b^{t^*}), j^{t^*}}) \sqcup \mathbf{1}_{b^{t^*}, i^{t^*}}) - f(\mathbf{x}^{t^*}) \\
& \leq f((\mathbf{x}^{t^*} \setminus \mathbf{1}_{y(b^{t^*}), j^{t^*}}) \sqcup \mathbf{1}_{b^{t^*}, i^{t^*}}) - f(\mathbf{x}^{t^*} \setminus \mathbf{1}_{y(b^{t^*}), j^{t^*}}) \\
& \leq f((\mathbf{x}^1 \sqcup \mathbf{1}_{b^{t^*}, i^{t^*}}) - f(\mathbf{x}^1) \\
& \leq f(\mathbf{x}^2) - f(\mathbf{x}^1).
\end{aligned}
$$

Using the monotonicity of f, we get the first inequality. Then the second is due to orthant submodularity. Because we greedily choose $\mathbf{x}^t$ for $t \in \{1, 2\}$, the third inequality holds. Similarly, we have

$$
f((\mathbf{x}^{t^*} \setminus \mathbf{1}_{y(b^{t^*}), j^{t^*}}) \sqcup \mathbf{1}_{b^{t^*}, i^{t^*}}) - f(\mathbf{x}^{t^*}) \leq f(\mathbf{x}^1) - f(\mathbf{x}^0).
$$

Combining the above two formulas, we have

$$
f((\mathbf{x}^{t^*} \setminus \mathbf{1}_{y(b^{t^*}), j^{t^*}}) \sqcup \mathbf{1}_{b^{t^*}, i^{t^*}}) - f(\mathbf{x}^{t^*}) \leq \frac{1}{2} f(\mathbf{x}^2) = \frac{1}{2} f(\mathbf{x}^\lambda). \tag{27}
$$

Proof of Lemma 8:

Given a fixed $t \in \{\lambda, \ldots, t^*\}$, by greedy choice of t-th iteration and the assumption about $t^* + 1$, we have

$$
\frac{f(\mathbf{x}^t \backslash \bigsqcup_{y(b) \in \bar{y}(b)} \mathbf{1}_{y(b),j} \sqcup \mathbf{1}_{b,i}) - f(\mathbf{x}^t)}{w_b} \leq \rho_t, \tag{28}
$$

for m-swaps $(\bar{y}(b), b)$ with $b \in U(\mathbf{o}^t_{|U(\mathbf{x}^t \backslash \mathbf{x}^\lambda)|}) \backslash U(\mathbf{x}^t)$ and $\bar{y}(b) \in [U(\mathbf{x}^t)\backslash U(\mathbf{x}^*)]^m$. Due to $U(\mathbf{o}^t_{|U(\mathbf{x}^t \backslash \mathbf{x}^\lambda)|}) \backslash U(\mathbf{x}^t) = U(\mathbf{x}^*) \backslash U(\mathbf{x}^t)$, we have

$$
\sum_{\mathbf{1}_{b,i} \preceq (\mathbf{o}^t_{|U(\mathbf{x}^t \backslash \mathbf{x}^\lambda)|} \backslash \mathbf{x}^t)} w_b \leq B - w_{\mathbf{x}^\lambda}. \tag{29}
$$

Combining the above formula, we get

$$\begin{aligned}
&\sum_{\mathbf{1}_{b,i}\preceq(\mathbf{o}^t_{|U(\mathbf{x}^t\backslash\mathbf{x}^\lambda)|}\backslash\mathbf{x}^t)} [g((\mathbf{x}^t\backslash \bigsqcup_{y(b)\in\bar{y}(b)} \mathbf{1}_{y(b),j})\sqcup \mathbf{1}_{b,i})-g(\mathbf{x}^t)] \\
&= \sum_{\mathbf{1}_{b,i}\preceq(\mathbf{o}^t_{|U(\mathbf{x}^t\backslash\mathbf{x}^\lambda)|}\backslash\mathbf{x}^t)} [f((\mathbf{x}^t\backslash \bigsqcup_{y(b)\in\bar{y}(b)} \mathbf{1}_{y(b),j})\sqcup \mathbf{1}_{b,i})-f(\mathbf{x}^t)] \qquad (30)\\
&\leq (B-w_{\mathbf{x}^\lambda})\rho_t,
\end{aligned}$$

for $\mathbf{1}_{y(b),j}\preceq \mathbf{x}^t\backslash\mathbf{x}^\lambda$.

Proof of Lemma 9:
We introduce a framework of proof inspired by Sarpatwar et al. (2019) to get

$$\frac{g((\mathbf{x}^{t^*}\backslash \bigsqcup_{y(b^{t^*})\in\bar{y}(b^{t^*})}\mathbf{1}_{y(b^{t^*}),j^{t^*}})\sqcup \mathbf{1}_{b^{t^*},i^{t^*}})}{g(\mathbf{x}^*)} \geq \frac{1}{\alpha}(1-e^{-\beta}). \qquad (31)$$

Let $B_\lambda=0$ and $B_t=\sum_{\tau=\lambda+1}^{t} w_{b^\tau}$, for any $t\in\{\lambda+1,\ldots,t^*+1\}$. Define $B'=B_{t^*+1}=B_{t^*}+w_{b^{t^*}}$ and $B''=B-w_{\mathbf{x}^\lambda}$. By the assumption of case 2, we have $B'>B\geq B''$. For $j=1,\ldots,B'$, we define $\gamma_j=\rho_{t-1}$ when $j=B_{t-1}+1,\ldots,B_t$. Note that $g(\mathbf{x}^\tau)-g(\mathbf{x}^{\tau-1})=w_{b^{\tau-1}}\rho_{\tau-1}$, using the above definition, we obtain that

$$g(\mathbf{x}^t)=\sum_{\tau=\lambda+1}^{t}[g(\mathbf{x}^\tau)-g(\mathbf{x}^{\tau-1})]=\sum_{\tau=\lambda+1}^{t} w_{b^{\tau-1}}\rho_{\tau-1}=\sum_{j=1}^{B_t}\gamma_j, \qquad (32)$$

for each $t\in\{\lambda+1,\ldots,t^*\}$, and

$$\begin{aligned}
g((\mathbf{x}^{t^*}\backslash \bigsqcup_{y(b^{t^*})\in\bar{y}(b^{t^*})}\mathbf{1}_{y(b^{t^*}),j^{t^*}})\sqcup \mathbf{1}_{b^{t^*},i^{t^*}}) &= \sum_{\tau=\lambda+1}^{t^*+1}[g(\mathbf{x}^t)-g(\mathbf{x}^{t-1})] \\
&= \sum_{\tau=\lambda+1}^{t^*+1} w_{b^{\tau-1}}\rho_{\tau-1}=\sum_{j=1}^{B'}\gamma_j.
\end{aligned} \qquad (33)$$

Using $g(\mathbf{x}^*)\leq\alpha[g(\mathbf{x}^t)+\frac{(B-w_{\mathbf{x}^\lambda})}{\beta}\rho_t]$ and (32), we have the following equalities

$$\begin{aligned}
g(\mathbf{x}^*) &\leq \alpha \min_{t\in\{\lambda+1,\ldots,t^*\}}\{g(\mathbf{x}^t)+\frac{B''}{\beta}\rho_t\} \\
&\leq \alpha \min_{t\in\{\lambda+1,\ldots,t^*\}}\{\sum_{j=1}^{B_t}\gamma_j+\frac{B''}{\beta}\gamma_{B_{t+1}}\}.
\end{aligned} \qquad (34)$$

From (33), (34) and Lemma 4, we obtain that

$$
\begin{aligned}
&\frac{g((\mathbf{x}^{t^*} \setminus \bigsqcup_{y(b^{t^*})\in\bar{y}(b^{t^*})} \mathbf{1}_{y(b^{t^*}),j^{t^*}}) \sqcup \mathbf{1}_{b^{t^*},i^{t^*}})}{g(\mathbf{x}^*)} \\
&\geq \frac{\sum_{j=1}^{B'} \gamma_j}{\alpha \min_{t\in\{\lambda+1,\dots,t^*\}} \{\sum_{j=1}^{B_t} \gamma_j + \frac{B''}{\beta}\gamma_{B_t+1}\}} \\
&= \frac{\sum_{j=1}^{B'} \gamma_j}{\alpha \min_{t\in\{1,\dots,B'\}} \{\sum_{j=1}^{t-1} \gamma_j + \frac{B''}{\beta}\gamma_t\}} \\
&\geq \frac{1}{\alpha}(1-(1-\frac{\beta}{B''})^{B'}) \\
&\geq \frac{1}{\alpha}(1-e^{-\frac{\beta B'}{B''}}) \\
&\geq \frac{1}{\alpha}(1-e^{-\beta}).
\end{aligned} \tag{35}
$$

Using $1-\frac{1}{\alpha}(1-e^{-\beta})-r \geq 0$ and (35), we have

$$
\begin{aligned}
f(\mathbf{x}^{t^*}) &= f(\mathbf{x}^{\lambda}) + g(\mathbf{x}^{t^*}) \\
&= f(\mathbf{x}^{\lambda}) + g((\mathbf{x}^{t^*} \setminus \bigsqcup_{y(b^{t^*})\in\bar{y}(b^{t^*})} \mathbf{1}_{y(b^{t^*}),j^{t^*}}) \sqcup \mathbf{1}_{b^{t^*},i^{t^*}}) \\
&- [g((\mathbf{x}^{t^*} \setminus \bigsqcup_{y(b^{t^*})\in\bar{y}(b^{t^*})} \mathbf{1}_{y(b^{t^*}),j^{t^*}}) \sqcup \mathbf{1}_{b^{t^*},i^{t^*}}) - g(\mathbf{x}^{t^*})] \\
&= f(\mathbf{x}^{\lambda}) + g((\mathbf{x}^{t^*} \setminus \bigsqcup_{y(b^{t^*})\in\bar{y}(b^{t^*})} \mathbf{1}_{y(b^{t^*}),j^{t^*}}) \sqcup \mathbf{1}_{b^{t^*},i^{t^*}}) \\
&- [f((\mathbf{x}^{t^*} \setminus \bigsqcup_{y(b^{t^*})\in\bar{y}(b^{t^*})} \mathbf{1}_{y(b^{t^*}),j^{t^*}}) \sqcup \mathbf{1}_{b^{t^*},i^{t^*}}) - f(\mathbf{x}^{t^*})] \\
&\geq f(\mathbf{x}^{\lambda}) + \frac{1}{\alpha}(1-e^{-\beta})g(\mathbf{x}^*) - rf(\mathbf{x}^{\lambda}) \\
&= \frac{1}{\alpha}(1-e^{-\beta})f(\mathbf{x}^*) + (1-\frac{1}{\alpha}(1-e^{-\beta})-r)f(\mathbf{x}^{\lambda}) \\
&\geq \frac{1}{\alpha}(1-e^{-\beta})f(\mathbf{x}^*).
\end{aligned} \tag{36}
$$

Proof of Lemma 10:

Recall our settings in proof of Lemma 6: For a fixed iteration step $t \geq \lambda + 1$ in KM-KM, the ground set of Greedy Algorithm(f, G) is $G = \{v_1, \ldots, v_{|U(\mathbf{x}^t \backslash \mathbf{x}^\lambda)|}\} = U(\mathbf{x}^t \backslash \mathbf{x}^\lambda)$ in a fixed order as we mentioned earlier. Each $b \in U(\mathbf{o}^t_{|U(\mathbf{x}^t \backslash \mathbf{x}^\lambda)|}) \backslash U(\mathbf{x}^t)$ will be mapped to $\bar{y}(b) \in [U(\mathbf{x}^t) \backslash U(\mathbf{x}^*)]^m$. Let $p = |\bar{y}(b)| \in [m]$ for each such b, then write $\bar{y}(b) = \{v_{q_1}, \ldots, v_{q_p}\}$, where $1 \leq q_1 < \ldots q_p \leq m$. We have

$$
\begin{aligned}
& f((\mathbf{x}^{t^*} \backslash \bigsqcup_{y(b^{t^*}) \in \bar{y}(b^{t^*})} \mathbf{1}_{y(b^{t^*}), j^{t^*}}) \sqcup \mathbf{1}_{b^{t^*}, i^{t^*}}) - f(\mathbf{x}^{t^*}) \\
= \quad & f((\mathbf{x}^{t^*} \backslash \bigsqcup_{l=q_1}^{q_p} \mathbf{1}_{v_l, j_l}) \sqcup \mathbf{1}_{b^{t^*}, i^{t^*}}) - f(\mathbf{x}^{t^*}) \\
= & - [f(\mathbf{x}^{t^*} \backslash \mathbf{1}_{v_{q_1}, j_{q_1}} \sqcup \mathbf{1}_{v_{q_1}, j_{q_1}}) - f(\mathbf{x}^{t^*} \backslash \mathbf{1}_{v_{q_1}, j_{q_1}})] \\
& - [f((\mathbf{x}^{t^*} \backslash \bigsqcup_{l=q_1}^{q_2} \mathbf{1}_{v_l, j_l}) \sqcup \mathbf{1}_{v_{q_2}, j_{q_2}}) - f(\mathbf{x}^{t^*} \backslash \bigsqcup_{l=q_1}^{q_2} \mathbf{1}_{v_l, j_l})] \\
& \qquad \qquad \ldots \\
& - [f((\mathbf{x}^{t^*} \backslash \bigsqcup_{l=q_1}^{q_p} \mathbf{1}_{v_l, j_l}) \sqcup \mathbf{1}_{v_{q_p}, j_{q_p}}) - f(\mathbf{x}^{t^*} \backslash \bigsqcup_{l=q_1}^{q_p} \mathbf{1}_{v_l, j_l})] \\
& + f((\mathbf{x}^{t^*} \backslash \bigsqcup_{l=q_1}^{q_p} \mathbf{1}_{v_l, j_l}) \sqcup \mathbf{1}_{b^{t^*}, i^{t^*}}) - f(\mathbf{x}^{t^*} \backslash \bigsqcup_{l=q_1}^{q_p} \mathbf{1}_{v_l, j_l}) \\
\leq \quad & [f(\mathbf{x}^{t^*} \backslash \mathbf{1}_{v_{q_1}, j_{q_1}} \sqcup \mathbf{1}_{v_{q_1}, j'}) - f(\mathbf{x}^{t^*} \backslash \mathbf{1}_{v_{q_1}, j_{q_1}})] \\
& + [f((\mathbf{x}^{t^*} \backslash \bigsqcup_{l=q_1}^{q_2} \mathbf{1}_{v_l, j_l}) \sqcup \mathbf{1}_{v_{q_2}, j'}) - f(\mathbf{x}^{t^*} \backslash \bigsqcup_{l=q_1}^{q_2} \mathbf{1}_{v_l, j_l})] \\
& \ldots \\
& + [f((\mathbf{x}^{t^*} \backslash \bigsqcup_{l=q_1}^{q_p} \mathbf{1}_{v_l, j_l}) \sqcup \mathbf{1}_{v_{q_p}, j'}) - f(\mathbf{x}^{t^*} \backslash \bigsqcup_{l=q_1}^{q_p} \mathbf{1}_{v_l, j_l})] \\
& + f((\mathbf{x}^{t^*} \backslash \bigsqcup_{l=q_1}^{q_p} \mathbf{1}_{v_l, j_l}) \sqcup \mathbf{1}_{b^{t^*}, i^{t^*}}) - f(\mathbf{x}^{t^*} \backslash \bigsqcup_{l=q_1}^{q_p} \mathbf{1}_{v_l, j_l}) \\
\leq \quad & \sum_{l=q_1}^{q_p} [f(\mathbf{x}^{\tau-1} \sqcup \mathbf{1}_{v_l, j'}) - f(\mathbf{x}^{\tau-1})] + [f((\mathbf{x}^{\tau-1} \sqcup \mathbf{1}_{b^{t^*}, i^{t^*}}) - f(\mathbf{x}^{\tau-1})] \\
\leq \quad & (p+1)[f(\mathbf{x}^\tau) - f(\mathbf{x}^{\tau-1})] \\
\leq \quad & (m+1)[f(\mathbf{x}^\tau) - f(\mathbf{x}^{\tau-1})]
\end{aligned} \tag{37}
$$

for $\tau \in 1, \ldots, \lambda$. The first inequality is due to pairwise monotonicity, that is, $-f_{\mathbf{x}}((v,i)) \le f_{\mathbf{x}}((v,j))$ for $i \ne j \in [k]$. The second is due to orthant submodularity. Because we greedily choose $\mathbf{x}^t$ for $t \in \{1, \ldots, \lambda\}$, the third inequality holds. Combining the above λ formulas, we have

$$f((\mathbf{x}^{t^*} \setminus \mathbf{1}_{y(b^{t^*}),j^{t^*}}) \sqcup \mathbf{1}_{b^{t^*},i^{t^*}}) - f(\mathbf{x}^{t^*}) \le \frac{m+1}{\lambda} f(\mathbf{x}^{\lambda}). \tag{38}$$

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