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Minimum total coloring of planar graphs with maximum degree 8

Liting Wang[1] · Huijuan Wang[1] · Weili Wu[2]

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Abstract
We define G to be a planar graph with maximum degree Δ. Suppose $\Delta \geq 8$ and G has no adjacent p,q-cycles for some $p, q \in \{3, 4, 5, 6, 7, 8\}$, then G can be totally colored by $(\Delta + 1)$ colors.

Keywords Minimum total coloring · Planar graph · Maximum degree

1 Introduction

In this paper, all graphs considered are finite, simple and undirected. We refer the readers to Bondy and Murty (1982) for undefined notions and terminologies. Supposing that G is a graph, then V is used to denote the vertex set and $d(v)$ is used to denote the degree of v. Similarly, we respectively use F and $d(f)$ to denote the face set, the degree of f. Moreover, E is used to denote the edge set. Let Δ to be the maximum degree of a graph and δ to be the minimum degree. We respectively use i-vertex, i^+-vertex and i^--vertex to denote the vertex v when $d(v) = i$, $d(v) \geq i$, or $d(v) \leq i$. A i-face, i^+-face, or i^--face can be similarly defined. We use $(l_1, l_2, \ldots, l_k)$ to denote a k-face whose boundary vertices are consecutively l_1-vertex, l_2-vertex $\ldots l_k$-vertex. We use $n_k(f)$ to denote the number of k-vertices incident with f, use $n_k(v)$ to denote the number of k-vertices adjacent to v and use $f_k(v)$ to denote the number of k-faces incident with v. A k-total-coloring for G is coloring of $V \cup E$ that no two adjacent or incident elements in $V \cup E$ receive a same color by using k colors. If G has a k-total-coloring, then we say that G can be totally colored by k colors. And G is total-k-colorable if G can be totally colored by k colors. Suppose G has a k-total-coloring but does not have a $(k-1)$-total-coloring. Then k is the total chromatic number of G defined as χ''. It is obvious to find that the lower bound of χ'' is $\Delta + 1$. Behzad (1965)

✉ Huijuan Wang
sduwhj@163.com

1 School of Mathematics and Statistics, Qingdao University, Qingdao 266071, China

2 Department of Computer Science, University of Texas at Dallas, Richardson, TX 75080, USA

and Vizing (1968) posed the Total Coloring Conjecture (TCC) independently for the upper bound of χ''.

Conjecture 1 For any graph, $\Delta + 1 \leq \chi''(G) \leq \Delta + 2$.

Total Coloring Conjecture has attracted a lot of researchers' attention. But this conjecture is still unsolved even for planar graphs. Kostochka (1996) confirmed TCC with $\Delta \leq 5$. For planar graphs, the conjecture is open only when $\Delta = 6$ (see Kostochka 1996; Sanders and Zhao 1999). Some researchers found that it is possible for some specific graphs to prove that $\chi''(G) = \Delta + 1$. It is proved that it is a NP-complete problem to judge whether $\chi''(G) = \Delta + 1$ for a simple graph G by Sánchez-Arroyo (1989). However, if G is a planar graph with a large maximum degree, then it is possible to prove that $\chi''(G) = \Delta + 1$. It was proved that $\chi''(G) = \Delta + 1$ when G is a planar graph with $\Delta(G) \geq 9$ (see Borodin et al. 1997; Wang 2007; Kowalik et al. 2008). It is still an unsolved problem to judge whether a planar graph can be totally colored by $(\Delta + 1)$ colors for $\Delta = 6, 7$ and 8. There are many results obtained by adding restrictions for a planar graph with $\Delta(G) = 8$ in Du et al. (2009), Hou et al. (2008), Tan et al. (2009), Wang et al. (2014). Recently, a result for a planar graph with $\Delta(G) = 8$ has been proved in Wang et al. (2017), that is, suppose $\Delta \geq 8$ and G has no adjacent p,q-cycles for some $p, q \in \{3, 4, 5, 6, 7\}$, then G can be totally colored by $(\Delta + 1)$ colors. Next we generalize the result and get this following result.

Theorem 1 *Let G be a planar graph with maximum degree* $\Delta \geq 8$. *Suppose G has no adjacent p,q-cycles for some* $p, q \in \{3, 4, 5, 6, 7, 8\}$, *then G can be totally colored by* $(\Delta + 1)$ *colors.*

2 Reducible configurations

Since Theorem 1 was proved for $\Delta \geq 9$ in Kowalik et al. (2008). We only need to prove the theorem for $\Delta = 8$ in this paper. Let $G = (V, E)$ to be a minimal counterexample to Theorem 1, that is to say, the number of $|V| + |E|$ is as small as possible. So every proper subgraph of G has a 9-total-coloring.

Lemma 1 (Borodin et al. 1997)

(a) *G is* 2*-connected.*
(b) *Suppose* u_1u_2 *is an edge of G and* $d(u_1) \leq 4$. *Then* $d(u_1) + d(u_2) \geq \Delta + 2 = 10$.
(c) *Suppose* G_{82} *is a proper subgraph of G and it is induced by all the edges joining* 8*-vertices to* 2*-vertices. Then* G_{82} *is a forest.*

Lemma 2 (Chang et al. 2013) *G cannot contain subgraph isomorphic to the configurations depicted in Fig. 1. A vertex is marked by* • *if all neighbors are depicted in G and* $7 - v$ *denotes the vertex whose degree is seven.*

Lemma 3 (Xu et al. 2014) *Suppose* $v \in V$, $d(v) = 8$ *and* $d \geq 6$. *If v is consecutively adjacent to* $v_1, v_2, \ldots, v_8$, *then let v be incident with* $f_1, f_2, \ldots, f_8$ *and* f_j $(1 \leq j \leq 7)$ *be incident with* v_j *and* v_{j+1}. *As for* f_8, *it is incident with* v_8 *and* v_1. *If* $d(v_1) = 2$ *and it is adjacent to v and* u_1. *Then G cannot contain the following configurations (see Fig. 2):*

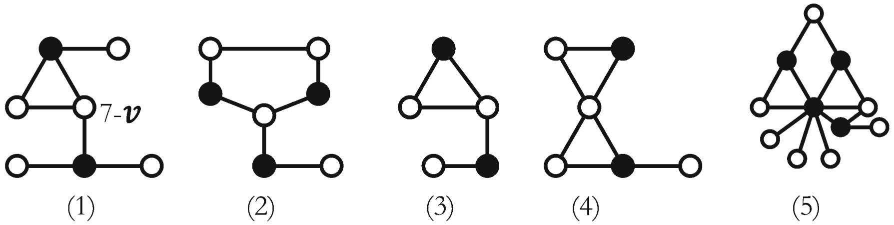

Fig. 1 Reducible configurations of Lemma 2

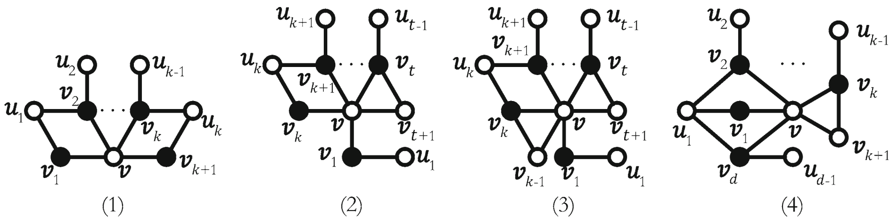

Fig. 2 Reducible configurations of Lemma 3

(1) *There exists an integer* k $(2 \leq k \leq 7)$ *such that* $d(f_j) = 4$ $(1 \leq j \leq k)$, $d(v_{k+1}) = 2$ *and* $d(v_i) = 3$ $(2 \leq i \leq k)$.
(2) *There exist two integers* k *and* t $(2 \leq k < t \leq 7)$ *such that* $d(v_k) = 2$, $d(v_i) = 3$ $(k + 1 \leq i \leq t)$, $d(f_t) = 3$ *and* $d(f_j) = 4$ $(k \leq j \leq t - 1)$.
(3) *There exist two integers* k *and* t $(3 \leq k \leq t \leq 7)$ *such that* $d(v_i) = 3$ $(k \leq i \leq t)$, $d(f_{k-1}) = d(f_t) = 3$ *and* $d(f_j) = 4$ $(k \leq j \leq t - 1)$.
(4) *There exist two integers* k $(2 \leq k \leq d - 2)$ *and* d, *such that* $d(v_d) = d(v_i) = 3$ $(2 \leq i \leq k)$, $d(f_k) = 3$ *and* $d(f_j) = 4$ $(0 \leq j \leq k - 1)$.

Lemma 4 (Wang et al. 2017) *Suppose* v *is a* 6*-vertex of* G. *If* v *is incident with a* 3*-cycle* (u, v, w) *where* u *or* w *is a* 4*-vertex, then* $n_{4^-}(v) = 1$.

Lemma 5 (Shen and Wang 2009) *G cannot contain* $(4^-, 6, 6)$*-cycles.*

3 Discharging

We will use discharging method to accomplish the proof of Theorem 1. By Euler's formula $|V| - |E| + |F| = 2$, we obtain

$$\sum_{v \in V}(2d(v) - 6) + \sum_{f \in F}(d(f) - 6) = -6(|V| - |E| + |F|) = -12 < 0$$

We define $\omega(x)$ to be the original charge. Let $\omega(v) = 2d(v) - 6$ for each $v \in V$ and $\omega(f) = d(f) - 6$ for each $f \in F$. So $\sum_{v \in V \cup F} \omega(x) < 0$. We define $\omega(x \to y)$ to be the amount of total charge which is transferred from x to y. We will give suitable discharging rules and distribute original charge to receive a new charge. We have two rounds of discharging rules. After the first round of discharging, we get a new charge of $x \in V \cup F$ denoted as $\omega^*(x)$. After the second round of discharging, we get a new charge of $x \in V \cup F$ denoted as $\omega'(x)$. If there exist no discharging rules for $x \in V \cup F$, then $\omega'(x) = \omega^*(x) = \omega(x)$. It is obvious that the total charge of G is unchangeable in the process of redistributing charge. So we have $\sum_{x \in V \cup F} \omega'(x) = \sum_{x \in V \cup F} \omega(x) = -6\chi(\Sigma) = -12 < 0$. We can get a contradiction by proving that $\sum_{x \in V \cup F} \omega'(x) \geq 0$.

These are the first round of discharging rules:

R1. Every 8-vertex sends 1 to its each adjacent 2-vertex.

R2. Suppose f is a face incident with v and $d(v) = 4$ or 5. If $d(f) = 5$, then $\omega(v \to f) = \frac{1}{3}$. If $d(f) = 4$, then $\omega(v \to f) = \frac{1}{2}$. At the end, v sends spare charge to its incident 3-faces evenly.

R3. If a 6-vertex and a 7^+-vertex is incident with a same 3-face, then the 7^+-vertex sends $\frac{5}{4}$ to the 3-face.

R4. Every 3-face receives $\frac{d(f)-6}{d(f)}$ from its adjacent 7^+-faces.

If the charge of a 5^--face is still negative after the first round of discharging rules, in other words, we have $\omega^*(f) < 0$, then we carry on the second round discharging:

R5. If $\omega^*(f) < 0$, then f receives $|\frac{\omega*(f)}{n_{6^+}(v)}|$ from each incident 6^+-vertices which does not send any charge to f.

Lemma 6 *Suppose v is a vertex incident with the face f.*

1. If $d(v) = 6$, then we have
$$\omega(v \to f) \leq \begin{cases} \frac{5}{4}, & if\, d(f) = 3\ and\ n_4(f) = 1, \\ \frac{11}{10}, & if\, d(f) = 3\ and\ n_5(f) \geq 1, \\ 1, & if\, d(f) = 3\ and\ n_{6^+}(f) = 3, \\ \frac{7}{8}, & if\, d(f) = 3,\ n_{5^-}(f) = 0\ and\ n_{7^+}(f) = 1, \\ \frac{1}{2}, & if\, d(f) = 3\ and\ n_{7^+}(f) = 2, \\ \frac{2}{3}, & if\, d(f) = 4\ and\ n_{3^-}(f) = 1, \\ \frac{1}{2}, & if\, d(f) = 4\ and\ n_{3^-}(f) = 0, \\ \frac{1}{3}, & if\, d(f) = 5. \end{cases}$$

2. If $d(v) \geq 7$, then we have
$$\omega(v \to f) \leq \begin{cases} \frac{3}{2}, & if\, d(f) = 3\ and\ n_{3^-}(f) = 1, \\ \frac{5}{4}, & if\, d(f) = 3\ and\ n_{3^-}(f) = 0, \\ 1, & if\, d(f) = 4\ and\ n_{3^-}(f) = 2, \\ \frac{3}{4}, & if\, d(f) = 4,\ n_{3^-}(f) = 1\ and\ n_4(f) = 1, \\ \frac{2}{3}, & if\, d(f) = 4,\ n_{3^-}(f) = 1\ and\ n_{5^+}(f) = 3, \\ \frac{1}{2}, & if\, d(f) = 4\ and\ n_{3^-}(f) = 0, \\ \frac{1}{3}, & if\, d(f) = 5. \end{cases}$$

Proof Suppose f is incident with v and $d(f) \geq 4$. Then it is easy to know that Lemma 6 is right by R2 and R5. If $d(v) \geq 7$ and $d(f) = 3$, then f is incident with at most one 3^--vertex, so $\omega(v \to f) \leq \frac{3}{2}$. If f is not incident with a 3-face, then $\omega(v \to f) \leq \frac{3-\frac{1}{2}}{2} = \frac{5}{4}$. Now we consider the case where $d(v) = 6$ and $d(f) = 3$ noted as (u, v, w). It is easy to find that the vertex u and the vertex w is equivalent. By lemma 1 (b), 6-vertex is not adjacent to 3^--vertices. If $d(u) = 4$, then $d(w) \geq 7$

by Lemma 5. So $\omega(v \to f) \leq 3 - \frac{5}{4} - \frac{1}{2} = \frac{5}{4}$. If $d(u) = 5$ and $d(w) = 6$, then $\omega(v \to f) \leq \frac{3-\frac{4}{5}}{2} = \frac{11}{10}$. Suppose $d(u) = d(w) = 5$. If u is incident with five 3-faces, then w is incident with at least two 6^+-faces. So $\omega(v \to f) \leq 3 - \frac{4}{5} - \frac{4}{3} \leq \frac{11}{10}$. If u is incident with four 3-faces, then u and u are incident with at least one 6^+-face. So $\omega(v \to f) \leq 3 - 1 \times 2 \leq \frac{11}{10}$. Suppose $d(u) = d(w) = 6$. Then $\omega(v \to f) \leq \frac{3}{3} = 1$. If $d(u) \geq 7$ and $d(w) \geq 6$, then the u sends $\frac{5}{4}$ to f by R4, so $\omega(v \to f) \leq \frac{3-\frac{5}{4}}{2} = \frac{7}{8}$. If $d(u) = d(w) \geq 7$, then $\omega(v \to f) \leq 3 - \frac{5}{4} \times 2 = \frac{1}{2}$. □

Lemma 7 *Suppose $d(v) = 8$ and v is consecutively adjacent to $v_1, v_2, \ldots, v_8$. Let v be incident with $f_1, f_2, \ldots, f_8$ and f_j $(1 \leq j \leq 7)$ be incident with v_j and v_{j+1}. As for f_8, it is incident with v_8 and v_1. If $d(v_1) = d(v_t) = 2$ $(t \geq 3)$ and $d(v_i) \geq 3$ $(2 \leq i \leq t-1)$, then we have $\sum_{i=1}^{t-1} \omega(v \to f_i) \leq \frac{5}{4}t - \frac{9}{4}$.*

Proof By Lemma 2, we know that $\min\{d(f_1), d(f_{t-1})\} \geq 4$. Firstly, suppose $d(f_1) = 4$ and $d(f_{t-1}) = 4$. If $\min\{d(f_2), d(f_3), \ldots, d(f_{t-2})\} \geq 5$, then $t \geq 4$, so $\sum_{i=1}^{t-1} \omega(v \to f_i) \leq 1 \times 2 + \frac{1}{3}(t-3) \leq \frac{5}{4}t - \frac{9}{4}$. If $\min\{d(f_2), d(f_3), \ldots, d(f_{t-2})\} = 4$ and $\max\{d(f_2), d(f_3), \ldots, d(f_{t-2})\} = 5$, then $\sum_{i=1}^{t-1} \omega(v \to f_i) \leq t - 2 + \frac{1}{3} \leq \frac{5}{4}t - \frac{9}{4}$. If $\max\{d(f_2), d(f_3), \ldots, d(f_{t-2})\}=\min\{d(f_2), d(f_3), \ldots, d(f_{t-2})\} = 4$, then $\sum_{i=1}^{t-1} \omega(v \to f_i) \leq t - 3 + \frac{3}{4} \times 2 \leq \frac{5}{4}t - \frac{9}{4}$ by Lemma 3. Suppose $\min\{d(f_2), d(f_3), \ldots, d(f_{t-2})\} = 3$ and $\max\{d(f_2), d(f_3), \ldots, d(f_{t-2})\} = 4$. Whether $d(f_2) = 3$ or $d(f_2) = 4$, we have $\omega(v \to f_1) + \omega(v \to f_2) \leq \max\{1 \times 2, \frac{3}{4} + \frac{5}{4}\} = 2$ by Lemma 3. Similarly, $\omega(v \to f_{t-2}) + \omega(v \to f_{t-1}) \leq \max\{1 \times 2, \frac{3}{4} + \frac{5}{4}\} = 2$. Moreover, v sends more charge to 3-faces than 4-faces, so we assume that v is incident with 3-faces as more as possible. Hence, $\sum_{i=1}^{t-1} \omega(v \to f_i) \leq 2 \times 2 + \frac{5}{4} \times (t-5) \leq \frac{5}{4}t - \frac{9}{4}$. Suppose $\max\{d(f_2), d(f_3), \ldots, d(f_{t-2})\}=\min\{d(f_2), d(f_3), \ldots, d(f_{t-2})\} = 3$, then f_j $(2 \leq j \leq t-2)$ receives at most $\frac{5}{4}$ from v by Lemma 3. Hence, $\sum_{i=1}^{t-1} \omega(v \to f_i) \leq \frac{3}{4} \times 2 + \frac{5}{4} \times (t-3) \leq \frac{5}{4}t - \frac{9}{4}$. Secondly, suppose $\min\{d(f_1), d(f_{t-1})\} = 4$ and $\max\{d(f_1), d(f_{t-1})\} \geq 5$. If $\max\{d(f_2), d(f_3), \ldots, d(f_{t-2})\} \geq 4$, then $\sum_{i=1}^{t-1} \omega(v \to f_i) \leq 1 \times 2 + \frac{1}{3} + \frac{3}{2} + \frac{5}{4} \times (t-5) \leq \frac{5}{4}t - \frac{9}{4}$. Otherwise, if $\max\{d(f_2), d(f_3), \ldots, d(f_{t-2})\}=\min\{d(f_2), d(f_3), \ldots, d(f_{t-2})\} = 3$, then $\sum_{i}^{t-1} \omega(v \to f_i) \leq \frac{3}{4} + \frac{1}{3} + \frac{3}{2} + \frac{5}{4} \times (t-4) \leq \frac{5}{4}t - \frac{9}{4}$. Finally, suppose $d(f_1) \geq 5$ and $d(f_{t-1})\} \geq 5$. Then $\sum_{i=1}^{t-1} \omega(v \to f_i) \leq \frac{1}{3} \times 2 + \frac{3}{2} \times 2 + \frac{5}{4} \times (t-5) \leq \frac{5}{4}t - \frac{9}{4}$. □

In the rest of this paper, we can check that $\omega'(x) \geq 0$ for every $x \in V \cup F$ which is a contradiction to our assumption. Let $f \in F$. If $d(f) \geq 7$, then $\omega'(f) \geq \omega(f) - \frac{d(f)-6}{d(f)} \times d(f) = 0$ by R4. If f is a 6-face, then $\omega'(f) = \omega(f) = 0$. Suppose $d(f) \leq 5$. If $n_{6^+}(f) \geq 1$, then $\omega'(f) \geq 0$ by R5. Otherwise, if $n_{6^+}(f) = 0$, then $n_5(f) = d(f)$. If $d(f) = 3$ and f is noted as (u_1, u_2, u_3), then $d(u_1) = d(u_2) = d(u_3) = 5$. By R2, 4^+-face receives at most $\frac{1}{2}$ from incident 4-vertices or 5-vertices. Suppose $f_3(u_i) \leq 3$ $(i = 1, 2, 3)$. Then $\omega(u_i \to f) \geq 1$, so $\omega'(f) \geq (3-6) + 1 \times 3 = 0$. Suppose there exists $f_3(u_i) \geq 4$. Without loss of generality, we assume that $f_3(u_3) \geq 4$. Then we have $f_3(u_1) \leq 4$ and $f_3(u_2) \leq 4$. Otherwise, $f_3(u_1) = 5$ or $f_3(u_2) = 5$, then for any integers $p, q \in \{3, 4, 5, 6, 7, 8\}$, there exists a vertex incident with adjacent p-cycles and q-cycles. So it is a contradiction to the condition of Theorem 1. If $f_3(u_1) = 4$, then u_1 is incident with a 9^+-face and u_2 is incident with at least two 6^+-faces, so $\omega(u_1 \to f) \geq 1$ and $\omega(u_2 \to f) \geq 1$. Consequently, $\omega'(f) \geq (3-6)+\frac{4}{5}+1+\frac{4}{3} > 0$. Similarly, we know that if $f_3(u_2) = 4$, then $\omega'(f) > 0$. Suppose $f_3(u_1) = f_3(u_2) = 3$. Then u_1 and u_2 is incident with at least one 6^+-face, so $\omega(u_i \to f) \geq \frac{4-\frac{1}{2}}{3} = \frac{7}{6}$, $(i = 1, 2)$. Consequently, $\omega'(f) \geq (3-6) + \frac{4}{5} + \frac{7}{6} \times 2 > 0$. If $d(f) = 4$, then $\omega'(f) \geq (4-6) + \frac{1}{2} \times 4 = 0$ by R2. If $d(f) = 5$, then $\omega'(f) \geq (5-6) + \frac{1}{3} \times 5 > 0$ by R2. So for every $f \in F$, we prove that $\omega'(f) \geq 0$. Next, we consider that $v \in V$. Suppose $d(v) = 2$. Then it is clear that $\omega(v) = -2$, so $\omega'(v) = -2 + 1 \times 2 = 0$ by R1. If $d(v) = 3$, then $\omega'(v) = \omega(v) = 0$. Suppose $d(v) = 4$ or $d(v) = 5$. Then $\omega'(v) = 0$ by R2.

If v is a 6^+-vertex and it is consecutively adjacent to $v_1, v_2, \ldots, v_d$. Let v be incident with $f_1, f_2, \ldots, f_d$ and f_j $(1 \leq j \leq d-1)$ be incident with v_j and v_{j+1}. As for f_d, it is incident with v_d and v_1. Suppose $d(v) = 6$. Then v is not incident with 3^--vertices by Lemma 1 (b) and v is incident with at most two 3-faces incident with a 4-vertex by Lemma 4. Clearly, $\omega(v) = 2d(v) - 6 = 6$. Hence, if $f_3(v) \leq 3$, then $\omega'(v) \geq 6 - (\frac{5}{4} \times 2 + \frac{11}{10} \times 1 + \frac{2}{3} \times 3) > 0$ by R4. Suppose $f_3(v) = 4$. If $f_{5^+}(v) \geq 1$, then $\omega'(v) \geq 6 - (\frac{5}{4} \times 2 + \frac{11}{10} \times 2 + \frac{2}{3} + \frac{1}{3}) > 0$. If $f_4(v) = 2$, then another three boundary vertices of each two 4-faces are adjacent to v, that is, all vertices of the two 4-faces are 4^+-vertices. Hence, $w'(v) \geq 6 - (\frac{5}{4} \times 2 + \frac{11}{10} \times 2 + \frac{1}{2} \times 2) > 0$. Suppose $f_3(v) \geq 5$. If v is adjacent to a 5-vertex v_0 and f is a 3-face incident with v and v_0, then $f_3(v_0) \leq 3$, so $\omega(v_0 \to f) \geq 1$ and $\omega(v \to f) \leq 1$. Suppose $f_3(v) = 5$. If $f_{5^+}(v) = 1$, then $\omega'(v) \geq 6 - (\frac{5}{4} \times 2 + 1 \times 3 + \frac{1}{3}) > 0$. If $f_4(v) = 1$, then another three boundary vertices of the 4-faces are adjacent to v, that is, the 4-face is incident with four 4^+-vertices. Hence, $\omega'(v) \geq 6 - (\frac{5}{4} \times 2 + 1 \times 3 + \frac{1}{2}) = 0$.

Suppose $f_3(v) = 6$, that is, $d(f_1) = d(f_2) = \ldots = d(f_6) = 3$. By Lemma 4, v is incident with at most one 4-vertex. So we may assume that $d(v_6) = 4$, then $d(v_1) \geq 7$ and $d(v_5) \geq 7$ by Lemma 5. Suppose $f_{6^+}(v_6) = 2$. Then $\omega(v_6 \to f_5) \geq 1$ and $\omega(v_6 \to f_6) \geq 1$, so $\omega(v \to f_5) \leq 1$ and $\omega(v \to f_6) \leq 1$. Therefore, $\omega'(v) \geq 6 - 1 \times 6 = 0$. Otherwise, $f_{5^-}(v) \geq 3$. Let f_l be the 5^--face incident with v_6 except f_5 and f_6. Suppose $d(f_l) = 5$. Then it is a contradiction to the condition of Theorem 1. Suppose $d(f_l) = 4$. Then v_6 is adjacent to v_4 and v_1 is adjacent to v_3. So we know that $f_{6^+}(v_6) = 1$ and $\omega(v_6 \to f_i) \geq \frac{2-\frac{1}{2}}{2} = \frac{3}{4}$ $(i = 5, 6)$. Therefore, $\omega(v \to f_i) \leq 3 - \frac{5}{4} - \frac{3}{4} \leq 1$ $(i = 5, 6)$, and $\omega'(v) \geq 6 - 1 \times 6 = 0$. Suppose $d(f_l) = 3$. Then each of the boundary vertices of f is adjacent to v. If v_6 is adjacent to v_4 and v_1 is adjacent to v_4, then $d(v_4) \geq 7$ by Lemma 5. So $\omega(v_4 \to f_4) = \frac{5}{4}$ and $\omega(v_5 \to f_4) = \frac{5}{4}$, then $\omega(v \to f_4) \leq \frac{1}{2}$ and $\omega'(v) \geq 6 - (\frac{5}{4} \times 2 + 1 \times 3 + \frac{1}{2}) = 0$. If v_6 is adjacent to v_3 and v_1 is adjacent to v_3, then $d(v_3) \geq 7$ by Lemma 5. Suppose $d(v_2) \geq 6$ and $d(v_4) \geq 6$. Then $\omega(v \to f_i) \leq \frac{3-\frac{5}{4}}{2} = \frac{7}{8}$ $(i = 1, 2, 3, 4)$. Hence, $\omega'(v) \geq 6 - (\frac{5}{4} \times 2 + \frac{7}{8} \times 4) = 0$. Suppose $d(v_2) = 5$ or $d(v_4) = 5$. Without of generality, assume that $d(v_4) = 5$. Then $\omega(v_4 \to f_3) \geq 1$ and $\omega(v_4 \to f_4) \geq 1$. So $\omega(v \to f_3) \leq 3 - (1 + \frac{5}{4}) = \frac{3}{4}$ and $\omega(v \to f_4) \leq 3 - (1 + \frac{5}{4}) = \frac{3}{4}$. Therefore, $\omega'(v) \geq 6 - (\frac{5}{4} \times 2 + 1 \times 2 + \frac{3}{4} \times 2) = 0$.

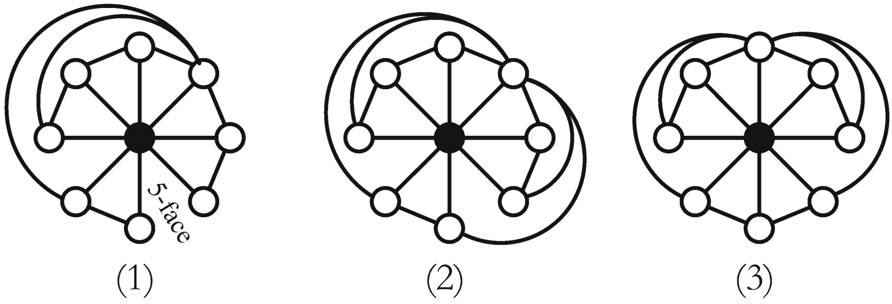

Fig. 3 $n_2(v) = 0$ and $f_3(v) = 6$

Suppose $d(v) = 7$. Then $\omega(v) = 2d(v) - 6 = 8$. Clearly, we have $f_3(v) \le 6$ and $n_{2^-}(v) = 0$ by Lemma 1 (b). Suppose each of the 3-faces incident with v is not incident with a 3-vertex. If $f_3(v) = 6$, then $f_{9^+}(v) = 1$, so $\omega'(v) \ge 8 - \frac{5}{4} \times 6 > 0$. If $f_3(v) = 5$, then each of the 4-faces incident with v is incident with at most one 3^--vertex. So $\omega'(v) \ge 8 - (\frac{5}{4} \times 5 + \frac{3}{4} \times 2) > 0$. If $f_3(v) \le 4$, then $\omega'(v) \ge 8 - (\frac{5}{4} \times 4 + 1 \times 3) = 0$. Suppose v is incident with at least one 3-face which is incident with a 3-vertex. Then each of the 4-faces is incident with at most one 3^--vertex. By Lemma 2, v is incident with at most two 3-faces which is incident with a 3-vertex. If $f_3(v) = 6$, then $f_{9^+}(v) = 1$, so $\omega'(v) \ge 8 - (\frac{3}{2} \times 2 + \frac{5}{4} \times 4) = 0$ by Lemma 6. Suppose $f_3(v) = 5$. If $f_{5^+}(v) \ge 1$, then $\omega'(v) \ge 8 - (\frac{3}{2} \times 2 + \frac{5}{4} \times 3 + \frac{2}{3} + \frac{1}{3}) > 0$ by Lemma 6. Otherwise, suppose $f_4(v) = 2$. If there exist two 3-faces incident with a 3-vertex, then each of the 4-faces incident with v is not incident with 3^--vertices. So $\omega'(v) \ge \min\{8 - (\frac{3}{2} \times 2 + \frac{5}{4} \times 3 + \frac{3}{4} + \frac{1}{2}), 8 - (\frac{3}{2} \times 1 + \frac{5}{4} \times 4 + \frac{3}{4} \times 2\} = 0$. Suppose $f_3(v) \le 4$. Then $\omega'(v) \ge 8 - (\frac{3}{2} \times 2 + \frac{5}{4} \times 2 + \frac{3}{4} \times 3) > 0$ by Lemma 6. If $d(v) = 8$, then $\omega(v) = 2 \times 8 - 6 = 10$ and $f_3(v) \le 6$. We will consider the following cases by discussing the number of $n_2(v)$ by Lemmas 6 and 7.

Case 1. $n_2(v) = 0$. Suppose $f_3(v) = 6$. If $f_{5^+}(v) \ge 2$ or $f_{6^+}(v) \ge 1$, then $\omega'(v) \ge 10 - (\frac{3}{2} \times 6 + 1) = 0$ by Lemma 6. Otherwise, $f_{5^+}(v) \le 1$ and $f_{6^+}(v) = 0$. Suppose $f_5(v) = 1$ and $f_4(v) = 1$. Then there exists only one case that satisfies the condition of Theorem 1. We show this case in Fig. 3(1). It is clear that another three boundary vertices of each 4-faces are adjacent to v, and v is incident with at least one 3-face which is not incident with a 3-vertex by Lemma 2. If the 4-face is incident with at most one 3-vertex, then $\omega'(v) \ge 10 - (\frac{3}{2} \times 5 + \frac{5}{4} + \frac{3}{4} + \frac{1}{3}) > 0$. Otherwise, the 4-face is incident with two 3-vertex, then v is incident with at least two 3-faces each of which is not incident with a 3-vertex by Lemma 2. Hence, $\omega'(v) \ge 10 - (\frac{3}{2} \times 4 + \frac{5}{4} \times 2 + \frac{3}{4} + \frac{1}{3}) > 0$. Suppose $f_4(v) = 2$. Then there exist only two cases that satisfies the condition of Theorem 1. We show these cases in Fig. 3 (2) and (3). In Fig. 3 (2), v is incident with at least four 3-faces each of which is adjacent to a 8^+-face. By R4, if there exists a 8^+-face adjacent to a 3-face, then 8^+-face sends $\frac{1}{4}$ to its adjacent 3-face, so each of the 3-face adjacent to a 8^+-face receives at most $\frac{3-\frac{1}{4}}{2} = \frac{11}{8}$ from the boundary vertices. There exist at most one 4-face incident with two 3-vertices in Fig. 3 (2). By Lemma 2, v is incident with at least one 3-face which is not incident with a 3-vertex, so $\omega'(v) \geq 10-(\frac{3}{2}+\frac{11}{8}\times 4+\frac{5}{4}+1+\frac{3}{4})=0$. In Fig. 3 (3), v is incident with at least four 3-faces each of which is adjacent to a 8^+-face. By Lemma 2, v is incident with at most one 4-face incident with two 3-vertices. If each of the two 4-faces is incident with at most one 3-vertex, then $\omega'(v) \geq 10-(\frac{3}{2}\times 2+\frac{11}{8}\times 4+\frac{3}{4}\times 2)=0$. Otherwise, v is incident with one 4-face which is incident with two 3-vertices, then there exist at least three 3-faces each of which is not incident with a 3-vertex by Lemma 2. Hence, $\omega'(v) \geq 10-(\frac{3}{2}\times 3+\frac{5}{4}\times 3+1+\frac{3}{4})=0$. Suppose $f_3(v)=5$. Then by the condition of Theorem 1, we have $f_{5^+}(v) \geq 1$, so $\omega'(v) \geq 10-(\frac{3}{2}\times 5+1\times 2+\frac{1}{3}\times 2) > 0$.

Case 2. $n_2(v)=1$. After transferring charge from v to 2-vertex, the remaining charge of v is $2\times 8-6-1=9$.

Case 2.1. Suppose the 2-vertex is incident with a 3-cycle. It is clear that $f_3(v) \leq 6$ and each of the 3-faces is not incident with a 3-vertex by Lemma 2. So v is incident with at most one 3-face that receives $\frac{3}{2}$ from v. If $f_3(v)=6$, then by the condition of Theorem 1, we know that $f_{6^+}(v) \geq 1$ or $f_{5^+}(v) \geq 2$, so $\omega'(v) \geq 9-(\frac{3}{2}+\frac{5}{4}\times 5) > 0$ by Lemma 6. Suppose $f_3(v)=5$. If $f_4(v)=3$, then there are at least two $(2^+, 4^+, 4^+, 8)$-faces between the three 4-faces by Lemma 2. Hence, $\omega' \geq 9-(\frac{3}{2}+\frac{5}{4}\times 4+1+\frac{3}{4}\times 2)=0$. If $f_4(v) \leq 2$, then we have $\omega'(v) \geq 9-(\frac{3}{2}+\frac{5}{4}\times 4+1\times 2+\frac{1}{3}) > 0$. Suppose $f_3(v)=4$. If $f_4(v)=4$, then there exist at least two $(2^+, 4^+, 4^+, 8)$-faces between the four 4-faces by Lemma 2. Hence, $\omega' \geq 9-(\frac{3}{2}+\frac{5}{4}\times 3+1\times 2+\frac{3}{4}\times 2) > 0$. If $f_4(v) \leq 3$, then $\omega'(v) \geq 9-(\frac{3}{2}+\frac{5}{4}\times 3+1\times 3+\frac{1}{3}) > 0$. If $f_3(v) \leq 3$, then $\omega'(v) \geq 9-(\frac{3}{2}+\frac{5}{4}\times 2+1\times 5)=0$.

Case 2.2. Suppose the 2-vertex is not incident with a 3-cycle. Then $f_3(v) \leq 6$. Suppose $f_3(v)=6$. Then the six 3-faces are consecutively adjacent and $f_{9^+}(v)=1$, so there exist at least four $(4^+, 4^+, 8)$-faces between the six 3-faces by Lemma 3. Consequently, $\omega'(v) \geq 9-(\frac{3}{2}\times 2+\frac{5}{4}\times 4+1\times 1) > 0$ by Lemma 6. Suppose $f_3(v)=5$. Then $f_{6^+}(v) \geq 1$ by the condition of Theorem 1. If $f_4(v)=2$, then another three the boundary vertices of each 4-faces are adjacent to v. So v is incident with at least two $(4^+, 4^+, 8)$-faces and one $(2^+, 4^+, 4^+, 8)$-face by Lemma 3. Hence, $\omega'(v) \geq 9-(\frac{3}{2}\times 3+\frac{5}{4}\times 2+1\times 1+\frac{3}{4}\times 1) > 0$. If $f_4(v)=1$, then v is incident with at least one $(4^+, 4^+, 8)$-face by Lemma 3. Hence, $\omega'(v) \geq 9-(\frac{3}{2}\times 4+\frac{5}{4}\times 1+1\times 1+\frac{1}{3}\times 2) > 0$. If $f_4(v)=0$, then $\omega'(v) \geq 9-(\frac{3}{2}\times 5+\frac{1}{3}\times 3) > 0$. Suppose $f_3(v)=4$. Then we have $f_4(v) \leq 3$ according to the condition of Theorem 1. If $f_4(v)=3$, then v is incident with at least two $(4^+, 4^+, 8)$-faces. Hence, $\omega'(v) \geq 9-(\frac{3}{2}\times 2+\frac{5}{4}\times 2+1\times 3+\frac{1}{3}\times 1) > 0$. If $f_4(v) \leq 2$, then $\omega'(v) \geq 9-(\frac{3}{2}\times 4+1\times 2+\frac{1}{3}\times 2) > 0$. Suppose $f_3(v)=3$. If v is incident with a 5^+-face, then $\omega'(v) \geq 9-(\frac{3}{2}\times 3+1\times 4+\frac{1}{3}) > 0$. Otherwise, $f_4(v)=5$, then v is incident with at least three $(2^+, 4^+, 4^+, 8)$-faces. Hence, $\omega'(v) \geq 9-(\frac{3}{2}\times 3+1\times 2+\frac{3}{4}\times 3) > 0$. If $f_3(v) \leq 2$, then $\omega'(v) \geq 9-(\frac{3}{2}\times 2+1\times 6)=0$.

Case 3. $n_2(v)=2$. Then $2\times 8-6-2=8$ and there are four cases where 2-vertices are located. We show these cases in Fig. 4. In Fig. 4(1), $\omega'(v) \geq 8-(\frac{5}{4}\times 8-\frac{9}{4}) > 0$ by Lemma 7. In Fig. 4 (2), $\omega'(v) \geq 8-[(\frac{5}{4}\times 3-\frac{9}{4})+(\frac{5}{4}\times 7-\frac{9}{4})]=0$. In Fig. 4 (3), $\omega'(v) \geq 8-[(\frac{5}{4}\times 4-\frac{9}{4})+(\frac{5}{4}\times 6-\frac{9}{4})]=0$. In Fig. 4 (4), $\omega'(v) \geq 8-2\times(\frac{5}{4}\times 5-\frac{9}{4})=0$ by Lemma 7.

Case 4. $n_2(v)=3$. Then $2\times 8-6-3=7$ and there are five cases where 2-vertices are located. We show these cases in Fig. 5. In Fig. 5(1), $\omega'(v) \geq 7-(\frac{5}{4}\times 7-\frac{9}{4}) > 0$

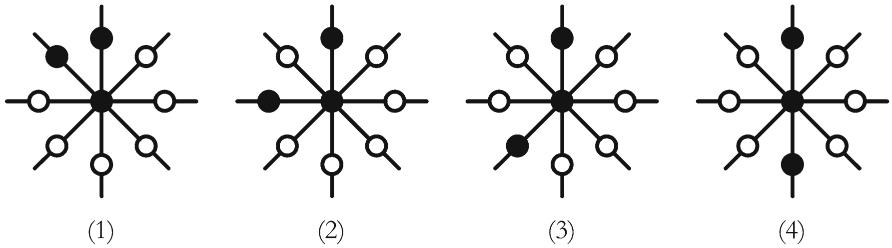

Fig. 4 $n_2(v) = 2$

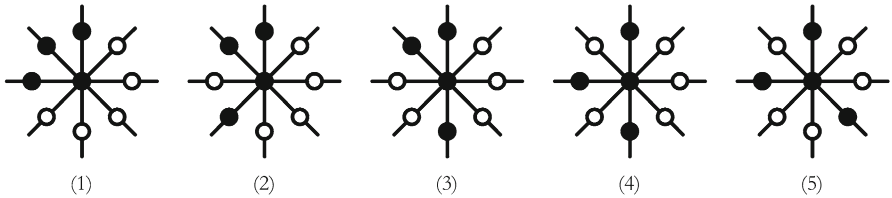

Fig. 5 $n_2(v) = 3$

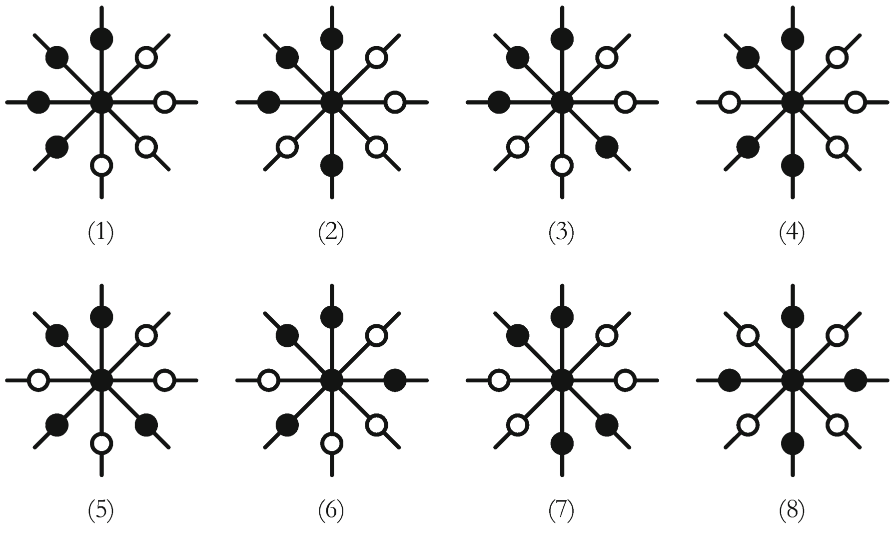

Fig. 6 $n_2(v) = 4$

by Lemma 7. In Fig. 5 (2), $\omega'(v) \geq 7 - [(\frac{5}{4} \times 3 - \frac{9}{4}) + (\frac{5}{4} \times 6 - \frac{9}{4})] > 0$. In Fig. 5 (3), $\omega'(v) \geq 7 - [(\frac{5}{4} \times 4 - \frac{9}{4}) + (\frac{5}{4} \times 5 - \frac{9}{4})] > 0$. In Fig. 5 (4), $\omega'(v) \geq 7 - [2 \times (\frac{5}{4} \times 3 - \frac{9}{4}) + (\frac{5}{4} \times 5 - \frac{9}{4})] = 0$. In Fig. 5 (5), $\omega'(v) \geq 7 - [2 \times (\frac{5}{4} \times 4 - \frac{9}{4}) + (\frac{5}{4} \times 3 - \frac{9}{4})] = 0$ by Lemma 7.

Case 5. $n_2(v) = 4$. Then $2 \times 8 - 6 - 4 = 6$ and there are eight cases where 2-vertices are located. We show these cases in Fig. 6 In Fig. 6(1), $\omega'(v) \geq 6 - (\frac{5}{4} \times 6 - \frac{9}{4}) > 0$ by Lemma 7. In Fig. 5 (2) and (4), $\omega'(v) \geq 6 - [(\frac{5}{4} \times 5 - \frac{9}{4}) + (\frac{5}{4} \times 3 - \frac{9}{4})] > 0$. In Fig. 6 (3) and (7), $\omega'(v) \geq 6 - 2 \times (\frac{5}{4} \times 4 - \frac{9}{4}) > 0$. In Fig. 6 (5) and (6), $\omega'(v) \geq$

$6-[2\times(\frac{5}{4}\times3-\frac{9}{4})+(\frac{5}{4}\times4-\frac{9}{4})]>0$. In Fig. 6 (8), $\omega'(v)\geq6-4\times(\frac{5}{4}\times3-\frac{9}{4})=0$ by Lemma 7.

Case 6. $n_2(v)\geq5$. Suppose $n_2(v)=5$. Then $2\times8-6-5=5$ and $f_3(v)\leq2$. If $f_3(v)=2$, then $f_{6^+}(v)\geq4$ by Lemma 2. Consequently, $\omega'(v)\geq5-\frac{3}{2}\times2-1\times2=0$ by Lemma 6. Suppose $f_3(v)=1$. Then $f_{6^+}(v)\geq3$ and $f_4(v)\leq4$. If $f_4(v)=4$, then each of the four 4-faces is a $(2^+,4^+,4^+,8)$-face. Hence, $\omega'(v)\geq5-(\frac{3}{2}\times1+\frac{3}{4}\times4)>0$. If $f_4(v)\leq3$, then $\omega'(v)\geq5-(\frac{3}{2}\times1+1\times3+\frac{1}{3})>0$. Suppose $f_3(v)=0$. Then $f_{6^+}(v)\geq2$. If $f_4(v)=6$, then each of the six 4-faces is a $(2^+,4^+,4^+,8)$-face. So $\omega'(v)\geq5-\frac{3}{4}\times6>0$. If $f_4(v)=5$, then v is incident with at least four $(2^+,4^+,4^+,8)$-faces. So $\omega'(v)\geq5-(1\times1+\frac{3}{4}\times4+\frac{1}{3}\times1)>0$. If $f_4(v)\leq4$, then $\omega'(v)\geq5-(1\times4+\frac{1}{3}\times2)>0$. Suppose $n_2(v)=6$. Then $f_3(v)\leq1$ and $2\times8-6-6=4$. If $f_3(v)=1$, then $f_4(v)\leq2$ and $f_{6^+}(v)\geq5$. Hence, $\omega'(v)\geq4-(\frac{3}{2}+1\times2)>0$. If $f_3(v)=0$, then $f_{6^+}(v)\geq4$. So $\omega'(v)\geq4-1\times4=0$. Suppose $n_2(v)\geq7$. Then $f_3(v)=0$ and $f_{6^+}(v)\geq6$, so $\omega'(v)\geq10-8-1\times2=0$.

In summary, we prove that $\omega'(x)\geq0$ for each $x\in V\cup F$. Therefore, $\sum_{x\in V\cup F}\omega'(x)\geq0$. We get a contradiction and accomplish the proof of Theorem 1.

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Data availability Enquiries about data availability should be directed to the authors.

Declarations

Competing interests The authors declare no competing interests.

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