Journal of Combinatorial Optimization (2023) 45:75
https://doi.org/10.1007/s10878-023-01008-7

Distance magic labeling of the halved folded n-cube

Yi Tian[1,2] · Na Kang[3] · Weili Wu[4] · Ding-Zhu Du[4] · Suogang Gao[1,5]

Accepted: 16 February 2023 / Published online: 7 March 2023

Abstract

Hypercube is an important structure for computer networks. The distance plays an important role in its applications. In this paper, we study a magic labeling of the halved folded n-cube which is a variation of the n-cube. This labeling is determined by the distance. Let G be a finite undirected simple connected graph with vertex set $V(G)$, distance function ∂ and diameter d. Let $D \subseteq \{0, 1, \ldots, d\}$ be a set of distances. A bijection $l : V(G) \to \{1, 2, \ldots, |V(G)|\}$ is called a D-magic labeling of G whenever $\sum_{x \in G_D(v)} l(x)$ is a constant for any vertex $v \in V(G)$, where $G_D(v) = \{x \in V(G) : \partial(x, v) \in D\}$. A $\{1\}$-magic labeling is also called a distance magic labeling. We show that the halved folded n-cube has a distance magic labeling (resp. a $\{0, 1\}$-magic labeling) if and only if $n = 16q^2$(resp. $n = 16q^2 + 16q + 6$), where q is a positive integer.

Keywords D-magic labeling · Distance-regular graph · Halved folded n-cube · Network · Incomplete tournament

✉ Suogang Gao
sggaomail@163.com

1 School of Mathematical Sciences, Hebei Normal University, Shijiazhuang 050024, People's Republic of China

2 School of Big Data Science, Hebei Finance University, Baoding 071051, People's Republic of China

3 School of Mathematics and Science, Hebei GEO University, Shijiazhuang 050024, People's Republic of China

4 Department of Computer Science, University of Texas at Dallas, Richardson, TX 75080, USA

5 Hebei International Joint Research Center for Mathematics and Interdisciplinary Science, Shijiazhuang 050024, People's Republic of China

1 Introduction

Hypercube is an important structure for computer networks (Bettayeb 1995). Many combinatorial structural properties are studied in order to enhance its various applications (Day and Al-Ayyoub 1997; Hsieh and Chang 2012; Lin et al. 2016; Zhao et al. 2016). Especially, the distance plays an important role (Bose et al. 1995). In this paper, we study a magic labeling of the halved folded n-cube which is a variation of the n-cube. This labeling is determined by the distance. And the magic labeling has its applications in incomplete tournament and in efficient addressing systems in communication networks, ruler models and radar pulse codes (Anholcer et al. 2016, 2015; Bloom and Golomb 1977, 1978; Chebotarev and Agaev 2013; Prajeesh et al. 2019).

Let G be a finite undirected simple connected graph with vertex set $V(G)$, distance function ∂ and diameter d. In the early 1960 s, Sedláček (1963) introduced the notion of magic labeling of G. The concept is motivated by the construction of magic squares (Anholcer et al. 2016; Prajeesh et al. 2019). In 1994, Vilfred (1994) introduced the distance magic labeling as follows.

Definition 1 A bijection $l : V(G) \to \{1, 2, \ldots, |V(G)|\}$ is called a *distance magic labeling* of G if $\sum\limits_{x \in N(v)} l(x)$ is a constant for every vertex $v \in V(G)$, where $N(v) = \{x \in V(G) : \partial(x, v) = 1\}$.

In 2013, O'Neal and Slater (2013) generalized the distance magic labeling to the D-magic labeling.

Definition 2 For a graph G and a set of distances $D \subseteq \{0, 1, \ldots, d\}$, the set $G_D(v) = \{x \in V(G) : \partial(x, v) \in D\}$ is called *D-neighborhood* of v. By a *D-magic labeling* of G, we mean a bijection $l : V(G) \to \{1, 2, \ldots, |V(G)|\}$ with the property that there exists a constant k such that $w(v) = \sum\limits_{x \in G_D(v)} l(x) = k$ for any vertex $v \in V(G)$, where $w(v)$ is the weight of v. The graph G admitting a D-magic labeling is called a *D-magic graph*.

Obviously, a $\{1\}$-magic labeling of G is precisely a distance magic labeling of G.

When studying a D-magic labeling, distance-regular graphs are a natural class of graphs to consider. For the definition of a distance-regular graph, we refer to Section 2. Until now, D-magic labelings of some distance-regular graphs have been studied. Simanjuntak and Anuwiksa (2020) characterized strongly regular graphs which are D-magic graphs, for all possible distances sets D. Gregor and Kovář (2013) proved that the n-cube with $n \equiv 2 \pmod 4$ is a $\{j\}$-magic graph for every odd j, where $1 \leq j \leq n$. Later, Cichacz et al. (2016) proved that if the n-cube is a $\{1\}$-magic graph, then $n \equiv 2 \pmod 4$. Anuwiksa et al. (2019) obtained some special distances sets D for which the n-cube with $n \equiv 2 \pmod 4$ has D-magic labelings. They also presented the necessary and sufficient condition for the n-cube to be a $\{0, 1\}$-magic graph. Tian et al. (2021) showed the necessary and sufficient condition for the folded n-cube to be a $\{1\}$-magic graph and a $\{0, 1\}$-magic graph, respectively. Recently, Miklavič and Šparl (2021) provided a sufficient condition for a Hamming graph to be a $\{1\}$-magic graph and classified $\{1\}$-magic folded n-cubes by showing the necessary and sufficient condition for the folded n-cube to be a $\{1\}$-magic graph.

In this paper, we focus on a halved folded n-cube with even $n \geq 8$ which is a distance-regular graph. For its definition, we refer to Section 2. We show the necessary and sufficient condition for the halved folded n-cube to be a $\{1\}$-magic graph and a $\{0, 1\}$-magic graph, respectively. The difference between this paper and other references on the D-magic labeling is that one has to choose a non-square matrix such that the corresponding map is bijective, where $D = \{1\}$ or $D = \{0, 1\}$; see Notation 1. The main result is as follows.

Theorem 1 *For even $n \geq 8$, let $\frac{1}{2}FQ_{n-1}$ denote a halved folded n-cube. Then the following* (i), (ii) hold:

(i) $\frac{1}{2}FQ_{n-1}$ has a distance magic labeling if and only if $n = 16q^2$, where q is a positive integer;
(ii) $\frac{1}{2}FQ_{n-1}$ has a $\{0, 1\}$-magic labeling if and only if $n = 16q^2 + 16q + 6$, where q is a positive integer.

The paper is organized as follows. Section 2 gives some definitions, basic notations and some facts used in this paper. Section 3 presents the proof of Theorem 1.

2 Preliminaries

In this section, we review some definitions, basic notations and some facts.

Recall that G is a finite undirected simple connected graph with vertex set $V(G)$, distance function ∂ and diameter d. For $v \in V(G)$ and $i \in \{0, 1, \dots, d\}$, let $G_i(v) = \{x \in V(G) : \partial(x, v) = i\}$. We define $G_{-1}(v) = \emptyset$ and $G_{d+1}(v) = \emptyset$. Particularly, $N(v) := G_1(v)$ is called the *neighborhood* of v, that is, two vertices u and v are *adjacent* whenever $u \in N(v)$. Furthermore, $|N(v)|$ is called the *degree* of v. The graph G is *regular* if $|N(v)|$ is a constant for every $v \in V(G)$. By $N[v] := N(v) \cup \{v\}$ we denote the *closed neighborhood* of v.

The graph G is called a *distance-regular graph* when for all $i \in \{0, 1, \dots, d\}$ there are constants c_i, a_i, b_i such that for any vertices x and y with $\partial(x, y) = i$, we have $|G_{i-1}(x) \cap G_1(y)| = c_i$, $|G_i(x) \cap G_1(y)| = a_i$ and $|G_{i+1}(x) \cap G_1(y)| = b_i$. Observe that G is regular with degree $b_0 = a_i + b_i + c_i$ and $|G_i(x)| = \frac{b_0 b_1 \dots b_{i-1}}{c_1 c_2 \dots c_i}$ which is determined by i for $0 \leq i \leq d$ (see Brouwer et al. 1989, p. 127). If $d = 2$, then G is a strongly regular graph. For more information on distance-regular graphs, we refer to Brouwer et al. (1989).

Now we recall the definitions of a folded n-cube and a halved folded n-cube with even $n \geq 2$.

Let $\mathbb{F}_2^{n-1}$ be an $(n-1)$-dimensional column vector space over $\mathbb{F}_2$. Let $\boldsymbol{e_i}$ $(1 \leq i \leq n-1)$ denote the vector in $\mathbb{F}_2^{n-1}$ such that the i-th component is 1 and the others are 0, and let $\mathbf{1} = \boldsymbol{e_1} + \cdots + \boldsymbol{e_{n-1}}$.

Definition 3 Simó and Yebra (1997) The *folded n-cube*, denoted by FQ_{n-1}, is a graph with vertex set consisting of all vectors in $\mathbb{F}_2^{n-1}$; two vertices $\boldsymbol{u}$ and $\boldsymbol{v}$ are adjacent whenever $\boldsymbol{u} \in \{\boldsymbol{v} + \boldsymbol{e_i} : 1 \leq i \leq n-1\} \cup \{\boldsymbol{v} + \mathbf{1}\}$.

By (van Dam et al. 2016, p. 23), FQ_{n-1} is a bipartite graph for even n and its halved graph is the *halved folded n-cube*. For the definition of the halved graph, we refer to (Brouwer et al. 1989, p. 25). Inspired by (Brouwer et al. 1989, pp. 264–265), the halved folded n-cube can also be described in $\mathbb{F}_2^{n-1}$ as follows.

Definition 4 Let $n \geq 2$ be an even number. The *halved folded n-cube*, denoted by $\frac{1}{2}FQ_{n-1}$, is a graph with vertex set consisting of all vectors in $\mathbb{F}_2^{n-1}$ that contains an even number of 1's; two vertices $\boldsymbol{u}$ and $\boldsymbol{v}$ are adjacent whenever $\boldsymbol{u} \in \{\boldsymbol{v} + \boldsymbol{e_i} + \boldsymbol{e_j} : 1 \leq i < j \leq n-1\} \cup \{\boldsymbol{v} + \boldsymbol{e_i} + \boldsymbol{1} : i = 1, \ldots, n-1\}$.

By Definition 4, we have $|V(\frac{1}{2}FQ_{n-1})| = 2^{n-2}$ and

$$\begin{aligned} N(\boldsymbol{0}) =& \{\boldsymbol{e_1} + \boldsymbol{e_2}, \ldots, \boldsymbol{e_1} + \boldsymbol{e_{n-1}}, \boldsymbol{e_2} + \boldsymbol{e_3}, \ldots, \boldsymbol{e_2} + \boldsymbol{e_{n-1}}, \ldots, \\ & \boldsymbol{e_{n-2}} + \boldsymbol{e_{n-1}}, \boldsymbol{e_1} + \boldsymbol{1}, \ldots, \boldsymbol{e_{n-1}} + \boldsymbol{1}\}, \end{aligned}$$

where $\boldsymbol{0} := (0, \ldots, 0)^{\mathrm{T}} \in V(\frac{1}{2}FQ_{n-1})$. In what follows, we use $\boldsymbol{v} \oplus N(\boldsymbol{0})$ to denote the set of all vectors $\boldsymbol{v} + \boldsymbol{u}$, where $\boldsymbol{u}$ runs through $N(\boldsymbol{0})$. Then by Definition 4, for any $\boldsymbol{v} \in V(\frac{1}{2}FQ_{n-1})$ we obtain

$$N(\boldsymbol{v}) = \boldsymbol{v} \oplus N(\boldsymbol{0}), \tag{1}$$

and thus

$$N[\boldsymbol{v}] = \boldsymbol{v} \oplus N[\boldsymbol{0}]. \tag{2}$$

By (Brouwer et al. 1989, p. 265), $\frac{1}{2}FQ_{n-1}$ with $n \geq 6$ is a distance-regular graph of diameter $d = \lfloor \frac{n}{4} \rfloor$ and its eigenvalues are $\theta_j = 2(\frac{n}{2} - 2j)^2 - \frac{n}{2}$ $(0 \leq j \leq d)$. Particularly, $\frac{1}{2}FQ_{n-1}$ with $n = 6$ is the complete graph K_{16} which is obviously $\{0, 1\}$-magic, but not $\{1\}$-magic (Miller et al. 2003). So we assume even $n \geq 8$ for the rest of this paper.

Inspired by Anuwiksa et al. (2019); Gregor and Kovář (2013), we give the following definitions.

Definition 5 A subset $A \subseteq \mathbb{F}_2^{n-2}$ is said to be *balanced* if for every $i \in \{1, 2, \ldots, n-2\}$

$$|\{\boldsymbol{v} \in A : \boldsymbol{v}_i = 1\}| = \frac{|A|}{2},$$

where $\boldsymbol{v}_i$ denotes the i-th component of $\boldsymbol{v}$.

Note that balance is invariant under translation. It means that a set $A \subseteq \mathbb{F}_2^{n-2}$ is balanced if and only if $\boldsymbol{u} \oplus A$ is balanced, where $\boldsymbol{u} \in \mathbb{F}_2^{n-2}$.

Definition 6 Suppose H is an $(n-2) \times m$ matrix with entries in $\mathbb{F}_2$. Let $r_i(H)$ denote the number of 1's in the i-th row of H for $i \in \{1, 2, \ldots, n-2\}$. The matrix H is said to be *balanced* whenever the set of its columns is balanced, that is, $r_i(H) = \frac{m}{2}$ for every $i \in \{1, 2, \ldots, n-2\}$.

Definition 7 For graph $\frac{1}{2}FQ_{n-1}$, let $D \subseteq \{0, 1, \ldots, \lfloor \frac{n}{4} \rfloor\}$ be a set of distances. A bijection $f : V(\frac{1}{2}FQ_{n-1}) \to \mathbb{F}_2^{n-2}$ is said to be *D-neighbor balanced* if the set $f(G_D(\boldsymbol{v})) = \{f(\boldsymbol{u}) : \boldsymbol{u} \in G_D(\boldsymbol{v})\}$ is balanced for every $\boldsymbol{v} \in V(\frac{1}{2}FQ_{n-1})$. If $D = \{1\}$ (resp. $D = \{0, 1\}$), then a D-neighbor balanced bijection is also called *neighbor balanced* (resp. *closed neighbor balanced*).

For every regular graph, we may equivalently consider labelings with labels starting from 0 instead of 1 (Gregor and Kovář (2013)). In fact, we work with labels (as well as with the image of vertices of $\frac{1}{2}FQ_{n-1}$ under the above map f) in the $(n-2)$-dimensional vector space $\mathbb{F}_2^{n-2}$ over $\mathbb{F}_2$, that is, in their binary representation. Using the arguments similar to Propositions 2.1 and 3.1 for the n-cube in Gregor and Kovář (2013), we get the following lemma for $\frac{1}{2}FQ_{n-1}$.

Lemma 1 *For a set of distances D of $\frac{1}{2}FQ_{n-1}$, let $f : V(\frac{1}{2}FQ_{n-1}) \to \mathbb{F}_2^{n-2}$ be a bijection. If f is D-neighbor balanced, then f is a D-magic labeling of $\frac{1}{2}FQ_{n-1}$.*

Proof For every vertex $\boldsymbol{v} \in V(\frac{1}{2}FQ_{n-1})$, we obtain (with the arithmetics in $\mathbb{N}$)

$$\begin{aligned}\sum_{\boldsymbol{u} \in (\frac{1}{2}FQ_{n-1})_D(\boldsymbol{v})} f(\boldsymbol{u}) &= \sum_{i=1}^{n-2} |\{f(\boldsymbol{u}) : (f(\boldsymbol{u}))_i = 1\}| 2^{i-1} \\ &= \frac{|f((\frac{1}{2}FQ_{n-1})_D(\boldsymbol{v}))|}{2}(2^{n-2}-1) \qquad (by Definition\ 5) \\ &= \frac{|(\frac{1}{2}FQ_{n-1})_D(\boldsymbol{v})|}{2}(2^{n-2}-1).\end{aligned}$$

Since $\frac{1}{2}FQ_{n-1}$ is a distance-regular graph, $|(\frac{1}{2}FQ_{n-1})_D(\boldsymbol{v})|$ is independent of the choice of $\boldsymbol{v}$. Therefore, f is a D-magic labeling of $\frac{1}{2}FQ_{n-1}$. □

From now on, we adopt the following notational convention.

Notation 1 *Let N be an $(n-2)\times(n-2)$ matrix with entries in $\mathbb{F}_2$ and let $M = \begin{pmatrix} \boldsymbol{0} & N \end{pmatrix}$ be an $(n-2) \times (n-1)$ matrix with entries in $\mathbb{F}_2$. For $\frac{1}{2}FQ_{n-1}$, we define a map*

$$f : V(\frac{1}{2}FQ_{n-1}) \to \mathbb{F}_2^{n-2}$$

by $f(\boldsymbol{v}) = M\boldsymbol{v}$ for every $\boldsymbol{v} \in V(\frac{1}{2}FQ_{n-1})$. For $\boldsymbol{v} \in V(\frac{1}{2}FQ_{n-1})$, let v_1 be the first component of $\boldsymbol{v}$ and write $\boldsymbol{v} := \begin{pmatrix} v_1 \\ \tilde{\boldsymbol{v}} \end{pmatrix}$.

Lemma 2 *With reference to Notation 1, f is a bijection if and only if the rank of M is $n-2$.*

Proof Suppose f is a bijection. Then for any $\boldsymbol{u}, \boldsymbol{v} \in V(\frac{1}{2}FQ_{n-1})$ if $f(\boldsymbol{u}) = f(\boldsymbol{v})$, then $\boldsymbol{u} = \boldsymbol{v}$. It follows that the system of equations $M(\boldsymbol{u} - \boldsymbol{v}) = \boldsymbol{0}$ has only the trivial solution. Then the system of equations $N(\tilde{\boldsymbol{u}} - \tilde{\boldsymbol{v}}) = \mathbf{0}$ has only the trivial solution. Thus the rank of N is $n-2$, and hence the rank of M is $n-2$.

Conversely, since $|V(\frac{1}{2}FQ_{n-1})| = |\mathbb{F}_2^{n-2}| = 2^{n-2}$, it suffices to show that f is injective. Suppose $f(\boldsymbol{u}) = f(\boldsymbol{v})$ for $\boldsymbol{u}, \boldsymbol{v} \in V(\frac{1}{2}FQ_{n-1})$. Then $M(\boldsymbol{u} - \boldsymbol{v}) = \mathbf{0}$. It follows that $N(\tilde{\boldsymbol{u}} - \tilde{\boldsymbol{v}}) = \mathbf{0}$. Since the rank of M is $n-2$, N is invertible. Thus $\tilde{\boldsymbol{u}} = \tilde{\boldsymbol{v}}$. Combining this with the fact that the number of 1's in both $\boldsymbol{u}$ and $\boldsymbol{v}$ is even, we obtain $u_1 = v_1$. Therefore, $\boldsymbol{u} = \boldsymbol{v}$, and hence f is injective. □

3 Proof of Theorem 1

In this section, we give the necessary and sufficient condition for $\frac{1}{2}FQ_{n-1}$ to be a $\{1\}$-magic graph and a $\{0, 1\}$-magic graph, respectively.

Let B be the $(n-1) \times \frac{n(n-1)}{2}$ matrix

$$B = \left(\boldsymbol{e_1} + \boldsymbol{e_2}, \ldots, \boldsymbol{e_1} + \boldsymbol{e_{n-1}}, \boldsymbol{e_2} + \boldsymbol{e_3}, \ldots, \boldsymbol{e_2} + \boldsymbol{e_{n-1}}, \ldots, \boldsymbol{e_1} + \mathbf{1}, \ldots, \boldsymbol{e_{n-1}} + \mathbf{1}\right) \tag{3}$$

and let $B^* = \begin{pmatrix} \mathbf{0} & B \end{pmatrix}$, where $\mathbf{0} := (0, \ldots, 0)^{\mathrm{T}} \in V(\frac{1}{2}FQ_{n-1})$. Obviously, the above B (resp. B^*) corresponds to the set $N(\mathbf{0})$ (resp. $N[\mathbf{0}]$). Recall that $r_i(H)$ denotes the number of 1's in the i-th row of H for $i \in \{1, 2, \ldots, n-2\}$, where H is an $(n-2) \times m$ matrix with entries in $\mathbb{F}_2$.

Now we show the following lemma.

Lemma 3 *With reference to Notation 1, let B, B^* and r_i be as above. Suppose $r_i(M) = t$ for some $i \in \{1, \ldots, n-2\}$. Then*

$$r_i(MB^*) = r_i(MB) = \begin{cases} (t+1)(n-1-t) & \text{if } t \text{ is odd,} \\ (n-t)t & \text{otherwise.} \end{cases}$$

Proof Since $B^* = \begin{pmatrix} \mathbf{0} & B \end{pmatrix}$, we have $MB^* = \begin{pmatrix} \mathbf{0} & MB \end{pmatrix}$. Then $r_i(MB^*) = r_i(MB)$ for every $i \in \{1, \ldots, n-2\}$.

Suppose $r_i(M) = t$ for some $i \in \{1, \ldots, n-2\}$. Let

$$M = \begin{bmatrix} 0 & a_{11} & a_{12} & \cdots & a_{1,n-2} \\ 0 & a_{21} & a_{22} & \cdots & a_{2,n-2} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & a_{n-2,1} & a_{n-2,2} & \cdots & a_{n-2,n-2} \end{bmatrix},$$

where $a_{sj} \in \mathbb{F}_2$ for any $s \in \{1, \ldots, n-2\}$ and any $j \in \{1, \ldots, n-2\}$. Clearly, the i-th row of MB is

$$\begin{aligned} &(a_{i1}, a_{i2}, \ldots, a_{i,n-2}, a_{i1} + a_{i2}, a_{i1} + a_{i3}, \ldots, a_{i1} + a_{i,n-2}, a_{i2} + a_{i3}, \ldots, \\ &a_{i2} + a_{i,n-2}, \ldots, a_{i,n-3} + a_{i,n-2}, a_{i1} + \cdots + a_{i,n-2}, a_{i2} + \cdots + a_{i,n-2}, \\ &a_{i1} + a_{i3} + \cdots + a_{i,n-2}, \ldots, a_{i1} + \cdots + a_{i,n-3}). \end{aligned}$$

We now compute $r_i(MB)$ for the fixed i. Since $r_i(M) = t$, we have that the number of 1's in $(a_{i1}, a_{i2}, \ldots, a_{i,n-2})$ is t; the number of 1's in $(a_{i1}+a_{i2}, a_{i1}+a_{i3}, \ldots, a_{i1}+a_{i,n-2}, a_{i2}+a_{i3}, \ldots, a_{i2}+a_{i,n-2}, \ldots, a_{i,n-3}+a_{i,n-2})$ is

$$C_t^1 C_{n-2-t}^1 = t(n-2-t);$$

$a_{i1} + \cdots + a_{i,n-2} = 1$ if t is odd and $a_{i1} + \cdots + a_{i,n-2} = 0$ otherwise; the number of 1's in $(a_{i2} + \cdots + a_{i,n-2}, a_{i1} + a_{i3} + \cdots + a_{i,n-2}, \ldots, a_{i1} + \cdots + a_{i,n-3})$ is $n-2-t$ if t is odd and t otherwise. Therefore, if t is odd, then

$$r_i(MB) = t + t(n-2-t) + 1 + n - 2 - t = (t+1)(n-1-t);$$

if t is even, then $r_i(MB) = t + t(n-2-t) + 0 + t = (n-t)t$.

This completes the proof of the lemma. □

3.1 Proof of Theorem 1 (i)

In this subsection, we shall present the necessary and sufficient condition for $\frac{1}{2}FQ_{n-1}$ to be a $\{1\}$-magic graph. We first give the following lemma. It enables us to determine a neighbor balanced bijection by constructing the appropriate matrix.

Lemma 4 *With reference to Notation 1, the following statements are equivalent.*

(i) *f is neighbor balanced.*
(ii) *$n = 16q^2$, the rank of M is $16q^2 - 2$ and $r_i(M) \in \{8q^2 + 2q - 1, 8q^2 - 2q - 1, 8q^2 + 2q, 8q^2 - 2q\}$ for any given $i \in \{1, \ldots, n-2\}$, where q is a positive integer.*

Moreover, if (i)–(ii) hold, then f is a $\{1\}$-magic labeling of $\frac{1}{2}FQ_{n-1}$.

Proof By Notation 1 and (1), for any $\boldsymbol{v} \in V(\frac{1}{2}FQ^{n-1})$, we obtain

$$f(N(\boldsymbol{v})) = M\boldsymbol{v} \oplus MN(\mathbf{0}). \tag{4}$$

Combining (4) with Definition 7, we know that f is neighbor balanced if and only if the set $M\boldsymbol{v} \oplus MN(\mathbf{0})$ is balanced for every $\boldsymbol{v} \in V(\frac{1}{2}FQ^{n-1})$. Recall that the set $M\boldsymbol{v} \oplus MN(\mathbf{0})$ is balanced if and only if the set $MN(\mathbf{0})$ is balanced. Furthermore, note that the set $MN(\mathbf{0})$ is precisely the set of columns of the matrix MB, where B is from (3). It means that the set $MN(\mathbf{0})$ is balanced if and only if the matrix MB is balanced. Thus f is neighbor balanced if and only if the matrix MB is balanced.

$(i) \Longrightarrow (ii)$ Since f is neighbor balanced, f is a bijection. It follows from Lemma 2 that the rank of M is $n-2$. Fix any $i \in \{1, \ldots, n-2\}$, we assume $r_i(M) = t$. Obviously, $t > 0$. Next we show (ii) holds according to the parity of t.

Case (i) t is odd.

By Lemma 3, $r_i(MB) = (n-1-t)(t+1)$. By Definition 5 and since the $(n-2) \times \frac{n(n-1)}{2}$ matrix MB is balanced, we have $(n-1-t)(t+1) = \frac{n(n-1)}{4}$. Now we let $t = 2p - 1$ ($p \in \mathbb{Z}^+$) since t is odd. Then we have

$$n \pm \sqrt{n} = 4p. \tag{5}$$

Since both $4p$ and n are even, $\sqrt{n}$ is even. Now we let $\sqrt{n} = 2r$, where $r \geq 2$ as $n \geq 8$. Substituting it into (5), we obtain

$$2r^2 \pm r - 2p = 0. \tag{6}$$

By (6), r is even. Let $r = 2q$, where q is a positive integer. Then $n = (2r)^2 = (4q)^2 = 16q^2$, and hence the rank of M is $16q^2 - 2$. By (6), we have $p = 4q^2 \pm q$. Then $t = 2p - 1 = 2(4q^2 \pm q) - 1 = 8q^2 \pm 2q - 1$.

Case (ii) t is even.

By Lemma 3, $r_i(MB) = (n-t)t$. By Definition 5 and since the $(n-2) \times \frac{n(n-1)}{2}$ matrix MB is balanced, we have $(n-t)t = \frac{n(n-1)}{4}$. Now we let $t = 2p$ since t is even, where p is a positive integer. Then we obtain $n \pm \sqrt{n} = 4p$. By using arguments similar to the proof of case (i) above, we have $n = 16q^2$, the rank of M is $16q^2 - 2$ and $p = 4q^2 \pm q$, where q is a positive integer. Then $t = 8q^2 \pm 2q$.

By the arguments above, (ii) holds.

$(ii) \Longrightarrow (i)$ By Lemma 2 and since the rank of M is $n-2$, f is a bijection. To prove that f is neighbor balanced, it suffices to show the $(n-2) \times \frac{n(n-1)}{2}$ matrix MB is balanced. To do this, by Definition 6 and since $n = 16q^2$, it suffices to show

$$r_i(MB) = \frac{1}{2}\frac{n(n-1)}{2} = 64q^4 - 4q^2 \tag{7}$$

for every $i \in \{1, \ldots, n-2\}$.

Fix any $i \in \{1, \ldots, n-2\}$, we assume $r_i(M) = t$. Next we prove that (7) holds according to the parity of t.

Case (i) t is odd.

By Lemma 3, we obtain $r_i(MB) = (t+1)(n-1-t)$. It is easy to see that (7) holds if $t \in \{8q^2 + 2q - 1,\ 8q^2 - 2q - 1\}$.

Case (ii) t is even.

By Lemma 3, we obtain $r_i(MB) = t(n-t)$. It is easy to see that (7) holds if $t \in \{8q^2 + 2q,\ 8q^2 - 2q\}$.

By the arguments above, (7) holds, and hence (i) holds.

If (i)–(ii) hold, then by Lemma 1, f is a $\{1\}$-magic labeling of $\frac{1}{2}FQ_{n-1}$. □

We list the following lemma which will be used in the proof of Theorem 1 (i).

Lemma 5 *Simanjuntak and Anuwiksa (2020) If G is a regular graph admitting a $\{1\}$-magic labeling, then 0 is an eigenvalue of G.*

Now we prove Theorem 1 (i).

Proof of Theorem 1 (i) Suppose that $\frac{1}{2}FQ_{n-1}$ has a $\{1\}$-magic labeling. By Lemma 5, 0 is an eigenvalue of $\frac{1}{2}FQ_{n-1}$, that is, there exists $j \in \{0, 1, \ldots, \lfloor\frac{n}{4}\rfloor\}$ such that $\theta_j = 2(\frac{n}{2} - 2j)^2 - \frac{n}{2} = 0$. Thus

$$n = 4(\frac{n}{2} - 2j)^2. \tag{8}$$

We claim that $\frac{n}{2} - 2j$ is even. In fact, assume $\frac{n}{2} - 2j = p$, where p is an integer. Substituting it into (8), we have $n = 4p^2$. It follows that $\frac{n}{2} - 2j = 2p^2 - 2j$. So the claim holds. Moreover, by (8) and since $n \geq 8$, we have $\frac{n}{2} - 2j \geq 2$. Let $\frac{n}{2} - 2j = 2q$, where q is a positive integer. It follows from (8) that $n = 4(2q)^2 = 16q^2$.

Conversely, since $n = 16q^2$, where q is a positive integer, we construct a $(16q^2 - 2) \times (16q^2 - 1)$ matrix M with entries in $\mathbb{F}_2$ as follows:

$$\begin{array}{c} \\ 8q^2-2q-2 \\ 8q^2-2q-2 \\ 2q \\ 2q \\ 2 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 8q^2-2q-2 & 8q^2-2q-2 & 2q & 2q & 2 \end{array} \\ \begin{bmatrix} \mathbf{0} & I & J & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & J & I & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & J & \mathbf{0} & I & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & J & \mathbf{0} & \mathbf{0} & I & \mathbf{0} \\ \mathbf{0} & J & \mathbf{0} & \mathbf{0} & \mathbf{0} & I \end{bmatrix} \end{array},$$

where I denotes the identity matrix and J (resp. $\mathbf{0}$) denotes the all 1's (resp. 0's) matrix. Obviously, $r_i(M) = 8q^2 - 2q - 1$ for every $i \in \{1, \ldots, n-2\}$. Moreover, by elementary row operations over $\mathbb{F}_2$, M can be transformed to the following matrix

$$\begin{bmatrix} \mathbf{0} & I & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & J & I & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & J & \mathbf{0} & I & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & J & \mathbf{0} & \mathbf{0} & I & \mathbf{0} \\ \mathbf{0} & J & \mathbf{0} & \mathbf{0} & \mathbf{0} & I \end{bmatrix},$$

which has the same block representation as M and its rank is $16q^2 - 2$. So the rank of M is $16q^2 - 2$. Let $f : V(\frac{1}{2}FQ_{n-1}) \to \mathbb{F}_2^{n-2}$ be given by $f(\boldsymbol{v}) = M\boldsymbol{v}$. By Lemma 4, f is a $\{1\}$-magic labeling of $\frac{1}{2}FQ_{n-1}$. Therefore, $\frac{1}{2}FQ_{n-1}$ has a $\{1\}$-magic labeling. □

3.2 Proof of Theorem 1 (ii)

In this subsection, we will present the necessary and sufficient condition for the halved folded n-cube to be a $\{0, 1\}$-magic graph. We first give the following lemma. It enables us to determine a closed neighbor balanced bijection by constructing the appropriate matrix.

Lemma 6 *With reference to Notation 1, the following statements are equivalent.*

(i) f is closed neighbor balanced.

(ii) $n = 16q^2 + 16q + 6$, *the rank of* M *is* $16q^2 + 16q + 4$ *and* $r_i(M) \in \{8q^2 + 10q + 3,\ 8q^2 + 6q + 1,\ 8q^2 + 10q + 4,\ 8q^2 + 6q + 2\}$ *for any given* $i \in \{1, \ldots, n-2\}$, *where* q *is a positive integer.*

Moreover, if (i)–(ii) hold, then f *is a* $\{0, 1\}$*-magic labeling of* $\frac{1}{2}FQ_{n-1}$.

Proof By Notation 1 and (2), for any vertex $\boldsymbol{v} \in V(\frac{1}{2}FQ_{n-1})$, we obtain

$$f(N[\boldsymbol{v}]) = MN[\boldsymbol{v}] = M\boldsymbol{v} \oplus MN[\mathbf{0}]. \tag{9}$$

Combining (9) with Definition 7, we know that f is closed neighbor balanced if and only if the set $M\boldsymbol{v} \oplus MN[\mathbf{0}]$ is balanced for every $\boldsymbol{v} \in V(\frac{1}{2}FQ^{n-1})$. Recall that the set $M\boldsymbol{v} \oplus MN[\mathbf{0}]$ is balanced if and only if the set $MN[\mathbf{0}]$ is balanced. Moreover, note that the set $MN[\mathbf{0}]$ is precisely the set of columns of the matrix MB^*. It means that the set $MN[\mathbf{0}]$ is balanced if and only if the matrix MB^* is balanced. Thus f is closed neighbor balanced if and only if the matrix MB^* is balanced.

(i) $\Longrightarrow$ (ii) Since f is closed neighbor balanced, f is a bijection. It follows from Lemma 2 that the rank of M is $n-2$. Fix any $i \in \{1, \ldots, n-2\}$, we assume $r_i(M) = t$. Obviously, $t > 0$. Next, we show (ii) holds according to the parity of t.

Case (i) t is odd.

By Lemma 3, $r_i(MB^*) = (n-1-t)(t+1)$. By Definition 5 and since the $(n-2) \times (\frac{n(n-1)}{2} + 1)$ matrix MB^* is balanced, we have $(n-1-t)(t+1) = \frac{n(n-1)+2}{4}$. Now we let $t = 2p - 1$ ($p \in \mathbb{Z}^+$) since t is odd. Then we have

$$n \pm \sqrt{n-2} = 4p. \tag{10}$$

Since both $4p$ and n are even, $\sqrt{n-2}$ is even. Now we let $\sqrt{n-2} = 2r$, where $r \geq 2$ as $n \geq 8$. Substituting it into (10), we obtain

$$2r^2 \pm r + 1 - 2p = 0. \tag{11}$$

It means that r is odd. Let $r = 2q + 1$, where q is a positive integer. Then $n = (2r)^2 + 2 = (4q+2)^2 + 2 = 16q^2 + 16q + 6$, and hence the rank of M is $16q^2 + 16q + 4$. By (11), we obtain $p = 4q^2 + 5q + 2$ or $4q^2 + 3q + 1$. Then $t = 2p - 1 = 8q^2 + 10q + 3$ or $8q^2 + 6q + 1$.

Case (ii) t is even.

By using arguments similar to the proof of case (i) above, we have $n = 16q^2 + 16q + 6$, the rank of M is $16q^2 + 16q + 4$ and $t \in \{8q^2 + 10q + 4,\ 8q^2 + 6q + 2\}$, where q is a positive integer.

By the arguments above, (ii) holds.

(ii) $\Longrightarrow$ (i) By Lemma 2 and since the rank of M is $n-2$, f is a bijection. To prove that f is closed neighbor balanced, it suffices to show that the $(n-2) \times (\frac{n(n-1)}{2} + 1)$ matrix MB^* is balanced. To do this, by Definition 6 and since $n = 16q^2 + 16q + 6$, it suffices to show

$$r_i(MB^*) = \frac{1}{2}(\frac{n(n-1)}{2} + 1) = 64q^4 + 128q^3 + 108q^2 + 44q + 8 \tag{12}$$

for every $i \in \{1, \ldots, n-2\}$.

Fix any $i \in \{1, \ldots, n-2\}$, we assume $r_i(M) = t$. Next we prove that (12) holds according to the parity of t.

Case (i) t is odd. By Lemma 3, we obtain $r_i(MB^*) = (t+1)(n-1-t)$. It is easy to see that (12) holds if $t \in \{8q^2 + 10q + 3,\ 8q^2 + 6q + 1\}$.

Case (ii) t is even. By Lemma 3, we obtain $r_i(MB^*) = t(n-t)$. It is easy to see that (12) holds if $t \in \{8q^2 + 10q + 4,\ 8q^2 + 6q + 2\}$.

By the arguments above, (12) holds, and hence (i) holds.

If (i)–(ii) hold, then by Lemma 1, f is a $\{0, 1\}$-magic labeling of $\frac{1}{2}FQ_{n-1}$. □

To prove Theorem 1 (ii), we use the following lemma.

Lemma 7 *Anholcer et al. (2016) If G is a regular graph admitting a $\{0, 1\}$-magic labeling, then -1 is an eigenvalue of G.*

Now we prove Theorem 1 (ii).

Proof of Theorem 1 (ii) Suppose that $\frac{1}{2}FQ_{n-1}$ has a $\{0, 1\}$-magic labeling. By Lemma 7, -1 is an eigenvalue of $\frac{1}{2}FQ_{n-1}$, that is, there exists $j \in \{0, 1, \ldots, \lfloor \frac{n}{4} \rfloor\}$ such that $\theta_j = 2(\frac{n}{2} - 2j)^2 - \frac{n}{2} = -1$. Thus

$$n = 4(\frac{n}{2} - 2j)^2 + 2. \tag{13}$$

We claim that $\frac{n}{2} - 2j$ is odd. In fact, assume $\frac{n}{2} - 2j = p$, where p is an integer. Substituting it into (13), we have $n = 4p^2 + 2$. It follows that $\frac{n}{2} - 2j = 2p^2 + 1 - 2j$. So the claim holds. Moreover, by (13) and since $n \geq 8$, we have $\frac{n}{2} - 2j > 1$. Let $\frac{n}{2} - 2j = 2q + 1$, where q is a positive integer. It follows from (13) that $n = 4(2q + 1)^2 + 2 = 16q^2 + 16q + 6$.

Conversely, since $n = 16q^2 + 16q + 6$, where q is a positive integer, we construct a $(16q^2 + 16q + 4) \times (16q^2 + 16q + 5)$ matrix M with entries in $\mathbb{F}_2$ as follows:

$$\begin{array}{c} \\ 8q^2+4q \\ 8q^2+4q \\ 2q \\ 2q \\ 2q \\ 2q \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccccc} 1 & 8q^2+4q & 8q^2+4q & 2q & 2q & 2q & 2q & 4 \end{array} \\ \begin{bmatrix} \mathbf{0} & I & J & J & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & J & I & J & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & J & \mathbf{0} & I & J & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & J & \mathbf{0} & J & I & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & J & \mathbf{0} & J & \mathbf{0} & I & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & J & \mathbf{0} & J & \mathbf{0} & \mathbf{0} & I & \mathbf{0} \\ \mathbf{0} & J & \mathbf{0} & J & \mathbf{0} & \mathbf{0} & \mathbf{0} & I \end{bmatrix} \end{array},$$

where I denotes the identity matrix and J (resp. $\mathbf{0}$) denotes the all 1's (resp. 0's) matrix. Obviously, $r_i(M) = 8q^2 + 6q + 1$ for every $i \in \{1, \ldots, n-2\}$. Moreover, by elementary row operations over $\mathbb{F}_2$, M can be transformed to the following matrix

$$\begin{bmatrix} \mathbf{0} & I & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & J & I & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & J & \mathbf{0} & I & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & J & \mathbf{0} & J & I & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & J & \mathbf{0} & J & \mathbf{0} & I & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & J & \mathbf{0} & J & \mathbf{0} & \mathbf{0} & I & \mathbf{0} \\ \mathbf{0} & J & \mathbf{0} & J & \mathbf{0} & \mathbf{0} & \mathbf{0} & I \end{bmatrix}$$

which has the same block representation as M and its rank is $16q^2 + 16q + 4$. So the rank of M is $16q^2 + 16q + 4$. Let $f : V(\frac{1}{2}FQ_{n-1}) \to \mathbb{F}_2^{n-2}$ be given by $f(\boldsymbol{v}) = M\boldsymbol{v}$. By Lemma 6, f is a $\{0, 1\}$-magic labeling of $\frac{1}{2}FQ_{n-1}$. Therefore, $\frac{1}{2}FQ_{n-1}$ has a $\{0, 1\}$-magic labeling. $\square$

Acknowledgements The authors would like to thank the referees for giving this paper a careful reading and many valuable comments and useful suggestions.

Funding This work was supported by the National Natural Science Foundation of China (Grant 11971146), the Natural Science Foundation of Hebei Province (Grant A2017403010) and the Science and Technology Foundation of Hebei Education Department (Grant ZC2023012).

Data availability Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Declarations

Conflict of interest The authors declare that they have no conflicts of interest.

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