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The geometry of three-way decision

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Abstract
A theory of three-way decision concerns the art, science, and practice of thinking, problem solving, and information processing in threes. It explores the effective uses of triads of three things, for example, three elements, three parts, three perspectives, and so on. In this paper, I examine geometric structures, graphical representations, and semantical interpretations of triads in terms of basic geometric notions of dots, lines, triangles, circles, as well as more complex structures derived from these basic notions. I use examples from different disciplines and fields to illustrate the uses of these structures and their physical interpretations for triadic thinking, triadic computing, and triadic processing. Following the principles of triadic thinking, this paper blends together three common ways to think, namely, numerical thinking, textual thinking, and visual thinking.

Keywords Three-way decision · triadic thinking · Trilevel analysis · Visual thinking · Numerical thinking · Eight trigrams · Enneagram

1 Introduction

The theory of three-way decision has received much attention in recent years [66, 70, 124, 143]. In a nutshell, three-way decision is about thinking, problem solving, and information processing based on a triad of three things [136, 138]. As an introduction of the paper, I present a brief discussion of the necessary background of three-way decision as triadic thinking. To connect with the readers of this special issue, I examine some implications of three-way decision to artificial intelligence. Finally, I state the objectives of this paper and give a summary of the main results.

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In the paper, I have drawn examples from many different disciplines and fields. As a non-expert in some of these fields, my descriptions are, in most cases, at a surface level and may contain possible misunderstandings. My main focus is on the uses of triadic structures, patterns, and methodologies in these examples. The triadic ways to develop and to present the contents are the focal points. In comparison, the contents matter to a lesser degree and my brief descriptions are sufficient for the purpose of illustrating their uses of the principles and practices of triadic thinking. Moreover, although I appreciate the triadic approaches in these examples, I may not necessarily agree with the contents. In other words, triadic thinking provides powerful principles, effective methodologies, and useful tools, independent of specific contents.

1.1 Three-way decision as triadic thinking

Triadic thinking and triadic methods of three-way decision are motivated by and rely on a fundamental finding of cognitive science and cognitive psychology about our limited ability in processing a few units of information, due to the capacity of our short-term working memory [23, 40, 83]. Although there is no general agreement on the exact number of units, thinking in threes is a common theme and a frequent appearance across different cultures and various disciplines [9, 12, 31, 59]. For example, a study of Changizi and Shimojo [14] based on more than 100 writing systems over human history shows that the "number of strokes per characters is approximately three, independent of the number of characters in the writing system." Pogliani et al [97, 98] give extensive examples to demonstrate that we humans in general and scientists in specific have an intriguing preference for a ternary patterned reality. According to Merriam [82], words and numbers are major elements in human symbol systems and there is a dominance of the number three in rhetoric with numbers. Booker [10] shows that the number three has archetypal significance in storytelling, as expressed by the rule of three (i.e., use three elements). Watson [121] points out that a tripartite system, consisting of three grand ideas, ages, or principles, has been used by numerous authors in studies of intellectual history.

While some authors assume the existence of triadic patterns in nature, others suggest that triadic patterns are our creations for representing and understanding the world. Dundes [31] states insightfully: "Trichotomy exists but it is not the nature of nature. It is part of the nature of culture." The root of this "nature of culture" is perhaps our limited information processing capacity [134]. Furthermore, our extensive experiences of and exposure to triadic thinking confirm, reconfirm, and re-reconfirm the easiness, the power, and the value of thinking, working, and processing in threes. On the other hand, a comprehensive theory of triadic thinking still does not exist in its own right. This has motivated the introduction of a theory of three-way decision for thinking in threes [131, 132, 134]. To have a sense of the rapid developments on the theories, methods, and applications of three-way decision, the following is a sample list of newly emerged topics inspired by three-way decision and sample publications on each topic:

– Theories of three-way decision [45, 46, 51, 65, 102, 138, 139, 160],
– Three-way data analysis [69, 74, 122, 123, 125, 159, 161],
– Three-way cluster analysis [119, 144–148],
– Three-way approximations of fuzzy sets and shadowed sets [37, 149, 153, 157],
– Three-way concept analysis [73, 76, 99, 103, 108, 135, 162],
– Three-way concept learning [62, 84],
– Three-way decision in sentiment analysis [50, 158],
– Three-way conflict analysis [57, 58, 112, 137],
– Three-way hypotheses analysis and assessment [34],
– Three-way attribute reduction and feature selection [63, 75, 154–156],
– Three-way granular computing [33, 64, 127, 136],
– Three-way multi-criteria decision-making and three-way multi-attribute decision-making [67, 71, 92, 113, 151],
– Three-way decision support systems [128],
– Three-way decision in knowledge management [2],
– Sequential three-way decision [24, 47, 61, 100, 101, 126, 133, 152].

By adding a third option to thinking in twos (i.e., dichotomous, black/white, or all/nothing thinking), three-way decision is about thinking in threes (i.e., trichotomous, black/grey/white, or all/some/nothing). One of the fundamental notions of three-way decision is a triad of three things. There are many interpretations of a triad, which offers the universality, flexibility, and operability of three-way decision.

1.2 Three-way decision and artificial intelligence

In many situations, one of the advantages of triadic thinking is the avoidance of a simplification of using competitive or exclusive opposite twos. The addition of a third option moves towards balance, harmony, and completeness, which often leads to a new point of view, additional insights, and different approaches. As examples, I look at some implications of triadic thinking to artificial intelligence (AI).

Studies of AI commonly focus on its dual goal, as stated by Herbert Simon: "AI can have two purposes. One is to use the power of computers to augment human thinking. ... The other is to use a computer's artificial intelligence to understand how humans think." This dichotomous way to think unfortunately has led to a separation of machines and humans. Lebiere et al [60] point out: "... rather than benefit from a complementary relationship, these two goals have diverged, and the fields of AI and cognitive science have each matured as essentially separate disciplines. Artificial intelligence has become dedicated to the sole purpose of the creation of intelligent computer programs, irrespective of their relation to human cognitive processes." If we can learn something from triadic thinking, we may introduce a third option. In fact, there have been efforts towards a more general understanding of AI, which may be termed integrated human-machine intelligence [11], human-machine integrated intelligence, or human-machine symbiosis [38, 68, 134]. Research initiatives along this line include, to give three examples, cognitive computing [134], artificial general intelligence (AGI) [60], and human-machine collective intelligence [85].

Another example of triadic thinking in the context of artificial intelligence is a classification of three types of artificial intelligence: a) artificial narrow intelligence (ANI), b) artificial general intelligence (AGI), and c) artificial super intelligence (ASI) [39]. As a third example, Peeters et al [95] identify three different perspectives on the future impact of artificial intelligence (AI) on human society: (1) the technology-centric perspective, (2) the human-centric perspective, and (3) the collective intelligence-centric perspective. There are many more examples of triadic thinking that are at work in artificial intelligence.

1.3 Objectives and summary

Among many ways to think, visual thinking, numerical thinking, and textual thinking stand out. Working together as a triad, they provide complementary perspectives and a more complete understanding. Images, symbols, numbers, and words are essential to human understanding and communication. Numbers are an important part of the language of science. Natural numbers, at least some of them, are often associated with special powers or meanings and they have both qualitative and quantitative interpretations. Studies on the theory of three-way decision explore the magic power of the number three, with respect to its literate and figurative meanings. One of the goals is to establish a mathematical and scientific basis of triadic thinking or thinking in threes.

As visual tools, graphical symbols, images, pictures, and figures often play an essential role in forming a profound understanding. They are intuitively appealing and attention-grabbing, capitalizing on our pattern-seeking instincts. They clarify, illustrate, and augment written ideas, leading to a simplification and a full understanding of complicated textual descriptions. Sometimes, they may convey meanings that may be difficult to describe fully in words. The main objectives of this paper are, therefore, to open up new avenues for research called the geometry of three-way decision. From a visual perspective, the geometry of three-way decision studies various geometric structures and patterns used in three-way decision, for example, trisegment lines of three-way decision, triangles of three-way decision, concentric tricircles of three-way decision, and many more. Numerical thinking (i.e., thinking in threes) and textual thinking (i.e., verbal description of the principles of three-way decision) are the two perspectives that have been explored in existing studies on three-way decision, the new visual thinking perspective enables us to form a triad of three perspectives. In this way, we study three-way decision by its own principle of thinking in threes.

The rest of the paper is organized as follows. Section 2 gives a brief description of my journey into three-way decision. Section 3 is an overview of the trisecting-acting-outcome (TAO) model that provides a general conception of three-way decision. Section 4 examines geometric structures and patterns of triads that underlie three-way decision. Section 5 looks at ways of triadic thinking embedded in more complex structures, including hexagons, the two classical Chinese figures of Hetu (the River Diagram) and Luoshu (the Luo Writing), the eight trigrams, and the enneagram.

2 How it all started: A personal journey

A reviewer recommended that I add a section on the historical development of the theory of three-way decision by recalling relevant basic ideas from three-valued logics, fuzzy sets, rough sets, and others. This is indeed a wonderful and an insightful suggestion. By giving some key ideas and milestones, a reader will have the necessary contexts to fully understand and appreciate the concept of three-way decision. In this section, I discuss the major works that have influenced my conception of three-way decision. To be consistent with the style of the rest of the paper, I will only give a conceptual level description. A reader may consult the references for more detailed mathematical formal formulations.

In the early 1980s, Pawlak [90, 91] introduced the theory of rough sets for reasoning about data based on descriptions of objects. When two objects have the same description, they are indiscernible from each other. This indiscernibility relation is an equivalence relation and objects with the same descriptions form equivalence classes. In this way, only equivalence classes and their unions are describable or definable sets. Consequently, an arbitrary set of objects may not be definable and has to be approximated by using definable sets. The lower approximation of a set is the greatest definable set inside the set, and the upper approximation is the least definable containing the set. In order words, the lower and upper approximations are two definable sets that approximate the given set from below and above. Alternatively, it is mathematically equivalent to use three definable sets to represent a rough set approximation. The positive region of the set is defined by the lower approximation, the boundary region by the difference of the upper and lower approximations, and the negative region by the set complement of the upper approximation.

The formulation with a pair of lower and upper approximations connects rough set theory to modal logics. The lower and upper approximations correspond to the necessity and possibility operators of modal logics. Pawlak rough set model corresponds to modal logic system S5. The formulation with three regions connects rough set theory to three-valued logics. The positive, negative, and boundary regions are characterized by the truth values of *true*, *false*, and a third *intermediate value* in three-valued logics [16]. A fundamental result of rough set theory is that the lower (upper) approximation of the union (intersection) of two sets cannot be computed from the lower (upper) approximations of the two sets. This non-truth-functional feature of rough set theory contributes uniquely to three-valued logics by calling for non-truth-functional logic connectives.

Rough set theory captures a type of uncertainty due to the indiscernibility of objects based on their descriptions. In this case, we know exactly whether an object is in or not in a given set. But we cannot precisely define the set based on descriptions of objects. In some other situations, another kind of uncertainty appears. Due to a lack of information, we only know statuses of some objects, namely, in or not in a set, and we do not know the statuses of the rest of objects. To model this type of uncertainty, in 1993 I introduced the notion of an interval set [129]. An interval set is a family of sets that are bounded by a pair of sets called the lower and upper bounds, in which the lower bound is a subset of the upper bound. Any set in the family may be the actual set, if the complete information is known. The lower and upper bounds of an interval set correspond to the rough set lower and upper approximations, although their respective semantics are different. Based on the lower and upper bounds, an interval set can be equivalently defined by three regions. The lower bound consists of these objects that we know are members of the set and, thus, defines the positive region; the complement of the upper bound consists of these objects that we know are not members of the set and, thus, defines the negative region. The boundary region is the difference of the upper bound and the lower bound, consisting of objects whose statuses we do not know. If we draw a correspondence between the three regions and three values in three-valued logics, interval set algebra is characterized by Kleene's three-valued logic, with the third intermediate value being interpreted as *unknown* [16].

In the early 1990s, we generalized Pawlak rough sets into decision-theoretic rough sets, namely, a probabilistic rough set model, based on the Bayesian decision procedure [141, 142]. A pair of thresholds on the conditional probability of an arbitrary object being in a set given that the object is in an equivalence class is used to construct probabilistic positive, negative, and boundary regions. The Pawlak rough sets are a special case in which the pair of thresholds is the pair of the two extreme values 1 and 0 of a probability function.

Decision rules induction from the pair of lower and upper approximations or the three regions is perhaps one of the most common applications of rough set theory. For the Pawlak rough sets, decision rules from the positive and negative regions are certain and the corresponding decisions are made free of errors. In contrast, decision rules from the probabilistic positive and negative regions are no longer certain and the corresponding decisions are made under some tolerance levels of errors. Consequently, there is a need to have a new interpretation of decision rules in probabilistic rough sets. Inspired by the ideas of sequential hypothesis testing introduced by Wald [118], that is, a hypothesis is accepted, rejected, or further tested, in 2009 I introduced the concept of three-way decision for interpreting decision rules in probabilistic rough sets [130, 131]. Rules from the positive, negative, and boundary regions are viewed as rules for acceptance, rejection, and non-commitment. It is understood that decisions of acceptance or rejection are made with a respective acceptable level of errors.

In three-valued logics, the third *intermediate value* denotes a truth value that is different from the the standard truth values, *true* and *false*. Depending on different interpretations of the third value, it is possible to build different three-valued logic systems [16]. However, the use of a single value to represent a wide range of possibilities is a weakness of three-valued logics. For this reason, more general many-valued logic systems have been proposed [42], in which the third value of three-valued logics is generalized into a set of values. A grand challenge of many-valued logics is the semantics of truth values and the validity of inference rules. A possible solution is to interpret many-valued logics in terms of three-valued logics by introducing the concepts of designated truth values and designated false values, corresponding to the standard truth values, *true* and *false*. The rest of the truth values correspond to the *intermediate value* of three-valued logics. In other words, a many-valued logic system is approximated by a three-valued logic system. The probabilistic rough sets are an example of this approach. Probabilistic rough set approximations are the results of a three-valued or three-way approximations of probabilistic logic. The designated truth values are probability values at or above one threshold, which defines the positive region. The designated false values are probability values at or below another threshold, which defines the negative region. The rest of probability values between the two thresholds define the boundary region.

In 1965, Zadeh [150] introduced the theory of fuzzy sets for representing concepts with gradually changing boundaries. An object is assigned a membership grade in the unit interval [0, 1] to indicate the degree to which the object is a member of a fuzzy set. The corresponding fuzzy logics are many-valued logics. Zadeh briefly described the concept of a three-way approximation of a fuzzy set, in a similar way as constructing a three-way approximation of a many-valued logic or constructing a three-way probabilistic rough set approximation. Specifically, if the membership grade of an object is at or above one threshold, the object is considered to be a member, if the membership grade is at or below another threshold, the object is considered to be a non-member, if the membership grade is between the two thresholds, the object has an indeterminate status.

In 1998, Pedrycz [93, 94] proposed the concept of shadowed sets as three-way approximations of fuzzy sets. The construction process of a shadowed set from a fuzzy set uses three actions, which is similar to Zadeh's method conceptually. If the fuzzy membership grade of an object is at or above one threshold, the membership grade is lifted to 1; if the membership grade is at or below another threshold, the membership grade is reduced to 0; if the membership grade is between the two thresholds, the membership grade is expanded into the unit interval [0, 1]. While Zadeh did not discuss how to determine the pair of thresholds, Pedrycz gave an optimization method for computing the pair of thresholds. According to the set of three values {1, [0, 1], 0}, it is straightforward to define the positive, boundary, and negative region of a shadowed set.

By observing the common notion of the three regions used in rough sets, interval sets, and three-way approximations of fuzzy sets, in 2012 I outlined of a theory of three-way decision [132], in which evaluation-based models were introduced. One model uses a single evaluation, of which probability functions and fuzzy membership functions are special cases. Another model uses a pair of evaluations, which, I found out later, has a close connection to the concepts of bipolarity [27], the evaluative space model [13], and bilattices [26].

These investigations have been focused more on classification and decision-making problems. They explore a narrow sense of three-way decision. Since 2012, I have been working on a wide sense of three-way decision, that is, a philosophy of thinking in threes, a methodology of working in threes, and a mechanism of processing in threes. The most recent results of the wide sense of three-way decision were given in the trisecting-acting-outcome (TAO) model [134, 138], which will be briefly reviewed in the next section. In the present paper, I will further explore the wide sense of three-way decision from a geometric point of view.

With respect to the narrow sense of three-way decision, there are formal mathematical models. These models provide ways to making three-way decisions in particular contexts. With respect to the wide sense, the investigations have been focused on the philosophical foundations and methodologies of three-way decision. Models of the wide sense are more of a conceptual nature and may be difficult to describe mathematically. The power of the philosophy and methodology of three-way decision lies in their universality, flexibility, and domain-independency. In addition to playing the role of guiding thinking, problem-solving, and information-processing, conceptual models of the wide sense guide us in building new concrete, formal, and operational models of the narrow sense.

3 A trisecting-acting-outcome (TAO) model

A fundamental philosophy of a theory of three-way decision is triadic thinking, that is, thinking in threes. A triad of three things is used to describe, represent, and process a whole in three parts. This section briefly reviews a trisecting-acting-outcome (TAO) model [136], in which we can observe many uses of threes.

3.1 An overview of the trisecting-acting-outcome (TAO) model

In three-way decision, we use a triad of three parts to represent a whole. Depending on different interpretations of a triad, we can formulate various models of three-way decision. One possible way to construct a triad is to decompose a whole into three parts. The trisecting-acting-outcome (TAO) model of three-way decision is formulated based on such an understanding of a triad.

Figure 1 depicts the basic three components of the TAO model, namely, trisecting the whole, acting upon the three parts, and optimizing the outcome. The middle triangle (A, B, C) denotes a trisection of the whole and connects two tripods. The upper tripod, consisting of the node “Whole” and three legs A, B, and C, describes the task of trisecting. There are two possible ways to interpret the tripod. A top-down reading suggests that the whole is decomposed into three parts; a bottom-up reading suggests that the three parts support the whole. The lower inverted tripod, consisting of the node “Strategies” and three legs A, B, and C, describes the task of acting. A set of strategies is applied to process the three parts. There are also two possible ways to interpret the tripod. A top-down reading from A, B, and C to strategies suggests that a trisection of the whole enables the formulation of a set of different strategies. We devise a set of strategies based on the trisection of the whole. A bottom-up reading suggests the applications of a pre-defined set of strategies to the trisection. In many cases, the trisection is, in fact, constructed based on a given set of strategies.

The dotted square represents an outcome of three-way decision determined by a combination of a trisection and

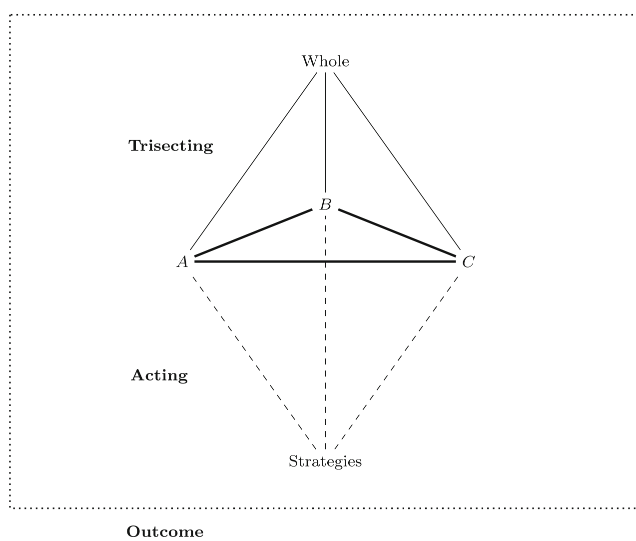


Fig. 1 A TAO model of three-way decision (adapted from [136])

a set of strategies in the square. The tasks of trisecting and acting are closely tied together. When the two tripods in Fig. 1 are put together, we have a triangular bipyramid with the base triangle (A, B, C) and consisting of three levels. This triangular bipyramid provides a basis for trilevel thinking and processing [139], leading to two basic modes of three-way decision [136]. A top-down trisection-driven mode starts with trisecting the whole and then searches for the most suitable strategies to process the three parts. A bottom-up action-driven mode starts with a pre-given set of strategies and searches for the most effective trisection. To produce a desired outcome, it may be necessary to search for the most effective combination of a trisection and a set of strategies. This often requires multiple iterations of the top-down and bottom-up modes, involving trisection-guided redesign of strategies and strategy-guided re-trisection of the whole.

The trisection (A, B, C) offers a total of seven ways to process the three parts. In Fig. 2, we arrange the seven ways into three levels [136]. The bottom level is an individual analysis in which each of the three parts is considered independently of the other two. This kind of analysis is indicated by a one-dimensional line in Fig. 1. For example, the solid line (Whole, A) only considers part A in the context of the "Whole" and the dashed line (A, Strategies) applies "Strategies" to part A. The other two parts B and C do not come into the picture. The middle level is a comparative analysis, in which the relationship between two parts is taken into consideration. This type of analysis is described by a two-dimensional triangle in Fig. 1. For example, when two parts A and B are considered together, we have a trisecting face given by the triangle (Whole, A, B) and an acting face given by the triangle (Strategies, A, B). In this case, parts A and B are considered together with respect to the "Whole" and "Strategies," respectively. The third part C does not play any role. The top level is an integrative analysis of all three parts, which is described by the three-dimensional trisecting pyramid (i.e., the upper one) and acting pyramid (i.e., the lower one).

The three levels of analysis correspond to one-, two-, and three-dimensional understanding of the whole. An analysis in an upper level may be viewed as an integration of relevant analyses in its lower level. An analysis of the lower may be viewed as a projection of its upper level. It is interesting to note that both the bottom level and the middle level consist of three entries. These two levels may also be viewed as two three-way methods.

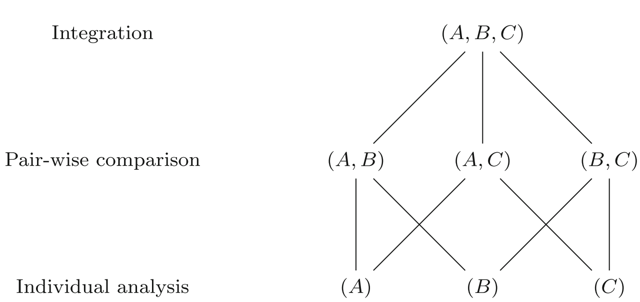


Fig. 2 Trilevel seven elements analysis (adapted from [136])

3.2 The TAO model explained by an example

As presented in earlier papers [136, 138, 139], there are many three-way decision approaches and examples from a wide-range and diverse disciplines that can be used to illustrate the ideas and value of the TAO framework. We explain basic notions and ideas of the TAO model based on a study of strategic decision-making by Schwenk [106].

By reviewing and abstracting from many descriptive models of strategic decision-making process, Schwenk [106] suggests a simplified three-stage model, consisting of problem identification, alternative generation, and evaluation/selection. For each stage, he lists three to four heuristics (i.e., cognitive simplification processes) that may operate. Table 1 summarizes the relevant information from Table 2 in Schwenk's paper.

In light of the TAO model, Schwenk's three-stage strategic decision-making model may be viewed as an instance of three-way decision as thinking in threes. The three stages form the trisection of the whole of a decision-making process. Cognitive simplification processes or heuristics that may operate in the three stages correspond to the set of strategies. For each cognitive process, Schwenk also provides possible undesirable effects. For example, in the problem identification stage, a decision-maker may rely on erroneous beliefs or hypotheses, leading to an ignorance or misinterpretation of evidence and information. As a double-edged sword, cognitive simplification makes complex strategic decision-making simpler in some situations and may also be harmful to organizations in some other situations. For the latter, it may be necessary to correct the undesirable effects of cognitive simplification processes.

In addition to the simplification processes in Table 1, Schwenk points out a number of more processes. This leads to a large number of combinations of cognitive simplification processes in the three stages, which may explain for the many different ways used by or styles of decision-makers. The three-stage conception of Schwenk takes into consideration of the trisecting and acting parts of the TAO model and provides a basis for systematically studying strategic decision-making. With respect to the outcome part of the TAO model, it may be constructive to investigate how to evaluate the effectiveness of a tristage model of strategic decision-making with cognitive simplification processes.

Table 1 Cognitive processes of three-stage model of Schwenk [106]

Stage I Goal formulation/ Problem definition	Stage II Strategic alternatives generation	Stage III Evaluation and selection
(1) Prior hypothesis bias	(1) Single outcome calculation	(1) Representativeness
(2) Adjustment and anchoring	(2) Inference of impossibility	(2) Illusion of control
(3) Escalating commitment	(3) Denying value trade-offs	(3) Devaluation of partially described alternatives
(4) Reasoning by analogy	(4) Problem sets	

4 Geometric structures and patterns in three-way decision

This section examines visual understanding of three-way decision through geometric interpretations of structures and patterns of three-way decision.

4.1 Representations and interpretations of triads

A central notion of three-way decision as triadic thinking is a triad of three things or items. There are many interpretations of a triad in different contexts and for various applications. In a graphical representation, we use either points or boxes to represent the three items and use lines to represent the relationships among the three items. We also examine other geometric representations, including triangles and circles.

4.1.1 Triad as a set of three dots

The simplest interpretation of a triad, given by the three nodes A, B, and C in the TAO model of Fig. 1, is three relatively independent and unconnected items, which is represented by three isolated dots in Fig. 3. Although there may be a sequential arrangement as in Fig. 3a or a layered arrangement as in Fig. 3b and c, we assume that these arrangements do not imply any particular order on the three items. By representing a triad simply as a set of three isolated dots, we temporarily ignore, for the sake of simplicity, some useful and important relationship, interaction, and dependency of the three items, which will be further explored later in the paper. This simplistic view allows us to focus mainly on triadic thinking with only three items, without the complication incurred by the relationships of the three. There is a large body of evidence to justify and support this specific type of ways to think, as demonstrated by various uses of the rule of three.

An important result from cognitive science and psychology is about our limited ability for processing information due to the capacity of short-term working memory. We humans can only process a few units of information in our working memory, typically ranging from two to seven units [23, 40, 83]. Another related result is our subitizing ability to tell immediately, without counting, the number of items presented to us when the number of items is small, typically fewer than six [53, 96]. For example, if we look at Fig. 3, we can easily tell that there are three dots without actually counting the dots. Although there are different opinions regarding the exact number, three seems to be a reasonable choice. A supporting evidence for three is that the first three numbers have been typically represented by series of identical marks in different languages [25]. For example, Chinese represents 1, 2, and 3 by one, two, and three horizontal lines, respectively. Roman numbers 1, 2, and 3 consist of one, two, and three vertical lines, respectively. Our subitizing ability enables to see the numerical sense of the identical marks immediately without counting. On the other hand, when a number is four or above, the pattern of series of identical marks no longer holds. We would have to count the number of identical marks, if the patterns of series identical marks were used.

In light of these cognitive findings, it is not surprising to find that triadic thinking is a universal practice across many cultures and different disciplines, manifesting in

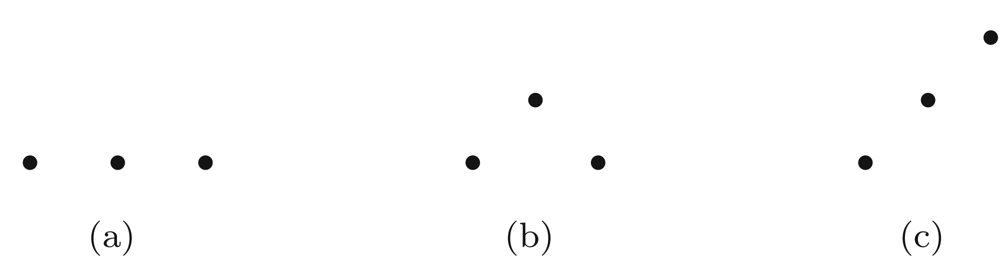


Fig. 3 Triads as three dots

many ways, shapes, and forms [9, 12, 31, 59, 97]. The rule of 3, as a practical cognitive simplification heuristics, tells us to focus on three things at a time. We can easily remember three things and prioritize three things [80]; we do not need to write them down or look them up [4]. When the rule is applied in our daily life, Meier [80] and Bailey [4] demonstrate that we can achieve more by simply deciding or choosing three things, goals, or tasks we want to accomplish at the beginning of each day and the start of each week. Booker [10] examines various uses of the rule of 3 in storytelling, in which things often appear in threes. Backman [3] explores the rule of 3 in writing through effective uses of three-part or triadic structures. Boer [9] and Gallo [35, 36] explain the effectiveness of the rule of 3 in public speeches. Brown [12] shows that trinitarianism, namely, thinking in threes, is widespread in the theory and practice of marketing. Pogliani et al [97] show that, when viewing the reality, there is an intriguing human preference for ternary patterns.

There are many other diverse uses of the rule of 3 for triadic thinking. We look at three more examples for different interpretations of the three. Three is perhaps the smallest number of items for us to form a pattern in our mind: Once is a chance, twice is a coincidence, thrice is a pattern. Sometimes, three is considered to be the maximum: The three strikes rule suggests a severe consequence/punishment of the third time. In some situations, three is used to denote figuratively many. We often repeat the same things thrice to show an emphasis of the importance, the priority, or the urgency, for example, "clarity, clarity, clarity" by Oliver Strunk for writing, "education, education, education" by Tony Blair when setting out his priorities for office, and "location, location, location" in real estate. It is interesting to note that this rhetoric device known as epizeuxis, in fact, taps into the above three senses of the number three: a) three as the minimum number of items to form a pattern, b) three as the maximum number of allowed appearances, and c) three as a figurative number of many. The resulting triad of three repetitions of the same word forms a powerful and memorable pattern.

4.1.2 Triad as a tripod

The simple interpretation and representation of a triad as three dots, although capturing an essential aspect of the threeness of a triad, does not take into consideration of the relationships between parts of the whole nor the interaction of the three items. To bring in these aspects, we use a tripod to represent a triad, following a suggestion by Merrell [81] for depicting and interpreting Pierce's signs with a tripod.

In the earlier discussion of the TAO model of three-way decision of Fig. 1, we identified two tripods, a trisection tripod (Whole, A, B, C) and a strategy tripod (Strategies, A, B, C). Figure 4a redraws the trisection tripod as a three-blade fan. The circle in the center represents a whole. The three lines describe the relationships between parts and whole: Reading outwards, the whole is trisected into three parts labelled by A, B, and C; reading inwards, the three parts support the whole. Relationships between different parts are realized through their connections to the whole. For example, A and B are related through the circle in the center. Finally, the tripod unites the three parts by the whole.

Figure 4b is a slightly modified version of the semiotic tripod used by Merrell [81] for interpreting Peirce's triad, Sign(Representamen)/Object/Interpretant. We use a circle instead of a dot in the center and add a dot at the outer end of each of the three lines. In the figure, R, I, and O represent, respectively, a sign (or a representamen), the interpretant, and the semiotic object. A tripod represents a genuine triadicity, with the center circle as a focal point tying all elements together. Each sign element connects to the other two through the center circle node. The interpretant plays a mediating role connecting the representamen and the object and the meaning arises from relations between the components of the sign.

We refer to the tripod of Fig. 4a as the tripod of three-way decision. The tripod provides a powerful geometric figure for describing a triad for triadic thinking. As another example for illustration, we give a 3V characterization of big data as depicted in Fig. 4c. In 2001, Laney [56] discussed a 3D conception of data management by introducing three aspects of data, namely, data volume,

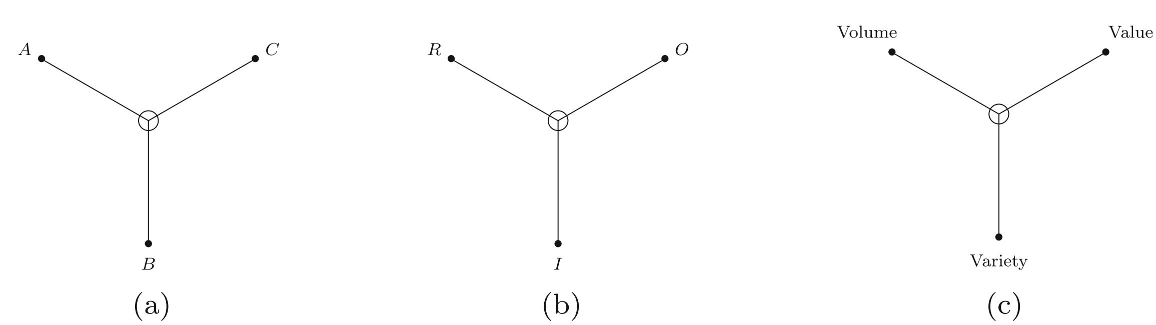


Fig. 4 Triads as tripods

velocity, and variety, which was later used as the 3V of big data. In light of the power of thinking in threes, the acceptance of the 3V conception is not a surprise. It is just another example of triadic thinking. What is interesting is that many proposals have been made by considering other aspects and features of big data by using words beginning with V, ranging from three Vs to ten Vs. The list of V-words includes, for example, Volume, Variety, Velocity, Veracity, Variability, Validity, Vulnerability, Viscosity, Volatility, Visualization, and Value. As predicated by the principle of triadic thinking, proposals with four or more Vs have not received the equal attention as the three Vs proposal. We have to make an effort to memorize and to sort when we are given four or more Vs.

The introduction of other Vs suggests that the three Vs of volume, velocity, and variety might not be the best choice. On the other hand, the principle of triadic thinking demands us to choose only three Vs to conceptualize big data. Among the list of V-words used in existing studies, we select Volume, Variety, and Value as the three Vs of big data. The word "big" qualifies various and different aspects of data analytics. Choosing the two Vs of Volume and Value seem to be an easy decision. Large volume is a constant, common, and unique feature of big data, which mainly focuses on the size of data. Deriving high value is a basic goal and objective of big data analytic. The choice of the third V needs some comments. We may take a broader meaning of "Variety" to capture the wide diversity, non-uniformity, non-monotony, or variance of different features of data, as well as methods for processing the data, as characterized by other Vs. For example, "Velocity" deals with a variety of speeds in which real-time data are changing and to be processed. "Veracity" concerns about a variety of inconsistencies or accuracy of data, and so on. By summarizing the discussion, we arrive at the triadic 3V conception of big data given in Fig. 4c.

By focusing on three Vs, we have a memorable and appealing characterization of big data. The inclusion of additional Vs, although covering more aspects, may not necessarily be advantageous. The discussions on the three Vs conception of big data are generally relevant when formulating a simple and memorable model in other contexts. To support the choice of three, we quote from Clayton [20]: "The emotional power of lists of three is so great that even when lists of four are used, we typically remember only three." Although Clayton specifically used lists of four elements, the statement remains valid for lists of more than four elements. I use three examples to further demonstrate the necessity and the power of triadicity in human understanding. The first example is taken from Clayton [20] on the common use of lists of three as powerful speech patterns by great speakers. He commented that Sir Winston Churchill's famous list of four, "blood, toil, tears and sweat," is often mis-quoted as or shortened into a list of three, "blood, sweat and tears." The second example is taken from Brown [12] on the production/sale/marketing three eras schema in marketing. Although the original paper by Keith [54] contained a fourth era of "marketing control," this fourth element has been forgotten in later studies. The third example is the concept of generations of programming languages. It is wide accepted that the development of programming languages is divided into three generations: 1) the first generation of machine languages, 2) the second generation of assembly languages, and 3) the third generation of high-level languages. Although two additional fourth and fifth generations have been introduced, they have not been well received nor commonly used. Talking about only up to three generations or eras is a common practice [12].

Two oversimplifies, three charms, and four alarms. Building a case with three good reasons is more effective than with either two or four [110]. A tripod has three supporting legs, which offers a good metaphor for model building. Tripartite models give us a great aesthetic pleasure of balance, harmony, and completeness. They are complex enough to allow for the necessary generality and flexibility on one hand and simple enough to not tax our brain on the other.

4.1.3 Triad as a trisegment line

Interpretations of a triad as a set of three dots or a tripod provide a general geometric understanding of three-way decision as thinking in threes. In many situations, we are interested in special arrangement of the three dots. The very first simple idea is to put three dots on a line, as shown in Fig. 5, capturing a sequential structure of the three dots. The interpretation of a triad as a trisegment line is a widely used way to express a simple view of the reality, such as space and time. The earlier discussed three-era schema and three-generation classification are typical examples of the trisegment line interpretation.

Figure 5a is an abstract geometric representation of a trisegment line. The line represents a continuous whole and

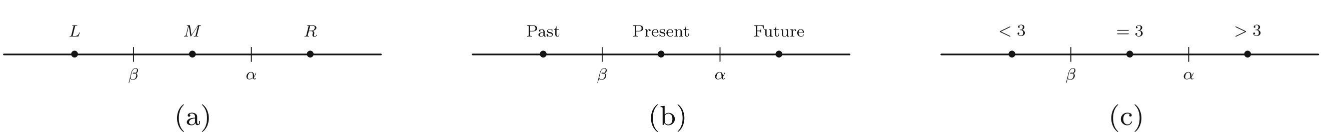


Fig. 5 Triads as trisegment lines

elements are ordered from left to right. Given a pair of thresholds, $\langle \alpha, \beta \rangle$, we divide the whole into three segments, which are denoted as three dots L (Left), M (Middle), and R (Right). Figure 5b is an example of trisegment time line, which figuratively depicts our everyday conception of time as a triad of the past, present, and future. Although we explicitly give a pair of thresholds for trisecting the line, it should be noted that such clear cuts may not be well-defined in practice. More often than not, we have rather blurred cuts and the three segments are given qualitatively, instead of quantitatively.

The trisegment line interpretation of a triad, in fact, serves as a basis for building an evaluation-based model of three-way decision [132]. We arrange a set of objects according to their evaluation status values (ESVs) and trisect the set based on a pair of thresholds. Suppose U is a set of objects. We assume that an evaluation function of objects is a mapping from U to the set of real numbers, namely, $e : U \longrightarrow \Re$. For $x \in U$, $e(x)$ is the evaluation status value of x. In general, we may consider a partially-ordered set or a totally-ordered set in place of $\Re$. Given a pair of thresholds $\langle \alpha, \beta \rangle$ with $\alpha \geq \beta$, we can trisect the set of objects. Depending on the decisions of objects with values equal to α or β, there are several ways for trisecting. Two possible ways are given below:

$$\begin{aligned} L^{(,\beta]}(e) &= \{x \in U \mid e(x) \leq \beta\}, \\ M^{(\beta,\alpha)}(e) &= \{x \in U \mid \beta < e(x) < \alpha\}, \\ H^{[\alpha,)}(e) &= \{x \in U \mid e(x) \geq \alpha\}, \end{aligned} \tag{1}$$

and

$$\begin{aligned} L^{(,\beta)}(e) &= \{x \in U \mid e(x) < \beta\}, \\ M^{[\beta,\alpha]}(e) &= \{x \in U \mid \beta \leq e(x) \leq \alpha\}, \\ H^{(\alpha,)}(e) &= \{x \in U \mid e(x) > \alpha\}, \end{aligned} \tag{2}$$

where L, M, and H indicates, respectively, low, medium, and high evaluation status values, and the superscripts give the ranges used to construct the corresponding segments.

Consider a special case of (2) where $\alpha = \beta = 0$. We have three subsets of U representing, respectively, negative, zero, and positive evaluation statuses. The triad of negative/zero/positive is, in fact, a powerful metaphor in our everyday life. While the negative and positive represent the two opposite and, sometimes, competitive extremes, zero represents the harmonic and tolerating golden middle. Taking the middle-way and avoiding extremes is a time-proven wisdom of life [43, 77]. Figure 5c explains three-way decision in terms of a trisegment line: Two or less is too small, four or more is too many, and three is the right number of things to think of effectively on a cognitive basis.

A division of a continuous whole into three parts represents a change from a quantitative understanding to a qualitative understanding. Changes within each line segment are of a quantitative nature. Once the changes are beyond some thresholds, they become qualitative changes. Examples of triadic qualitative representation, being easy-to-grasp, easy-to-understand, and easy-to-remember, are ubiquitous. There are low, middle, and high income families; there are low, normal, and high body temperatures or blood pressures; there are basic, standard, and luxury versions of a product; there are good, better, and best practices and values. In statistical analysis, we sometimes consider the middle and two tails of a distribution. Changes of stock prices are described in terms of declining, unchanged, and advancing. Our attitudes towards an issue are categorized into negative, neutral, and positive. An evaluation-based model of three-way decision captures the essential features of these examples. We may also point out that three-valued logics may serve as a basis for reasoning with triadic qualitative values.

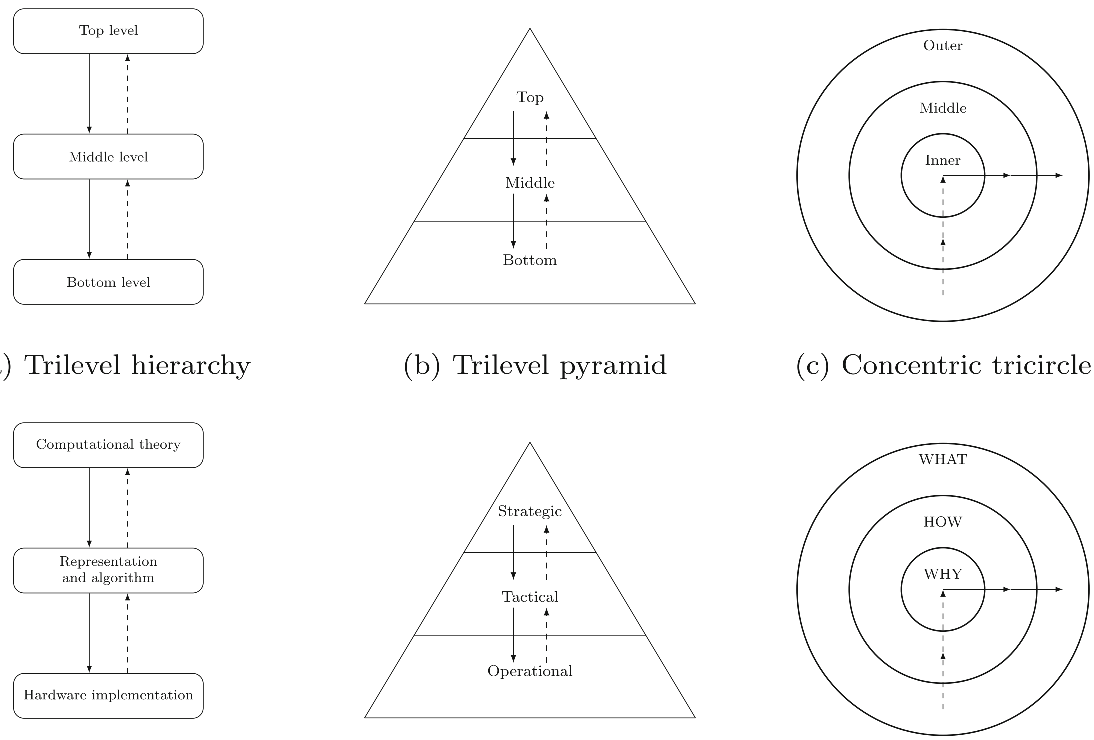
4.1.4 Triad as a trilevel structure

A trisegment line representation of a triad gives us a sequential sense of a set of three things. In some situations, it is useful to arrange the three things hierarchically to capture a sense of control and support. This aspect leads to trilevel thinking as one of the modes of three-way decision [139].

Figure 6 depicts three commonly used hierarchical representations of a triad, namely, trilevel hierarchies, trilevel pyramids, and concentric tricircles (or nested trilayer circles). Trilevel hierarchies are widely used to organize the world and ourselves. Examples of trilevel structures are three levels of governments, three levels of management, three levels of leadership, three levels of strategy, three levels of understanding, three levels of design, three levels of biodiversity, three levels of economics, and many others. There is typically a control-support relationship between levels and the top level or inner layer is indirectly related to the bottom level or outer layer through the middle level or layer. As indicated by a solid line, a higher level controls and determines its lower level; as indicated by a dashed line, a lower level supports its higher level. In some sense, a hierarchical control and subordinating brings in an order in a complex situation. Hierarchical structures play a crucial role in granular computing. Lower levels consist of smaller granules, representing lower granularities and more details, and higher levels consist of larger granules, representing higher granularities and more abstraction. This connection gives rise to three-way granular computing [136].

In what follows, I discuss general ideas and examples of the three types of trilevel structures in Fig. 6.

Trilevel hierarchies In Fig. 6a, we use three rectangles of the same size to convey a sense of the sameness of the values of the three levels, for example, each of the three levels



(a) Trilevel hierarchy (b) Trilevel pyramid (c) Concentric tricircle

(d) Marr's three levels (e) Anthony's trilevel pyramid (f) Sinek's Golden Circle

Fig. 6 Trilevel structures

is as important as the other two. The three levels simply describe or represent the same problem at different levels of abstraction, detail, or scale. They provide descriptions of the whole from three different perspectives. I use three examples to illustrate the uses of trilevel hierarchies.

The first example of trilevel thinking is Marr's [79] three-level understanding of an information processing system, which is given in Fig. 6d. The top level focuses on computational theory in abstract, the middle level concerns specific representations, algorithms, and processes, and the bottom level is about physical implementation. With the trilevel conception, it is possible to talk about computational theory without referring to particular representations and algorithms, and to talk about representations and algorithms without referring to specific machine hardware. A theory may determine and, at the same time, be supported by multiple representations and algorithms. Similarly, an algorithm may determine and, at the same time, be supported by multiple hardware implementations. A trilevel hierarchy enables us to separate our concerns, to focus on different problems at different levels, and to ask the right questions at the right levels. Marr argued that a full understanding of an information processing system depends crucially on understandings at the three levels. By iteratively exploring top-down control and bottom-up support, we increase the probability of finding the most appropriate theory, representations and algorithms, and implementations. Marr's three levels have influenced trilevel thinking in computer science, cognitive science, brain science, and several other fields.

The second example of trilevel thinking is Weaver's trilevel model of communication problems, consisting of technical problems of transferring symbols through a communication channel, semantics problems of understanding the transferred messages, and effectiveness problems of utilizing the understood messages [107]. It is interesting to note that Weaver's three levels are arranged according to an increasing order of complexity from bottom-up. Multi-level hierarchies with varying complexity and difficulty is a commonly used effective way to approach a complex problem.

The third example is the ANSI-SPARC architecture of database management systems consists of three levels, namely, the external level for user views of data, conceptual level for logical views of data, and internal level for physical data storage and retrieval [48]. Similar to Marr's three levels, the separation of the three views enables us to focus on a particular type of problem at a particular level. For example, we can investigate logical aspects of data without worrying about their physical storage and retrieval methods.

Trilevel pyramids In Fig. 6b, we depict a triad by a trilevel pyramid. Our preference for calling Fig. 6b a pyramid, rather than a triangle, is based on three reasons. First, three levels have different sizes and form a pyramidical structure, with a large base and small top. Second, we represent three things of a triad as the three levels in the pyramid. We do not associate any special meanings to neither the three edges nor the three corners of the triangle. We will discuss such triangle-based interpretations in the next subsection. Third, a pyramid provides a metaphor of a hierarchical organization that has a large number of members at the bottom and, as we move up the hierarchy, a small number of members at the top. While a trilevel hierarchy of Fig. 6a captures a sense of sameness of the three levels, a trilevel pyramid reflects a sense of difference of the three levels. It is perhaps the sense of difference that makes a trilevel pyramid a commonly used geometric figure. I use three examples to illustrate the use of trilevel pyramids as a representation of triads.

The first example is the organizational trilevel pyramid introduced by Anthony [1]. As shown by Fig. 6e, the three levels of planning and control are characterized by strategic decisions, tactical decisions, and operational decisions. As illustrated by a pyramid, the scales/impacts of and the number of decisions made are very different at the three levels: Many operational decisions are made at the bottom, some tactical decisions are made in the middle, and a few important strategic decisions are made at the top. Decisions at a lower level are guided by decisions at a higher level and, reversely, decisions at a lower level support decisions at a higher level. Although pyramids with more than three levels are also a common appearance in management science, Anthony's pyramid seems to be more popular, due to its simplicity and connections with various types of information systems. For example, corresponding to trilevel pyramid, it is possible to study information management systems at three levels [41].

The second example of trilevel pyramids is my re-interpretation of the four-level data/information/knowledge/wisdom (DIKW) hierarchy, in which the two levels of information and knowledge are combined into the middle information-knowledge level [139]. Under the interpretation, the bottom level concerns about raw unprocessed data, the middle information-knowledge level concerns about various types of knowledge embedded in data (i.e., information is considered to be a type of weak knowledge), and the top level concerns the wise use of knowledge. In some sense, the new DI-KW hierarchy corresponds closely to Anthony's pyramid. Operational decisions focus on data, tactical decisions are based on available information and knowledge distilled from data, and strategic decisions show the wisdom in actions.

The third example is the AIDA (Attention-to-Interest-to-Desire-to-Action) model of advertising, which is often referred to as AIDA pyramid or AIDA funnel. It may be commented that a trilevel funnel is an inverted pyramid, which provides another geometric representation of a triad. AIDA model works by drawing a customer's attention, rising the customer's interest and desire, and finally making the customer take a purchasing action. What lie between attention and action may be more than just interest and desire. This motivates many proposals that introduce more stages in the AIDA model [5, 117], which unfortunately makes the model complicated. On the opposite direction, it may be possible to combine the middle two levels into a single middle level based on the concept of the hierarchy of effects [5, 6]. By following the labels of the cognition/affect/conation (behaviour) hierarchy of effects [5, 6], corresponding to thinking/feeling/doing, I may suggest labeling the new middle level as the affect level. In this way, there is a simple attention/affect/action 3A pyramid of marketing.

Concentric tricircles Trilevel hierarchies and trilayer pyramids provide us with a view of a triad as a stack of or a stratum of three things. They are appropriate for a top-down and bottom-up understanding of a triad. In some situations, we want to consider three things according to a spatial metaphor of internal versus external or a container metaphor of containment. A circle with multiple layers or a family of concentric circles may serve this purpose.

Figure 6c is a trilayer circle characterized by an inner layer, an outer layer, and an in-between middle layer. Reading outwards, the inner layer determines or supports the middle layer and the middle determines or supports the outer layer. Reading inwards, the outer layer builds on the middle layer and the middle layer builds on the inner layer. The sizes of the three layers of this onion structure immediately offer two interpretations. The small-to-large sequence indicates a kind of growth from the small internal cores to large external shells. The large-to-small sequence indicates containment of a smaller layer by a larger layer. I use three examples to show the values of the trilayer circle metaphor.

The first example is the Golden Circle leadership model introduced by Sinek [109], which is given in Fig. 6f by adding a solid line and a dashed line. As indicated by the three layers of the Golden Circle, every organization and everyone of us should know three most important things: What we do, how we do, and why we do. In some sense, the sizes of three circles provide hints on our grasp of the three: We all know WHAT, some of us know HOW, and only very few of us can clearly articulate WHY. What differentiates great leaders and us ordinary people is that great leaders start from WHY and think, act, and communicate from the inside out, as indicated by the solid line in the figure. That is, a powerful and effective pattern of thinking, acting, and communicating is WHY-to-HOW-to-WHAT: Why we do leads to how we do, and how we do leads to what we do. On the other hand, many of us normally start from WHAT and seldom move into WHY, as indicated by the dashed line in the figure. The Golden Circle with three layers and two directions of movement provide a way leading to great leadership.

The second example is a recycle of the Golden Circle by Clear [21] in his three level model of behavior change. Corresponding to the trilayer circle of Fig. 6c, Clear's three labels of the three layers are Identity, Processes, and Outcome. The inner layer is about who we are, concerning changes of our identity as reflected by our beliefs, worldview, self-image, judgments of self and others. The middle layer is about how we do, concerning changes of our processes, namely, changing our habits and systems. The outer layer is about what we get, concerning changes of our outcomes, namely, the results of changes. The identity, processes, and outcomes layers are, respectively, about what we believe, what we do, and what we get. To build habits that last, we must be willing to make changes at all three layers. Changes at the three layers, although of different nature, are all useful. What makes the trilayer circle valuable is the two directions in which changes are made [21]. We can use Fig. 6c to explain Clear's model. The dashed line indicates an outcome-directed way to build habits. Most of us work this way by focusing on or starting with what we want to achieve. The solid line indicates an identity-directed way to build habits by focusing on who we wish to be. An internal search of the root of who we are will help us to know how we do, and knowing how we do will bear the fruits of what we get. For habits building, we should always focus on becoming the type of person, instead of on getting a particular outcome.

The third example is the commonly used trilayer circle description of a computer system by labeling three layers as hardware, system software, and application software. System software is supported by the hardware and is built on the top of and around the hardware; application software is supported by the system software and is built on the top of and around the system software.

4.1.5 Triad as a triangle

In trisegment lines and trilevel structures, two elements of a triad are not directly linked, but are connected through the third element. When a direct link between any pair is useful, we need to consider a triangle representation and interpretation of a triad, as given by the middle triangle (A, B, C) in the TAO model of Fig. 1. We can easily obtain a triangle from a trisegment line of Fig. 5 by lifting the middle point and connecting the two end points. In a triangle based interpretation of a triad, the triangle represents a

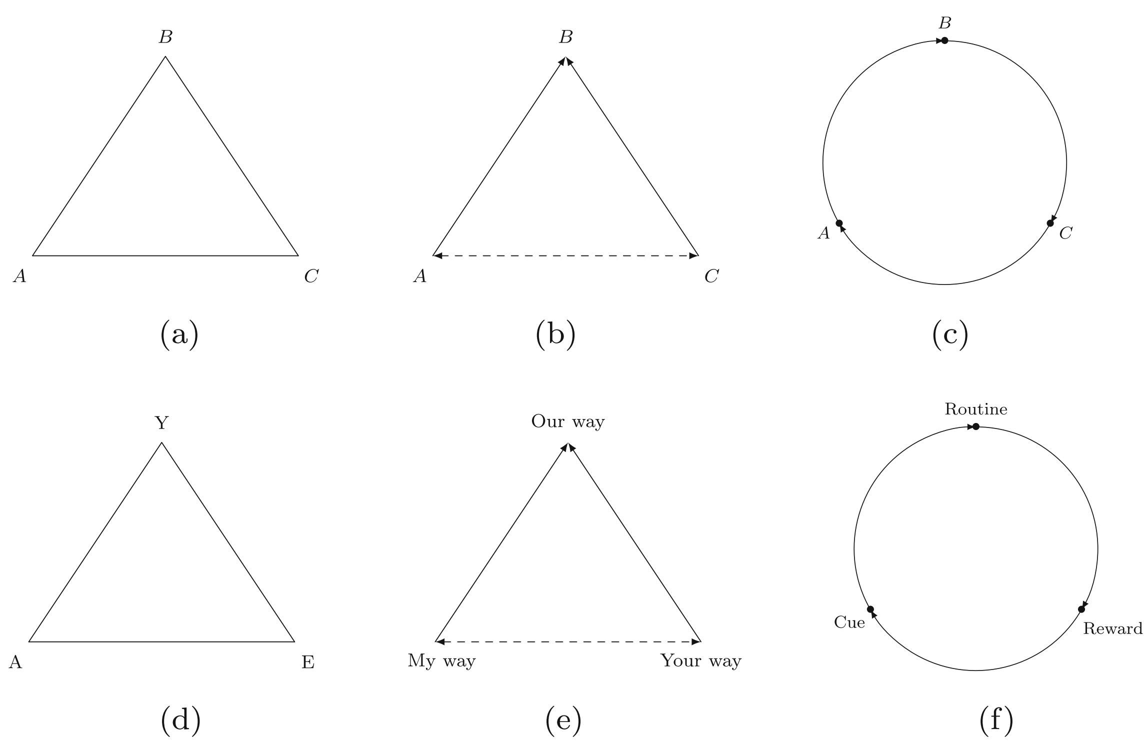


Fig. 7 Trisections as triangles

whole and the three corners represent the three elements of a triad. The geometric representation of a triad by a triangle is the most common one and appears everywhere. As examples for illustration, I will discuss three types of triangles in Fig. 7.

Three-way decision is about thinking based on a set or triplet of three elements. In Fig. 7a, we interpret the triangle as representing a whole, the three corners of the triangle as the three elements of a triad, and the three edges as the pairwise connections of the three elements. In the triangle, every element is connected to the other two elements, showing their dependencies, associations, and supports. If we consider only one edge, we see the relationship of two elements. For example, the edge (A, B) links A and B together. If we consider any two edges together, we see a direct relationship of the shared element with the two unshared elements, but not a direct relationship of the two unshared elements. For example, if we consider edges (A, B) and (A, C) together, A becomes the center of attention and may be explained in the contexts of B and C, but the relationship of B and C is not present directly. If we consider all three edges, each of the three corners appears exactly twice. Two types of relationship appear, a direct relationship given by an edge and an indirect relationship given by two edges through a third element. For example, for the pair (A, B), we have the direct relation (A, B) and an indirect relation through C, namely, (A, C) and (C, B). It is fair to say that a triangle interpretation of a triad offers us many possibilities to explore three-way decision.

For a more concrete triangle-based interpretation, I use Béziau's [8] abstract A-E-Y triangle of contrariety given in Fig. 7d. The corners of an A-E-Y triangle represent three states used to describe conditions, situations, or positions of something at specific times. Furthermore, the two states A and E represent opposite extremes and the third state Y represents somewhere in the middle. Typically, the three states are pairwise exclusive in the sense that at any time the thing must be in one of the three states. The abstract A-E-Y triangle has many interpretations when we attach specific meanings to the three corners. For example, in the quantificational triangle of contrariety, symbols A, E, and Y are interpreted, respectively, as All (i.e., universal affirmatives), None (i.e., universal negations), and Some but not all. Table 2 summarizes more examples of various interpretations of triangles of contrariety [8].

A triangle may be made more useful in three ways: a) introducing different types of edges through various line styles and colors, b) adding arrows to edges, and c) attaching annotations to lines. Figure 7b is an example of an arrowed triangle with two types of edges. I can use this triangle to explain Covey's [22] powerful notion of the 3rd alternative. As shown by Fig. 7e, the 1st Alternative is "My way" and the 2nd Alternative is "Your way." These two ways typically are in conflict of each other, as shown by the double arrowed dashed bottom edge of the triangle. By synergizing, we raise up to the 3rd Alternative of "Our way," as indicated by the two single-arrowed solid edges in the figure. This 3rd Alternative is a reconciliation of two competitive ways, leading to a higher and superior way to resolve the conflict. Martin [78] considered similar ideas and introduced the notion of integrative thinking. He defined integrative thinking as the "ability to face constructively the tension of opposing ideas and, instead of choosing one at the expenses of the other, generate a creative resolution of the tension in the form of a new idea that contains elements of the opposing ideas but is superior to each." Covey's 3rd alternative and Martin's integrated new idea are related to but different from the earlier discussed notion of middle way. The middle way is typically more about moderation and aversion of extremes, staying in the middle of two opposites. The two interpretations of a moderate middle and a new integrated or synergized middle provide a fuller

Table 2 Triangles of contrariety

Name	A	Y	E
Quantificational triangle of contrariety	All	Some	None
Space triangle of contrariety	Everywhere	Somewhere	Nowhere
Time triangle of contrariety	Always	Sometimes	Never
Alethic triangle of contrariety	Necessary	Contingent	Impossible
Deontic triangle of contrariety	Obligatory	Allowed	Prohibited
Directional triangle of contrariety	Left	Center	Right
Punctual triangle of contrariety	In advance	On time	Late
Moral triangle of contrariety	Bad	Right	Good
Ordering triangle of contrariety	$>$	$=$	$<$
Sign triangle of contrariety	$+$	0	$-$
Intensity triangle of contrariety	Low	Medium	High

picture of the notion of a third way in addition to commonly used two ways, forming a basis of three-way decision.

Consider now a particular arrowed triangle in which three arrowed edges form a circle. For example, A points at B, B points at C, and C points back at A. In other words, the three elements in a triad form a circular sequence or have a kind circular dependency. A circle figuratively give a sense of completeness. In this case, a circular triangle may be drawn as a trisegment circle as shown in Fig. 7c. As a concrete example, Fig. 7f depicts Duhigg's [30] cue/routine/reward habit loop. The three-step loop explains how our brain work: A cue triggers a habit and leads to a physical, mental, or emotional routine, the result of the routine produces a reward, and the loop starts again with a new cue. The continuous many repetitions of the loop give birth to habits.

4.1.6 Triad as a basis of a three-dimensional space

A three-dimensional (3D) space provides a formal and visual method to describe spatial information. A set of three linearly independent vectors forms a basis of a three-dimensional space so that any point in the space can be produced by a linear combination of the three basis vectors. In this way, a basis is the simplest characterization of the space. Due to the fact that we live in a 3D space, the 3D metaphor is universally used in different cultures and disciplines. For this reason, we may treat a triad as a basis of a 3D space, from the triad the whole space can be generated. That is, a triad consists of three basic elements from which all other elements can be produced. This interpretation of a triad with a generative power offers a new perspective on three-way decision as 3-dimensional thinking.

Figure 8a describes the 3-dimensional vector space with an orthogonal basis $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$, where $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ are three unit vectors along the three axes. A point p in the space is represented by a triplet (a, b, c), where a, b, and c are the projections of p on the three axes x, y, and z. In other words, p is composed by a units of x, b units of y, and c units of z, the vector representing p is written as $a\mathbf{x} + b\mathbf{y} + c\mathbf{z}$. The triplet is also referred to as the components of the vector defined by p. If we also use a, b, and c to label three points on the three axes, we immediately have a tripod (p, a, b, c) with p as the top and a, b, c as three legs, as shown by the dotted lines in Fig. 8a. This metaphor of decomposing one into three components or combining three components into one explains the underlying principle of 3-dimensional thinking, which is one mode of three-way decision. To further illustrate 3D thinking, I use two more examples to make it three.

The concept of guna is one of the key ideas in the Hindu worldview [72]. The word guna literally means a cord, a thread, a strand, or a rope, indicating that everything in the world is bound by gunas [116]. As an embracing notion, translations of the word guna are varying and in many, including quality, peculiarity, attribute, property, mode, merit, virtue, and so on [49, 72, 116]. The triple gunas, as primary qualities, modes of nature, or driving forces, are sattva (i.e., goodness, purity, light, harmony, superiority, etc.), rajas (i.e., passion, activity, motion, ambivalence, etc.), and tamas (i.e., darkness, decay, inertia, inertness, inactivity, inferiority, etc.) [19, 49, 72, 116]. With reference to the positive-neutral-negative triad, we may similarly explain the triple gunas: a) sattva is positive and is associated with goodness, truth, wholesomeness, health, etc., b) tamas is negative and is associated with darkness, ignorance, death, etc., and c) rajas can be either positive or negative, depending on a particular context [72].

Figure 8b labels the three dimensions in a 3D space by the triple gunas. According to the 3D thinking metaphor, in a similar that we define any point in the 3D vector space by three numbers, we can characterize all things and beings in the world by the three basic qualities of sattav, rajas, and tamas, in terms of their proportions, dispositions, and compositions. All things have mixtures of these three gunas with differing proportions. A harmony is a right proportion of the three gunas and an unbalanced state of being is a disproportion of the three gunas. The use of three gunas, rather than other numbers, provides another piece of evidence in support of three-way decision as thinking in threes. In passing, for the strength of triadic thinking, the

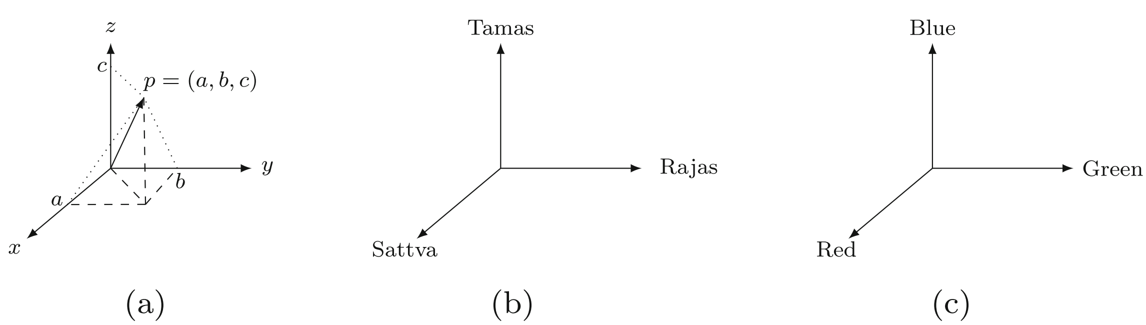


Fig. 8 3-dimensional spaces

combination of the literal meaning of the word guna (i.e., a strand or a cord) and the number three also reminds us of the metaphor that "a cord of three strands is not easily broken."

Colors play an important role in our everyday living and communications. Thinking in three colors is a very common practice. The triple gunas are represented as white (sattva, brightness), red (rajas, emotion and unrest), and black (tamas, darkness) [116]. A red-yellow-green traffic light tells us to stop when it is red, to proceed when it is green, to prepare to stop or to proceed if safe when it is yellow. In project management, the same tricolor system indicates that a project is ahead of schedule (i.e., green), is behind schedule (i.e., red), and needs attention (i.e., yellow). When thinking in three colors, we may decorate each item in a triad by a different color.

Instead of using three colors, there are several models for generating or representing various colors by using three primary additive colors [111]. For example, Fig. 8c is an illustration of a 3D RGB space based on the three primary colors, Red, Green, and Blue. Similar to an expression of a point in the 3D space, a color is a convex combination of the three primary colors, that is, a mixture of the three primary colors in a certain proportion. Although the 3D color space allows for an infinite number of possibilities, in practice we typically consider only a finite number of colors. In the RGB color model, combinations of pairs of primary colors of equal intensity produce three secondary colors: cyan (combination of green and blue), yellow (combination of red and green), and magenta (red and blue). A combination of a primary color and one of its two adjacent secondary colors of equal intensity produces a tertiary color. With the three primary colors and the three secondary colors, there are six tertiary colors. Although further combinations are possible, these first three levels of 12 colors are often sufficient.

The trichromatic theory of color vision is built and explained based on three types of long- (L), middle- (M), and short- (S) wavelength-sensitive cones in the retina [111]. It may be viewed as another example of thinking in threes. The common practice of using three colors for visual representation, 3D color spaces, and the trichromatic theory of color vision offer additional insights into three-way decision from a new color perspective.

4.2 Preference orderings of the threes

Discussions so far have been focused on the representations and interpretations of a triad based on relationships of the three things in the triad. When applying strategies to a triad to achieve a desirable outcome, we sometimes need to consider another type of relationships that reflect our preferences on the three things. As indicated in Fig. 9, there are three basic categories of preferences [134]: 1) unordered three (e.g., Fig. 9a), 2) partially ordered three (e.g., Fig. 9b–d), and 3) totally ordered three (e.g., Fig. 9e). Preference orderings of the three tell us how to prioritize our attention in the acting step of the TAO model of three-way decision, particularly when time and resources are scarce.

A preference ordering of the three things of a triad may be the same as or different from other structures discussed earlier. A triad is therefore jointly characterized by a relational structure (e.g., a trilevel hierarchy) and a preference ordering (e.g., a partially-ordered three). Triadic thinking depends on both structural information and preferential information of the three things in the triad.

When three things of a triad are of equal importance or it is not meaningful to impose a preference ordering, we will represent the preference ordering simply as three dots, as shown in Fig. 9a. For example, it is very common to describe the study of a field from philosophy, theory, and practice three perspectives. One may argue that theory is guided by its underlying philosophy and practice is guided by theory, forming a trilevel hierarchy in a structural description. However, it may be unwise to argue which one is more important and an unordered three seems to be more reasonable.

There are three possible types of partially-ordered threes. Figure 9b shows that, among three things, one is preferred to another one, and both of them are not compared with the third. Although we divide a whole into three parts, it may be only necessary to process two of them one after the other. For example, in medical decision-making we may divide a group of people into three classes with respect to a disease: (i) those who have the disease, (ii) those who possibly have the disease, and (iii) those who do not have the disease. For designing treatment and monitoring

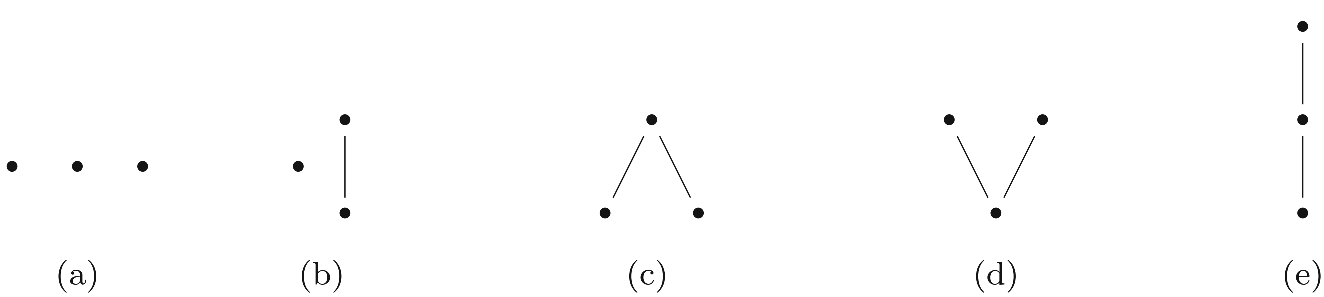


Fig. 9 Example configurations of partially-ordered and totally-ordered threes

plans, (i) is ordered before (ii) and (iii) is not taken into consideration. Figure 9c shows that one of the three is preferred to other two. The philosophy of the middle-way is a typical example of this structure. We prefer the moderate middle to the two opposite extremes. In many tests in medicine, a middle value indicates normal and a too low or a too high value indicates abnormal. The normal is preferred to abnormal. Figure 9d shows that two of the three are preferred to the third, which is the reverse of the Fig. 9c. Consider again the earlier medical decision-making example. Groups (i) and (iii) are associated with certainty or less uncertainty and group (ii) is associated with more uncertainty. Decision-making under certainty is easy and decision-making under uncertainty is difficult. From the point of view of uncertainty, we may prefer both (i) and (iii) to (ii). We humans generally prefer knowing, whether the result is positive or negative, to not knowing.

Figure 9e shows a totally ordered three, which is the opposite case of an unordered three. In this case, processing is easy, that is, one after another in a sequence. A totally ordered three in many situations is the same as the trilevel structure. For Marr's three levels, it is naturally to work in the sequence of computational theory, representations and algorithms, and hardware implementations. With reference to Sinek's Golden Circle, it is more effective to address WHY, HOW, and WHAT questions sequentially.

The three categories of unordered threes, partially ordered threes, and totally ordered threes also form a triad of preference structures, from non-preference to strong preferences. The two opposite ends of unordered threes and totally ordered threes are relatively simple and easy to work with, the middle of partially ordered threes is more complex and offers more ways to work with. We may draw a correspondence between unordered threes and totally ordered threes to chaos and order. Then the partially ordered threes lie at the edge between chaos and order. The three states may be metaphorically described as the shapeless gas, flowable liquid, and rigid solid, in which complexity, interesting things, and innovations happen in the liquid middle, namely, the edge of chaos [52]. This might suggest that three-way decision with partially ordered threes may produce useful models, as the cases of the middle way [77], Covey's 3rd alternative [22], and Martin's integrated third idea [78].

5 Explanations of richer structures based on triadic thinking

In this section, I will show, through three examples, that it is possible to construct and to explain more complex geometric structures by using the basic structures of three-way decision discussed in the last section.

5.1 Hexagon of a tripartition for set-theoretic three-way decision

Set-theoretic models of three-way decision are formulated and explained based on trisections of a universe of objects [140]. Depending on the properties of the three subsets in a trisection, it is possible to study different models. In this section, I only consider a tripartition based model.

Suppose that U is a set called the universe of objects. Three subsets $A, B, C \subseteq U$ form a trisection of U, written as $\langle A, B, C\rangle$, if they satisfy the property:

(i) $A \cup B \cup C = U$.

That is, the union of the three subsets covers the universe. A tripartition of U is a trisection of U that satisfies the following properties:

(ii) $A \neq \emptyset, \quad B \neq \emptyset, \quad C \neq \emptyset,$
(iii) $A \cap B = \emptyset, \quad A \cap C = \emptyset, \quad B \cap C = \emptyset.$

Property (ii) requires that all three subsets are nonempty. Property (iii) states that the three subsets are pairwise disjoint. A tricovering of U is a trisection of U and satisfies properties (i), (ii) and

(iv) $A \neq B, \quad A \neq C, \quad B \neq C.$

Property (iv) states that the three subsets are pairwise different. Given property (ii), property (iii) implies (iv). Thus, a tripartition is a special case of a tricovering.

For a tripartition $\langle A, B, C\rangle$, by taking the set complement of each subset, we have another trisection $\langle A', B', C'\rangle$, where $A' = U - A$ denotes the complement of A and so on. By the properties of a tripartition, the following additional relationships hold [8]:

(1) $A' = B \cup C, \quad B' = A \cup C, \quad C' = A \cup B,$
(2) $A = B' \cap C' \neq \emptyset, \quad B = A' \cap C' \neq \emptyset, \quad C = A' \cap B' \neq \emptyset,$
(3) $A \subseteq B', \quad A \subseteq C', \quad B \subseteq A', \quad B \subseteq C', \quad C \subseteq A', \quad C \subseteq B',$
(4) $A' \cup B' = U, \quad A' \cup C' = U, \quad B' \cup C' = U,$
(5) $A' \neq B', \quad A' \neq C', \quad B' \neq C'.$

It follows that the trisection $\langle A', B', C'\rangle$ is a tricovering of U.

We adopt the notion of a set-theoretical hexagon from Béziau [8], Dubois and Prade [28], and Ciucci et al [17] to describe a hexagon of three-way decision. By using the six subsets in the tripartion $\langle A, B, C\rangle$ and the tricovering $\langle A', B', C'\rangle$ as corners of a hexagon and their relationships as edges, we arrive at the hexagon of three-way decision, as depicted by Fig. 10. A dashed line connects two disjoint sets, a dotted line connects a set and its complement, a solid line connects two sets with a nonempty intersection, and an arrowed solid line connects a pair of sets such that one is

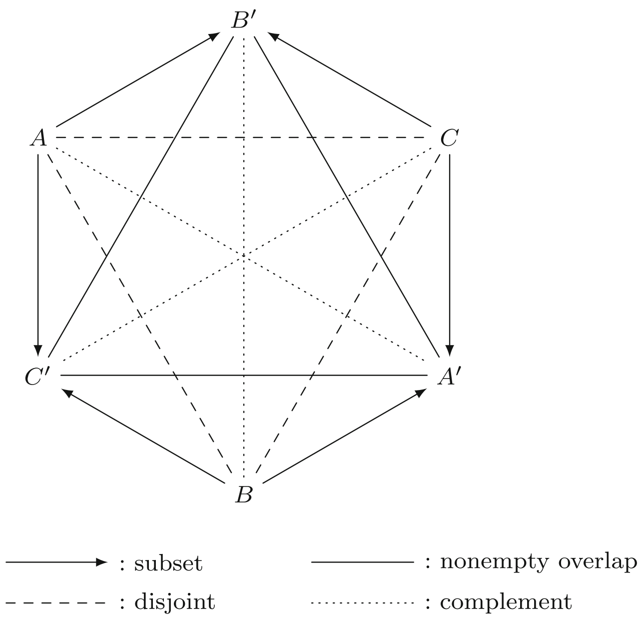


Fig. 10 Hexagon of a tripartition of a universe

a subset of the other. Triangles in the hexagon are useful triads.

The tripartition $\langle A, B, C\rangle$ is the middle triangle formed by three dashed lines. A tripartition can be equivalently defined by any pair of subsets from $\langle A, B, C\rangle$, as the third subset can be defined as the set complement of the union of the other two. For example, $\langle A, B, C\rangle$ can be equivalently written as $\langle A, B, (A \cup B)'\rangle$. Thus, a tripartition is related to the notion of an orthopair (i.e., a pair of disjoint sets) proposed by Ciucci [15]. For example, $\langle A, B, C\rangle$ can be equivalently defined by an orthopair $\langle A, B\rangle$. The tricovering $\langle A', B', C'\rangle$ is the middle triangle formed by three solid lines. According to property (4), any two subsets from the tricovering $\langle A', B', C'\rangle$ are sufficient to cover the universe U; moreover, the two subsets determine the third subset. For example, $A' \cup B' = U$ and $C' = (A' \cap B')'$. It is sufficient to define the tricovering $\langle A', B', C'\rangle$ by giving only two subsets from it.

There are six triangles corresponding to the six corners. The three triangles, with each formed by three solid lines, give three tricoverings $\langle A, B', C'\rangle$, $\langle A', B, C'\rangle$, and $\langle A', B', C\rangle$. In each of these three tricoverings, the two subsets with set complement operator $'$ are sufficient to cover the universe and the third subset is the intersection of these two subsets. For example, in the tricovering $\langle A, B', C'\rangle$, $B' \cup C' = U$ and $A = B' \cap C'$. The other three triangles, with each formed by two solid lines and one dashed line, do not produce tricoverings.

There are three rectangles in the hexagon: (A, B', A', B), (A, C, A', C'), and (B, C', B', C). From a rectangle, by combining two adjacent edges and one of the two diagonal lines, we can obtain four triangles, giving rise to four tricoverings. For example, from the rectangle (A, C, A', C') we have four tricoverings: $\langle A, C, C'\rangle$, $\langle A', C, C'\rangle$, $\langle A, A', C\rangle$, and $\langle A, A', C'\rangle$. There are a total of 12 such tricoverings. Each of the 12 tricoverings is defined by a subset from the tripartition $\langle A, B, C\rangle$, the complement of the subset, and a second different subset from $\langle A, B, C\rangle$ or the complement of the second subset. For example, tricovering $\langle A, C, C'\rangle$ consists of subset C, C's complement C', and a second subset A; tricovering $\langle A', C, C'\rangle$ consists of subset C, C's complement C', and A' (i.e., the complement of a second set A). In other words, each of these tricoverings can be simply characterized by a pair of complementary subsets linked by a dotted line in the hexagon in the context of a second subset.

The hexagon of a tripartition and triangles in the hexagon suggest different patterns and structures that may be used for modeling three-way decision. One can construct similar hexagons for other types of trisection of a universe. These hexagons have been used to connect three-way decision with different data analysis methods such as rough set analysis and formal concept analysis [18, 28, 138].

5.2 Sequences and cycles of the eight trigrams

The notion of Yinyang is deeply rooted in Chinese thought and culture. Wang [120] presented a very comprehensive analysis and discussion of the richness and multiplicity of the meanings and applications of Yinyang. In this paper, I examine specific ways of triadic thinking based on a very basic understanding of Yinyang. The focus is on triads embedded in or implicitly used in Yinyang pairs. I hope that such an examination of a triadic understanding of Yinyang may offer something new to the rich and multifaceted concept of Yinyang.

5.2.1 A three-step construction process of the eight trigrams

Yinyang represents a pair of mutually dependent, complementary, and transformable two opposites. Yin is negative, dark, and feminine; Yang is positive, bright, and masculine. Everything in the world contains or is explainable in terms of its Yin and Yang. Yin and Yang co-exist and one cannot exist without the other. The concept of Yinyang itself has its two sides. It is simple, straightforward, and earthly on the one hand, and complex, subtle, and heavenly on the other. The interaction, integration, and harmony of Yin and Yang give rise to Zhong (i.e., middle) or He (i.e., harmony). In this way, we have a triad consisting of Yin, Yang, and Zhong, with Zhong embedded in Yin and Yang as a middle position, a harmonized third state, a boundary of Yin and Yang. Examples of the (Yin, Yang, Zhong) triad are common appearances in Chinese thought and culture. For example, Yang represents the heaven, Yin represents the earth, and we humans in the middle connect the heaven and earth. Father symbolizes Yang, mother symbolizes Yin, and children in the middle tie together father and mother. Hot is Yang, cold is Yin, and the middle is a more comfortable warm zone. Mixing of Yin and Yang would produce Zhong, in the same way that mixing cold and hot water would produce warm water.

By combining Yin and Yang in structured ways, it is possible to produce useful patterns. Fig. 11 shows the construction of eight trigrams by a three-stage combination process of Yin and Yang, which is an example of thinking in threes. In the figure, a broken line represents Yin and an unbroken line represents Yang. We assume that the upwards direction represents the direction of the heaven (i.e., Yang) and the downwards direction represents the the direction of the earth (i.e., Yin). As we move up, we increase the amount of Yang and decrease the amount of Yin, and as we move down, we increase the amount of Yin and decrease the amount of Yang. Furthermore, in each step we add either a Yin line or a Yang line on the top of an existing pattern to form a new pattern.

In the first step, moving up from Taiji (the Great Ultimate) produces Yang, and moving down produces Yin, and the pair is called the Liangyi (two modes). In the second step, adding Yin and Yang lines on the top of each of the two modes produces Sixiang (four images). Adding a Yang line is equivalent to moving up and adding a Yin line is equivalent to moving down. For example, Greater Yang (i.e., (Yang, Yang)) is the result of moving up, which produces more Yang. The naming system of the first two levels, i.e., Yang, Yin, Greater Yang, Lesser Yin, Lesser Yang, and Greater Yin, suggests that the newly added Yin or Yang line has a higher weight. For example, by adding a Yin line on the top of Yang produces a Lesser Yin, in which Yin is dominating. Finally, in the third step, adding Yin and Yang lines on the top of each of the four images produces Bagua (eight trigrams). The eight trigrams are referred to as Qian (Heaven), Dui (Marshes), Li (Fire), Zhen (Thunder), Xun (Wind), Kan (Water), Gen (Mountains), and Kun (Earth).

Qian consists of only Yang lines, is of Yang nature, and is labeled as Yang. Kun consists of only Yin lines, is of Yin nature, and is labeled as Yin. For the rest of six trigrams, if a trigram contains only one Yang line, it is of Yang nature and is labeled as Yang; if a trigram contains only one Yin line, it is of Yin nature and is labeled as Yin. Therefore, Dui, Li, and Xun are of Yin nature and their labels are Yin; Zhen, Kan, and Gen are of Yang nature and their labels are Yang. It

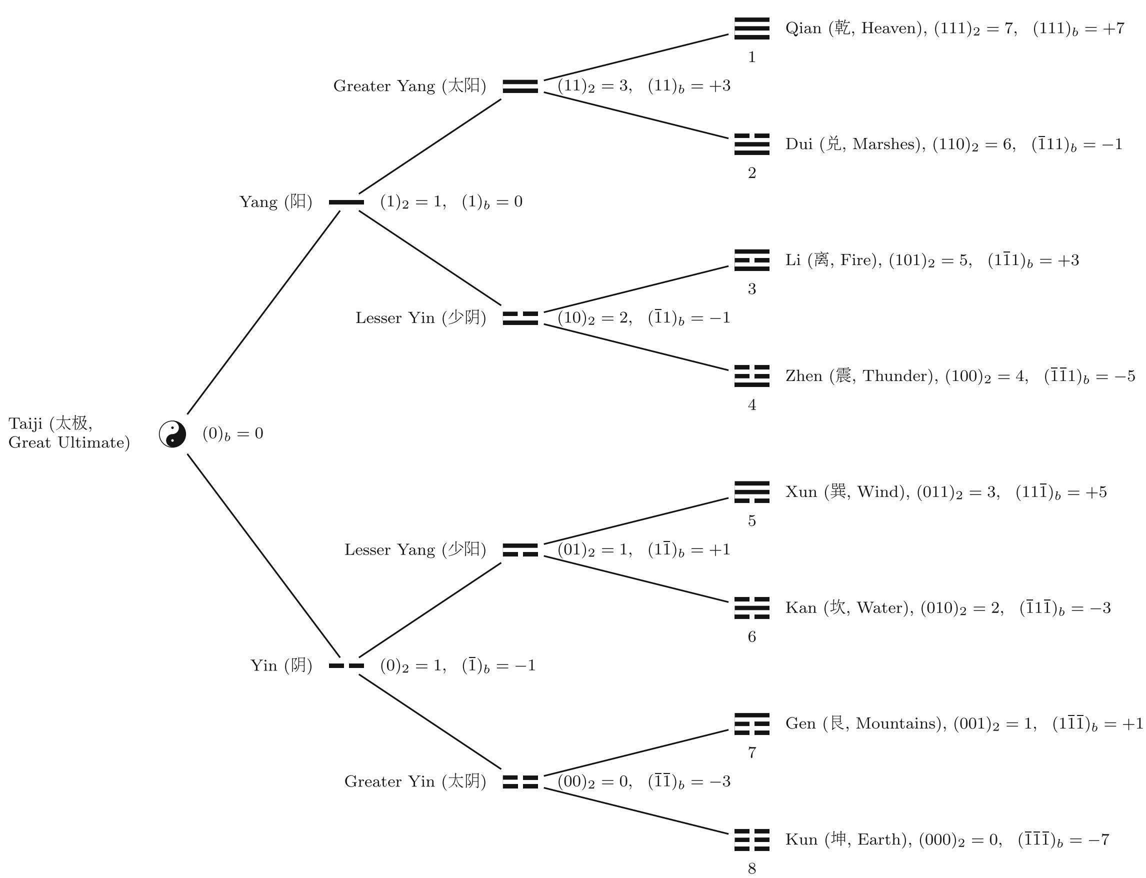


Fig. 11 Eight trigrams

is interesting to note that this rule of determining Yin/Yang nature/labels at the level three is not applicable to the levels one and two. The later added Yin/Yang lines do not have higher weights. This is also somewhat inconsistent with a common practice of interpreting the three Yin/Yang lines. Typically, eight trigrams offer a way of triadic thinking in which the top line is associated with heaven, the bottom line with earth, and the middle line with human beings. The three lines may have different weights.

The set of the eight trigrams consists of all possible trifold combinations of Yin and Yang. The eight trigrams are used as a means to categorize and study myriads of things. In the process of forming the archetypes of the eight trigrams, there have been many different ways to attaching meanings to them, including, for example, assigning numbers, connecting to natural objects and phenomena, and relating to spatial positions and directions.

5.2.2 Numerical interpretations of trigrams

In some specific sense, the eight trigrams simply serve a role of an index of categories of things. By using three lines, trigrams are easy-to-memorize and easy-to-grasp. As discussed earlier, the structure of a trigram is determined by both the number of Yin and Yang lines and their positions. It is therefore natural to attached numbers with their associated meanings to the eight trigrams.

By labeling the eight trigrams top-down with numbers 1 to 8 in Fig. 11, we have Qian-1, Dui-2, Li-3, Zhen-4, Xun-5, Kan-6, Gen-7, and Kun-8. The numbers are simply the positions of individual trigrams in the sequence and may be interpreted as the serial numbers of the trigrams. In Chinese numerical thinking, odd numbers are commonly known as Yang numbers and even numbers as Yin numbers. Unfortunately, the serial numbers of trigrams do not convey the information about the Yin/Yang nature of trigrams. For example, the serial number of Li is 3, but its nature is Yin instead of Yang.

It has been suggested that the eight trigrams can be viewed as a representation of binary numbers [88]. In Fig. 11, by interpreting Yin as 0 and Yang as 1 and reading a trigram bottom-up, we can produce a binary number as shown by $()_2$. Qian's binary number is $(111)_2 = 1 \times 2^2 + 1 \times 2^1 + 1 \times 2^0 = 7$, Dui's binary number is $(110)_2 = 1 \times 2^2 + 1 \times 2^1 + 0 \times 2^0 = 6$, and so on. In this way, we produce the same sequence of the eight trigrams, but according to the decreasing order of values, from 7 down to 0. Alternatively, we may interpret Yin as 1 and Yang as 0. Qian's binary number is $(000)_2 = 0$, Dui's binary number is $(001)_2 = 1$, ..., and Kun's binary number is $(111)_2 = 7$. The same sequence is obtained according to the increasing order from 0 up to 7. When constructing the binary number of a trigram bottom-up, the bottom Yin or Yang has a dominant contribution. This seems to be inconsistent with the assumption that the top Yin or Yang dominates. However, constructing binary numbers top-down by treating Yin as 0 and Yang as 1 will produce values 7, 3, 5, 1, 6, 2, 4, 0 for Qian, Dui, ..., and Kun, which gives a different sequence. In addition, it is difficult to assign a number to Taiji that is neither Yin nor Yang.

As mentioned earlier, a third Zhong is embedded in Yin and Yang to form a triad of (Yin, Yang, Zhong). The standard binary system may be insufficient for such a triadic thinking, as we have 0 and 1 for Yin and Yang and no symbol for Zhong. In addition, the obvious connotations of Yang for positive and Yin for negative are not explicitly present. To address these issues, I suggest a numerical interpretation of trigrams by using a lesser-known balanced binary number system [55, 104]. The system matches perfectly the triad of (Yin, Yang, Zhong) with three symbols, namely, $\bar{1} = -1$ for Yin, $1 = +1$ for Yang, and 0 for Zhong. Yin and Yang are clearly a pair of negative and positive numbers, and their sum is the $0 = (-1) + (+1)$ for Zhong. In Fig. 11, we use $()_b$ to denote a balanced binary number representation of a trigram. To be consistent with the assumption that the top Yin or Yang dominates, we construct a balanced binary number top-down. For example, for Qian, Dui, ..., to Kun, we have $(111)_b$, $(\bar{1}11)_b$, ..., to $(\bar{1}\bar{1}\bar{1})_b$. The same procedure for computing the decimal value of a binary number is used to compute the decimal value of a balanced binary number. For example, Qian is computed by $(111)_b = (+1)\times 2^2 + (+1)\times 2^1 + (+1)\times 2^0 = +7$, Dui is computed by $(\bar{1}11)_b = (-1)\times 2^2 + (+1)\times 2^1 + (+1)\times 2^0 = -1$, and so on.

With the balanced binary system, the $+$ and $-$ signs of the corresponding decimal values provide explicit and convenient way to show the Yin and Yang nature of a trigram, respectively. It is also possible to express the composition of Yin and Yang in each trigram. For example, Qian contains no Yin and only $+7$ units of Yang, Dui contains -4 units of Yin and $+3$ units of Yin and sum $(-4) + (+3) = -1$ shows the Yin nature of Dui, and so on. Unfortunately, the decimal values of Qian, Dui, ..., to Kun under the balanced binary number system do not confirm to the commonly used Yin/Yang labels of trigrams. On the other hand, it is possible to express Taiji as 0, which is the sum of Yin (-1) and Yang $(+1)$, -3 and $+3$, and so on. The complementary of Yin and Yang is explicitly present. Yin is -1 and Yang is $+1$ and their sum is 0 representing Taiji. A pair of trigrams is called an image complementary pair, if one is obtained from the other by flipping Yin lines to Yang lines and Yang lines to Yin lines. For the image complementary pair (Qian, Kun), Qian is $+7$, Kun is -7, and their sum is 0. For the complementary pair (Li, Kan),

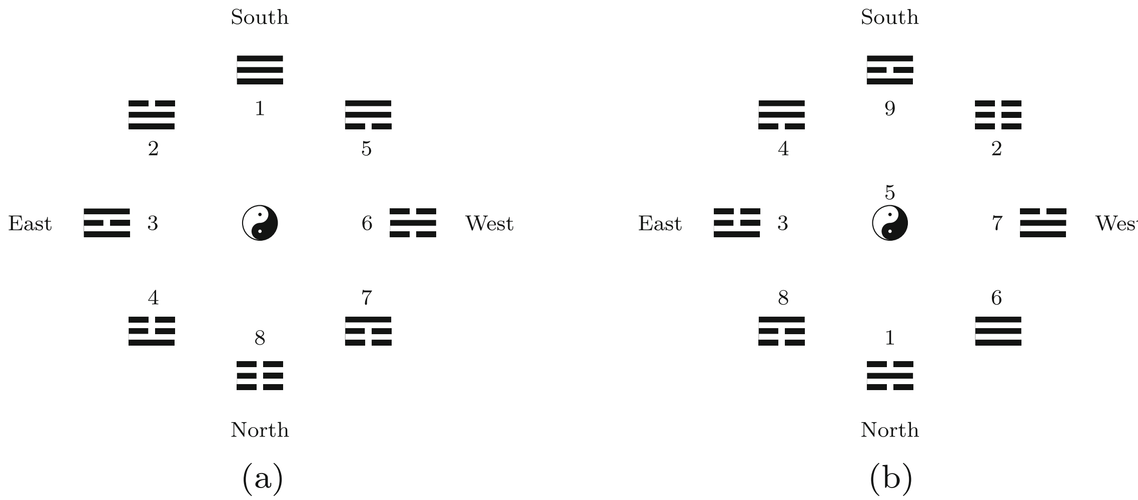


Fig. 12 Two cycles of the eight trigrams

Li is $+3 = (-2) + (+5)$, Kan is $-3 = (-5) + (+2)$, and their sum is 0, although their corresponding Yin/Yang nature/labels are reversed.

Everything contains a mixture of Yin and Yang and each of the four trigrams, Li, Gen, Zhen, Xun, contains both Yin and Yang lines. It might not be unreasonable to reinterpret the two image complementary pairs Li-Gen and Zhen-Xun regarding their Yin/Yang labels. For example, it might be unreasonable to interpret Li (fire) as Yang, instead of Yin. In this way, the serial numbers, in terms of even and odd, and the decimal values of the balanced binary numbers, in terms of negative and positive, would confirm to the Yin/Yang nature/labels of all eight trigrams. In addition, it would also confirm to the triadic understanding of the three lines in a trigram for representing heaven, human beings, and earth, with possibly different weights or importance.

5.2.3 Spatial interpretations of trigrams

There are two widely used circular arrangements or cycles of the eight trigrams, known as the Xiantian Tu (The Diagram of Before Heaven) and Houtian Tu (The Diagram of After Heaven) [87, 89, 114, 120]. To some extent, the two diagrams offer spatial interpretations of the eight trigrams, as shown in Fig. 12. Among many features and interpretations of the two cycles, it is possible to identify one of their most important functions, namely, a principle of and a way to positioning and orientation. In this aspect, triadic thinking can be observed. There are three levels for positioning and orientation, in a decreasing order of importance. The center of the cycle represents Zhong (middle, Taiji), which is the most important position. The four principal directions are east, south, west, and north. The secondary four directions are southeast, southwest, northwest, and northeast. Each of them is the middle of two adjacent principal directions. If we position ourselves at the center, the eight trigrams are our connections to the outside world. Alternatively, it is possible to label the four principal directions as left, upper, right, and lower, and the four secondary directions as upper-left, upper-right, lower-right, and lower-left.

Typically, the upper is superior and represents heaven and the lower is inferior and represents earth. Based on this understanding, there exists an easy way to explain and interpret the Before Heaven cycle by simply arranging the sequence in Fig. 11. As shown in Fig 12a, Qian consists of three Yang lines and takes the position of heaven, and Kun consists of three Yin lines and takes the position of earth. The three trigrams following Qian are put counterclockwise sequentially on the left half of the cycle, and the three trigrams preceding Kun are put counterclockwise in the decreasing order on the right half of the cycle. There are several interesting observations. Connecting the trigrams according to the increasing order of their serial numbers results in an S shape. The two trigrams of each of the four image complementary pairs are in the opposite sides, representing two opposite directions. Furthermore, the sums of their numbers are all 9. If the decimal values of the corresponding balanced binary numbers are used, the two trigrams have opposite signs and their sum is 0.

Other ways to construct and interpret cycles of the eight trigrams rely on two more diagrams known as Hetu (the River Diagram) and Luoshu (the Luo Writing), as shown in Fig. 13. In the two diagrams, black dots represent Yin (even) numbers and white dots represent Yang (odd) numbers. Hetu (Fig. 13a) is composed of ten numbers from 1 to 10, and Luoshu (Fig. 13c) is composed of nine numbers from 1 to 9. In both diagrams, number 5 is in the middle. The image of number 5 is a cross that is unique and very different from all other numbers; the images of other numbers are lines or

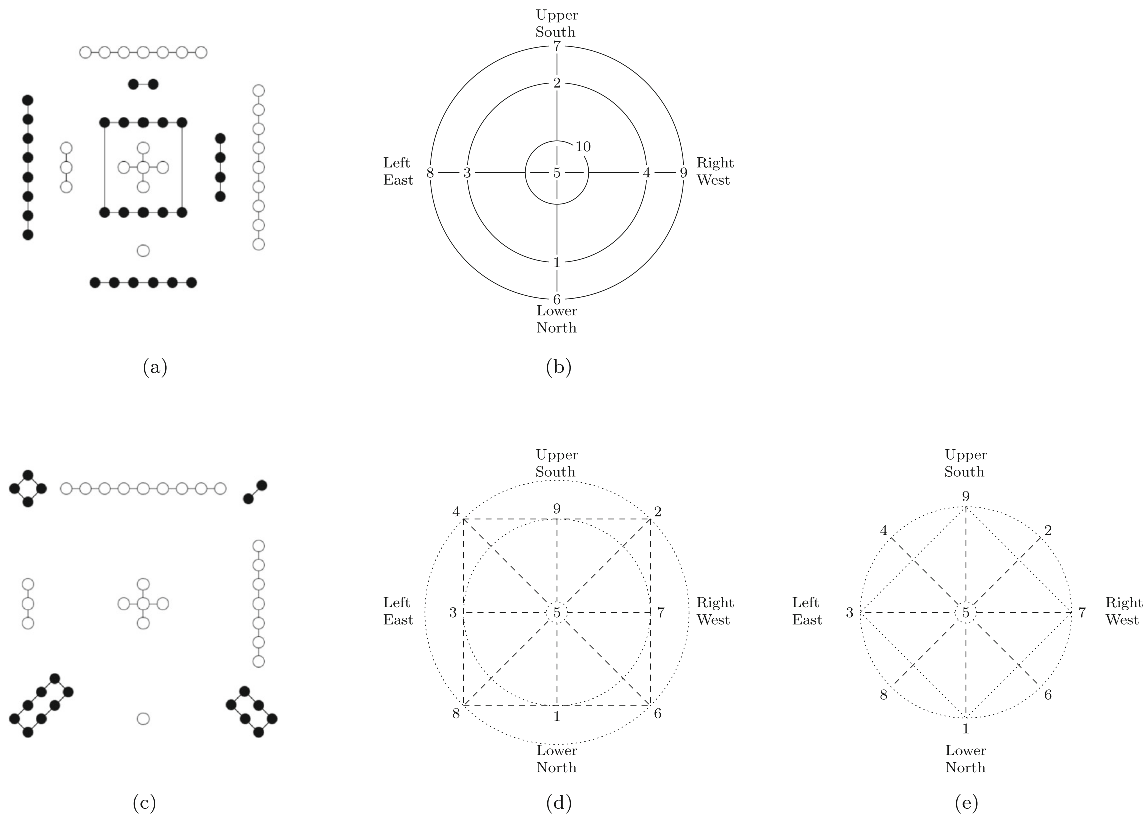


Fig. 13 Hetu and Luoshu

rectangles. The cross representation of number 5 captures truthfully five directions, namely, the center and the four principal directions. In fact, we can interpret the cross of five as two triads, one is the horizontal triad and the other is the vertical triad. That is, thinking in number 5 is a combination of two ways of triadic thinking. The horizontal triad may explain the principle of the middle way, emphasizing a superior position of the middle in an attempt to avoid biases towards either left or right. The vertical triad may explain our middle position for connecting heaven and earth.

Both Hetu and Luoshu may be interpreted based on concentric tricircles, as shown in Fig. 13. While a concentric tricircle is more obvious in Hetu [114], as indicated by Fig. 13b, for Luoshu a concentric tricircle is embedded in the magic square of Fig. 13d. In Fig. 13e, the middle circle is replaced by a square instead. In Hetu, the pair (5, 10) indicates the center, the four pairs, (3, 8), (2, 7), (4, 9), and (1, 6), denote the four principal directions. In order to denote the four secondary directions, we may rotate outer circle 45 degree either clockwise or counterclockwise. On the other hand, Luoshu directly provides the center and the eight directions. The number 5 is the center, 1, 3, 7, and 9 are the four neighbors for the principal directions, and 1, 2, ..., 9 are the eight neighbors for the principal and the secondary directions. This triad of a cell, four neighbors, and eight neighbors is commonly used in image processing and pattern recognition. In the magic square of Luoshu, if we add the three numbers horizontally, vertically, or diagonally, the sum is always 15. For each of the four lines passing the number 5, the sum of the two ending numbers is 10 and their average and the middle is 5. That is, the triad of three numbers clearly represents two opposites and a middle, which reflects the principles of the middle way.

To produce diagrams of the Before Heaven and After Heaven in Fig. 12, we may assign numbers according to Hetu or Luoshu. For example, the sequence of After Heaven according to Luoshu is given by Kan-1, Kun-2, Zhen-3, Xun-4, Zhong-5, Qian-6, Dui-7, Gen-8, Li-9. The result is the diagram of After Heaven in Fig. 12b. However, although a pair of trigrams of two opposite directions are number complementary with respect to number 10, i.e., their sum is 10, they are not necessarily image complementary. If we slightly change the number sequence of Before Heaven into Qian-1, Dui-2, Li-3, Zhen-4, Zhong-5, Xun-6, Kan-7, Gen-8, and Kun-9, then a pair of two trigrams for opposite directions is both number complementary (i.e., their sum is 10) and image complementary in the diagram of Before Heaven. If we are interested in building a simpler interpretation of a cycle of eight trigrams, it would be nice if we can make the three aspects of the eight trigrams to be consistent, namely, a numerical interpretation of the Yin/Yang nature of a trigram, the image complementary of pairs of trigrams, and the number complementary of pairs of trigrams.

A combination of two trigrams produces a hexagrams. The total of sixty-four hexagrams serve as the basic patterns and structures for interpreting Yijing (The Book of Changes). Each line in a hexagram has a specific meaning. In addition to thinking in two trigrams, the six lines in a hexagram are divided into three groups of two lines, offering another way of triadic thinking. Similar to the interpretations of a trigram, the positions of the lines in a hexagram have a great significance. For example, the bottom two lines associate with earth, the middle two lines associate with human beings, and the top two lines associate with heaven [120]. Thus, a hexagram is used as a trilevel hierarchy discussed in Section 4.1.4. Alternatively, the six lines in a hexagram can be read inside-out. The middle two lines represent the essential part or the central force of the hexagram, the second bottom and second top lines are, respectively, the middle of the lower and upper trigrams and represent the most desirable positions, and the bottom and the top lines represent, respectively, the beginning and the ending results. In this way, a hexagram is used similar to a concentric tricircle discussed in Section 4.1.4.

There are many views of the eight trigrams, the sixty-four hexagrams, and their sequences and cycles. In terms of these patterns and structures, there are roughly two schools of thought in interpreting the Yijing, namely, the school of images and numbers and the school of meanings and principles. Many existing studies, unfortunately, are not entirely consistent and compatible. To a certain degree, this has created some unnecessary complication and mystery of many fundamental concepts, for example, the eight trigrams, numerical interpretations of the eight trigrams, sequences and cycles of the eight trigrams. As a final note, I should point out that my brief discussion on a triadic understanding and interpretation of these basic concepts is not intended to untangle a whole complex web of many issues and should not be viewed as a search for a right solution. Instead, its main purpose is to bring our attention to triadic thinking involved. Following the earlier discussions on the third alternative and the third way, it is perhaps possible to integrate and combine the two schools of thought into a third school in which results from the complementary perspectives on numbers, images, meanings, and principles are made consistent under a holistic view. More thorough and systematic investigations of the third school of thought may be worthy of further efforts.

5.3 Enneagram

The enneagram, as shown in Fig. 14a, is a nine-pointed geometric figure that has many interpretations and serves many purposes, for example, as a representation of a spiritual worldview, a symbol of transformation, a tool for personal and spiritual reflection and growth, a system of personality typing, and many others [7, 32, 44, 86]. Based on the earlier discussion about the eight trigrams, it is also possible to study the enneagram from multiple angles suggested by its images, numbers, meanings, and principles. In this paper, I mainly look at the ways and modes of triadic thinking suggested by the enneagram. It may be appropriate to mention that the enneagram has received criticism. For example, the scientific validity of the enneagram as a system of personality typing has been questioned [32, 115]. Nevertheless, the power of the enneagram as a visual and metaphorical tool for organization, presentation,

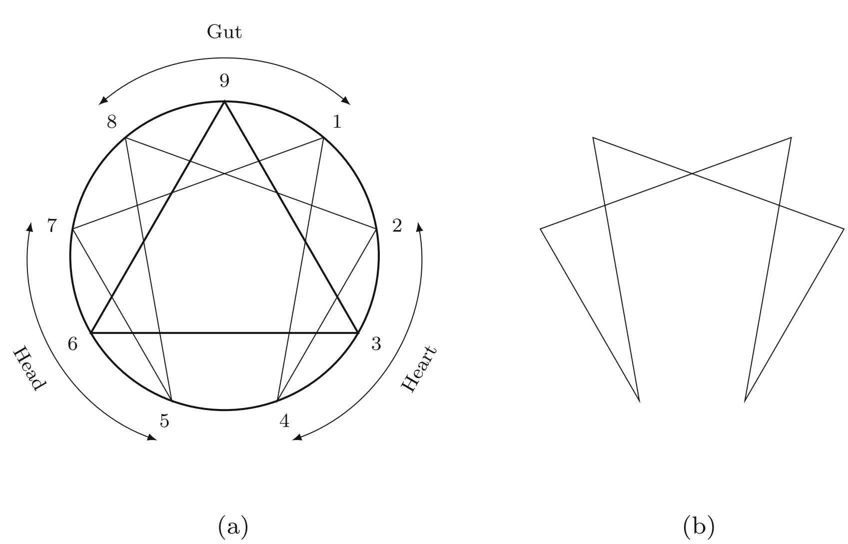


Fig. 14 The enneagram and the hexad

and reasoning is effective, powerful, and insightful. The enneagram describes patterns that help us to interpret the world and to manage our emotions in threes.

5.3.1 Triadic structures and triadic thinking in the enneagram

The enneagram itself may be viewed as a triadic structure consisting of three main components related to the three numbers 9, 3, and 7. The first component is a circle divided into nine parts of equal length, with one dividing point at the uppermost position. Starting with number 9 at the uppermost position, the nine dividing points are successively numbered from 9 down to 1 counterclockwise. The second component is the equilateral triangle that connects the three multiples of number 3, namely, 3, 6, and 9. The third component is a hexad, that is, a six-pointed figure in Fig. 14b, that connects the rest six numbers according to a specific order defined by number 7. If we divide numbers 1 to 6 by 7, the following sequence of repeating decimals of six repeating digits is produced:

$$
\begin{aligned}
1/7 &= 0.\overline{142857},\\
2/7 &= 0.\overline{285714},\\
3/7 &= 0.\overline{428571},\\
4/7 &= 0.\overline{571428},\\
5/7 &= 0.\overline{714285},\\
6/7 &= 0.\overline{857142}.
\end{aligned} \tag{3}
$$

It can be observed that the cycle of sequence of $1, 4, 2, 8, 5, 7$ is common in all six repeating decimals. The hexad in Fig. 14b is, in fact, constructed by connecting two adjacent numbers in the sequence of $1, 4, 2, 8, 5, 7$, which is structurally equivalent to a circle of the six numbers under the sequence. In summary, we can simply interpret the enneagram in term of three circles: A circle connecting nine numbers, a circle (represented by a triangle) connecting three numbers, and a circle (represented by a hexad) connecting the rest of six numbers.

The enneagram contains significant symbolism and its three components give rise to three principles and laws: The circle for the law of one or the law of wholeness, the triangle for the law of three, and the hexad for the law of seven [44]. A circle does not have a beginning nor an end, providing a metaphor of everlasting movement. The law of one emphasizes on both integration and division of a whole. The whole is divided into parts and the parts are integrated into the whole. It is interesting to note that number 9 is the largest single digit number, metaphorically representing a highest state and taking the uppermost position in the circle.

The law of three is thinking in threes. There are many interpretations and applications of this law. For example, there are three forces that guide everything in motion, that is, active, passive, and neutral forces [44], which has a close connection to the philosophy of the three gunas. Everything is created and works in accordance with a triadic form. According to Gurdjieff, a new arising can be born from existing ones in such way that "the higher blends with the lower in order to actualize the middle and thus becomes either higher for the preceding lower, or lower for the succeeding higher" [7]. A process of transformation typically requires three forces of affirmation, denial, and reconciliation. This reminds us of the philosophy of the middle way or the third alternative way.

The law of seven describes the path of movement of everything. A traversal of six numbers in the hexad starting from a number and coming back to the number vividly describes a typical path of how everything in the world develops. Suppose that we start with 1 and traverse the hexad in the increasing order of numbers. By considering the relative position of the next number relative to the current number, we have the following sequence:

lower-left, upper-right, upper-left, lower-right, upper-left, upper-right. (4)

The path is not a straight line, but consists of crisscrossed striving and failing of energies, as well as moving left and right, along the path. It might be worthy mentioning a kind of similarity to the earlier discussed construction process of the eight trigrams, in which moving up and down indicates, respectively, the increase of Yang and Yin, respectively.

5.3.2 Enneagram personality

One of the most popular applications of the enneagram is personality typing. I review briefly the basic ideas from Riso and Hudson [105] and Dueck [29], with an emphasis on the uses of triads. I examine an understanding of the nine types of enneagram personality at three levels.

At the first level, the nine numbers in the enneagram are grouped into three centers to represent three body or intelligence triads: (8, 9, 1) of the Gut (Instinctual, Body) triad, (5, 6, 7) of the Head (Intellectual, Thinking) triad, and (2, 3, 4) of the Heart (Emotional, Feeling) triad. The Gut triad is typically associated with anger, the Head triad with fear, and the Heart triad with shame. The middle numbers in the three triads, namely, 9, 6, and 3, are associated with the primary personality types that are usually blocked with instinct, thinking, or feeling. The remaining six numbers are associated with the secondary personality types that are somewhat more mixed and more connected with instinct, thinking, or feeling. The Enneagram's three triads therefore specify three fundamental psychological orientations.

At the second level, the nine basic personality types may be characterized individually. For example, Riso and Hudson [105] give the following descriptions:

2: the Helper (the encouraging, demonstrative, possessive type),
3: the Motivator (the ambitious, pragmatic, image-conscious type)
4: the Individualist (the sensitive, self-absorbed, depressive type)
5: the Investigator (the perceptive, cerebral, provocative type),
6: the Loyalist (the committed, dutiful, suspicious type),
7: the Enthusiast (the spontaneous, fun-loving, excessive type),
8: the Leader (the self-confident, assertive, confrontational type),
9: the Peacemaker (the pleasant, easygoing, complacent type),
1: the Reformer (the rational, idealistic, orderly type).

For each basic type, Riso and Hudson use only three basic and representative characteristics, instead of a full exhaustive list. This may be viewed as another example of the applications of the rule of three and the principles of triadic thinking.

At the third level, a basic personality type can be further explained with reference to two more basic types, forming again a triad of three basic types. Given a basic type, one of its two adjacent types on the enneagram circle is a wing of the type. While the basic type dominates, a wing may complement and add important, possibly contradictory, elements to the basic type. Given a triad consisting of a basic type and its two wings, there are three possibilities, that is, using one wing, using the dominant wing, or using both wings.

The frequent uses of triads in enneagram personality typing are perhaps not surprising, as they follow from the rule of three. As a tool and a metaphorical symbol, the enneagram with many embedded triads is useful, powerful, and instrumental. A fruitful direction of investigations is to pursue new scientific and wise uses of the enneagram.

6 Conclusion

The theory of three-way decision concerns thinking in threes, working with threes, and processing through threes. In this paper, by considering the geometry of three-way decision, I have made an effort to explore a new avenue of research. The results have given rise to a triad of three perspectives on three-way decision, namely, the visual, numerical, and textual perspectives. It may be commented that the triad of the three perspectives offers a powerful metaphor in a much wider context. The triad of three complementary perspectives has significant implications to, in general, our everyday living and to, in specific, the design and implementation of intelligent systems. The triad helps us to become a better integrative thinker by thinking visually, numerically, and textually. Effective communications, for example presenting an idea in a scientific paper, rely on a seamless integration of the three perspectives. The output from any intelligent system becomes more meaningful if it can be presented, explained, and communicated visually, numerically, and textually.

In the paper, I have intentionally used number three in many places, for example, a trilevel structure of sections, subsections, and sub-subsections, phrases of three words, three examples, and others. In fact, if one takes a closer look at any scientific paper, one would see the presence of triadic thinking. Many book titles have three words or contain three topics. We either consciously or subconsciously use threes. The goal of a theory of three-way decision is to systematically study what we have been doing and to search for a model of explanation.

In the introduction section, I have briefly discussed some implications of triadic thinking (i.e., three-way decision) to artificial intelligence. As future research, it is worthwhile to have a more in-depth study of this topic. Three-way decision is a human way and heuristics to approach complexity. It may also be used by intelligent systems. With human intelligence on one side and machine intelligence on the other, triadic thinking suggests their integration as the third alternative way. When a network of people and machines work together, collective and global intelligence may emerge. Finally, the geometric structures and patterns discussed in the paper may help us to expand the application domains of three-way decision, particularly in designing intelligent systems.

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