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Price of Anarchy for Highly Congested Routing Games in Parallel Networks

Riccardo Colini-Baldeschi[1] ·
Roberto Cominetti[2] · **Marco Scarsini**[1]

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Abstract We consider nonatomic routing games with one source and one destination connected by multiple parallel edges. We examine the asymptotic behavior of the price of anarchy as the inflow increases. In accordance with some empirical observations, we prove that under suitable conditions on the costs the price of anarchy is asymptotic to one. We show with some counterexamples that this is not always the case, and that these counterexamples already occur in simple networks with only 2 parallel links.

Keywords Nonatomic routing games · Price of Anarchy · Regularly varying functions · Wardrop equilibrium · Parallel networks · High congestion

1 Introduction

The study of network routing equilibria and their efficiency goes back to Pigou [24] who, in the first edition of his book, introduced his famous two-road example.

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✉ Riccardo Colini-Baldeschi
rcolini@luiss.it

Roberto Cominetti
roberto.cominetti@uai.cl

Marco Scarsini
marco.scarsini@luiss.it

1 Dipartimento di Economia e Finanza, LUISS, Viale Romania 32, 00197 Roma, Italy

2 Facultad de Ingeniería y Ciencias, Universidad Adolfo Ibáñez, Santiago, Chile

Wardrop [32] developed an equilibrium model where many players (vehicles on the road) choose a road in order to minimize their cost (travel time) and the influence of each one of them, singularly taken, is negligible. His concept of equilibrium has become the standard in the literature on nonatomic network games.

When drivers minimize their travel time ignoring the negative externalities imposed on other travelers, the collective outcome is typically inefficient, i.e., it is worse than the outcome that a benevolent planner would have achieved. Various measures have been proposed to quantify this inefficiency, among which the *price of anarchy* has been the most successful. Introduced by Koutsoupias and Papadimitriou [15] and given this name by Papadimitriou [22], it is the ratio of the worst social equilibrium cost and the minimum achievable cost.

The price of anarchy has been studied intensively and many interesting bounds have been established for different classes of cost functions. However, most of these results consider worst-case scenarios which need not be representative of practical situations. In a recent paper O'Hare et al. [20] show, both theoretically and with the aid of simulations, how the price of anarchy is affected by changes in the total inflow of players. Considering data for three different cities they have found that: *"In each city, it can be seen that there are broadly three identifiably distinct regions of behaviour: an initial region in which the Price of Anarchy is one; an intermediate region of fluctuations; and a final region of decay, which has a similar characteristic shape across all three networks. The similarities in this general behaviour across the three cities suggest that there may be common mechanisms that drive this variation."*

The core of the paper [20] is an analysis of the intermediate fluctuations. Here we will focus instead on the asymptotic behavior of the price of anarchy as the mass of players grows to infinity. We consider nonatomic congestion games with a single source and a single destination connected by multiple parallel edges. We show that for a large class of cost functions the price of anarchy is indeed asymptotic to one. Nevertheless, we also present counterexamples in which the lim sup is larger than 1 and it can even be infinite.

1.1 Contribution

The goal of this paper is twofold. On one hand we provide some positive results showing that under some conditions the price of anarchy for nonatomic parallel network games is indeed asymptotic to one. On the other hand, we present counterexamples where the lim sup of the price of anarchy is larger than one.

We first show that, for any single-source and single-destination network, the price of anarchy is asymptotic to one whenever the cost of at least one path is bounded. Then we focus on parallel graphs and we show that the price of anarchy is asymptotic to one for a large class of costs that we characterize in terms of regularly varying functions [3]. This class includes polynomial functions and functions that can be bounded by a pair of affine functions with the same slope.

Next, we present counterexamples where the behavior of the price of anarchy is periodic on a logarithmic scale, so that its lim sup is larger than one both as the mass of players grows unbounded and as it goes to zero. In another counterexample the lim sup of the price of anarchy is infinite. A further counterexample shows that the price of anarchy may not converge to one even for convex costs. An interesting point is that all the counterexamples concern a very simple parallel graph with just two edges, so that the bad behavior of the price of anarchy depends solely on the costs and not on the topology of the graph. This is in stark contrast with the results in [20], where the irregular behavior of the price of anarchy in the intermediate region of inflow heavily depends on the structure of the graph.

1.2 Related Literature

Wardrop's nonatomic model has been studied by Beckmann et al. [2] and many others. The formal foundation of games with a continuum of players came with Schmeidler [31] and then with Mas Colell [17]. Nonatomic congestion games have been studied, among others, by Milchtaich [18, 19].

Various bounds for the price of anarchy in nonatomic games have been reported under different conditions. In particular Roughgarden and Tardos [28] prove that, when the cost functions are affine, the price of anarchy in nonatomic games is at most $4/3$, irrespective of the topology of the network. The bound is sharp and is attained even in very simple networks. Specifically, for any positive flow there is a parallel network with two edges and affine costs such that the price of anarchy is exactly $4/3$. In this paper we show that the order of the quantifiers in this result is fundamental. Namely, for every parallel network with given affine costs the price of anarchy goes to 1 as the inflow grows large.

Several authors have extended this bound to larger classes of functions. Roughgarden [26] shows that if the class of cost functions includes the constants, then the worst price of anarchy is achieved on parallel networks with just two edges. In his paper he considers bounds for the price of anarchy when the cost functions are polynomials of degree at most d. Dumrauf and Gairing [9] do the same when the degrees of the polynomials are between s and d. Roughgarden and Tardos [29] provide a unifying result for the class of standard costs, i.e., costs c that are differentiable and such that $xc(x)$ is convex. Correa et al. [5] consider the price of anarchy for networks where edges have a capacity and costs are not necessarily convex, differentiable, or even continuous. In [7] they reinterpret and extend these results using a geometric approach. In [6] they consider the problem of minimizing the maximum latency rather than the average latency and provide results about the price of anarchy in this framework. The reader is referred to [27, 30] for a survey.

Recent papers have pointed out that in real situations the price of anarchy may substantially differ from the worst-case scenario [16, 33]. González Vayá et al. [13] deal with a problem of optimal schedule for the electricity demand of a fleet of plug-in electric vehicles. Without using the term, they show that the price of anarchy goes to one as the number of vehicles grows. Cole and Tao [4] study large Walrasian auctions and large fisher markets and show that in both cases the price of anarchy goes to one as the market size increases. Feldman et al. [11] define a concept of (λ, μ)-smoothness for sequences of games, and show that the price of anarchy in atomic congestion games converges to the price of anarchy of the corresponding nonatomic game, when the number of players grows. Patriksson [23] and Josefsson and Patriksson [14] perform sensitivity analysis of Wardrop equilibrium to some parameters of the model. Closer to the scope of our paper, Englert et al. [10] examine how the equilibrium of a congestion game changes when either the total mass of players is increased by ε or an edge that carries an ε fraction of the mass is removed. For polynomial cost functions they bound the increase of the equilibrium cost when a mass ε of players is added to the system. Other recent papers, such as [21, 25], have also raised questions about the practical utility of the worst case results about the price of anarchy.

2 The Model

Consider a finite directed multigraph $\mathscr{G} = (V, E)$, where V is a set of vertices and E is a set of edges. The multigraph $\mathscr{G}$ together with a source $s \in V$ and a destination $t \in V$, is called a network. A path P is a set of consecutive edges that go from source to destination. Call $\mathscr{P}$ the set of all paths. Each path P has a flow $x_P \geq 0$ and we call $\boldsymbol{x} = (x_P)_{P\in\mathscr{P}}$. The total flow from source to destination is denoted by $M \in \mathbb{R}_+$. A flow $\boldsymbol{x}$ is *feasible* if $\sum_{P\in\mathscr{P}} x_P = M$. Call $\mathscr{F}_M$ the set of feasible flows. For each edge $e \in E$ there exists a cost function $c_e(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$, which is assumed (weakly) increasing and continuous. Call $\boldsymbol{c} = (c_e)_{e\in E}$. This defines a *nonatomic congestion game* $\Gamma_M = (\mathscr{G}, M, \boldsymbol{c})$ where the number M is interpreted as the mass of players in the routing game.

The cost of a path P with respect to a flow $\boldsymbol{x}$ is the sum of the costs of its edges: $c_P(\boldsymbol{x}) = \sum_{e\in P} c_e(x_e)$, where

$$x_e = \sum_{\substack{P \in \mathscr{P}: \\ e \in P}} x_P.$$

A flow $\boldsymbol{x}^*$ is an *equilibrium flow* if for every $P, Q \in \mathscr{P}$ such that $x_P^* > 0$ we have $c_P(\boldsymbol{x}^*) \leq c_Q(\boldsymbol{x}^*)$. Denote $\mathscr{E}(\Gamma_M)$ the set of all such equilibrium flows.

For each flow $\boldsymbol{x}$ define the *social cost* associated to it as

$$C(\boldsymbol{x}) := \sum_{P\in\mathscr{P}} x_P c_P(\boldsymbol{x}) = \sum_{e\in E} x_e c_e(x_e),$$

and let $\mathsf{Opt}(\Gamma_M) = \min_{\boldsymbol{x}\in\mathscr{F}_M} C(\boldsymbol{x})$ be the *optimum cost* of Γ_M. Define also the *worst equilibrium cost* of Γ_M as $\mathsf{WEq}(\Gamma_M) = \max_{\boldsymbol{x}\in\mathscr{E}(\Gamma_M)} C(\boldsymbol{x})$. Actually, in the present setting the cost $C(\boldsymbol{x}^*)$ is the same for every equilibrium $\boldsymbol{x}^*$ (see [12]).

The *price of anarchy* of the game Γ_M is then defined as

$$\mathsf{PoA}(\Gamma_M) := \frac{\mathsf{WEq}(\Gamma_M)}{\mathsf{Opt}(\Gamma_M)}.$$

We are interested in the price of anarchy of this game, as $M \to \infty$. We will show that, under suitable conditions, it is asymptotic to one, and we will call *asymptotically well behaved* the congestion games for which this happens.

3 Asymptotically Well Behaved Congestion Games

3.1 General Result for Bounded Costs

The following general result shows that for any network the price of anarchy is asymptotic to one when at least one path has a bounded cost.

Theorem 1 *For each path $P \in \mathscr{P}$ denote*

$$c_P^\infty = \sum_{e \in P} c_e^\infty \quad \textit{with} \quad c_e^\infty = \lim_{z \to \infty} c_e(z)$$

and suppose that $B := \min_{P \in \mathscr{P}} c_P^\infty$ is finite. Then, $\lim_{M \to \infty} \mathsf{PoA}(\Gamma_M) = 1$.

Proof Let $\boldsymbol{x}^*$ be an equilibrium for Γ_M. Then if $x_P^* > 0$ we have

$$c_P(\boldsymbol{x}^*) = \min_{Q \in \mathscr{P}} c_Q(\boldsymbol{x}^*) \leq \min_{Q \in \mathscr{P}} c_Q^\infty = B$$

and therefore

$$\mathsf{WEq}(\Gamma_M) = \sum_{P \in \mathscr{P}} x_P^* c_P(\boldsymbol{x}^*) \leq \sum_{P \in \mathscr{P}} x_P^* B = MB.$$

It follows that

$$\mathsf{PoA}(\Gamma_M) \leq \frac{MB}{\mathsf{Opt}(\Gamma_M)},$$

so that it suffices to prove that $\mathsf{Opt}(\Gamma_M)/M \to B$. To this end denote $\Delta(\mathscr{P})$ the simplex defined by $\boldsymbol{y} = (y_P)_{P \in \mathscr{P}} \geq 0$ and $\sum_{P \in \mathscr{P}} y_P = 1$, so that

$$\begin{aligned} \frac{1}{M} \mathsf{Opt}(\Gamma_M) &= \min_{\boldsymbol{x} \in \mathscr{F}_M} \sum_{P \in \mathscr{P}} \frac{x_P}{M} c_P(\boldsymbol{x}) \\ &= \min_{\boldsymbol{y} \in \Delta(\mathscr{P})} \sum_{P \in \mathscr{P}} y_P \, c_P(M\boldsymbol{y}). \end{aligned}$$

Denote $\Phi_M(\boldsymbol{y}) = \sum_{P \in \mathscr{P}} y_P \, c_P(M\boldsymbol{y})$. Since the cost functions $c_e(\cdot)$ are non-decreasing, the family $\Phi_M(\cdot)$ monotonically increases with M towards the limit function $\Phi_\infty : \Delta(\mathscr{P}) \to \mathbb{R} \cup \{\infty\}$ defined as follows

$$\Phi_\infty(\boldsymbol{y}) = \sum_{P \in \mathscr{P} : y_P > 0} y_P \, c_P^\infty.$$

Now, a monotonically increasing family of functions epi-converges (see [1]) and since $\Delta(\mathscr{P})$ is compact it follows that the minimum $\min_{\boldsymbol{y} \in \Delta(\mathscr{P})} \Phi_M(\boldsymbol{y})$ converges as $M \to \infty$ towards

$$\min_{\boldsymbol{y} \in \Delta(\mathscr{P})} \Phi_\infty(\boldsymbol{y}).$$

Clearly this latter optimal value is B and is attained by setting $y_P > 0$ only on those paths P that attain the smallest value $c_P^\infty = B$, and therefore we conclude

$$\frac{1}{M} \mathsf{Opt}(\Gamma_M) = \min_{\boldsymbol{y} \in \Delta(\mathscr{P})} \Phi_M(\boldsymbol{y}) \to B,$$

as was to be proved. □

3.2 Parallel Graphs

In this section we examine the asymptotic behavior of the price of anarchy when the game is played on a parallel graph.

Let $\mathscr{G} = (V, E)$ be a parallel graph such that $V = \{s, t\}$ are the vertices and $E = \{e_1, e_2, \dots, e_n\}$ are the edges. For each edge $e_i \in E$ the function $c_i(\cdot)$ represents the cost function of the edge e_i. Call $\Gamma_M = (\mathscr{G}, M, \boldsymbol{c})$ the corresponding game. In the whole section we will deal with this graph.

3.2.1 Adding a Constant to Costs

First we prove a preservation result. We show that if the price of anarchy of a game converges to 1, then adding positive constants to each cost does not alter this asymptotic behavior.

Theorem 2 *Given a game $\Gamma_M = (\mathscr{G}, M, \boldsymbol{c})$ and a vector $\boldsymbol{a} \in [0, \infty)^n$, consider a new game $\Gamma_M^{\boldsymbol{a}}(\mathscr{G}, M, \boldsymbol{c}^{\boldsymbol{a}})$ where $c_i^{\boldsymbol{a}}(x) = a_i + c_i(x)$. If $c_i(\cdot)$ is strictly increasing and continuous with* $\lim_{M\to\infty} \mathsf{PoA}(\Gamma_M) = 1$, *then* $\lim_{M\to\infty} \mathsf{PoA}(\Gamma_M^{\boldsymbol{a}}) = 1$.

Proof If some $c_i(\cdot)$ remains bounded the conclusion follows from Theorem 1, so we focus on the case where $c_i(x) \to \infty$ as $x \to \infty$ for all i. In this case all the equilibrium flows x_i^* must diverge to ∞ as $M \to \infty$. In particular they will be all positive and the equilibrium is characterized by $c_i(x_i^*) = \lambda$ for some $\lambda \to \infty$ as $M \to \infty$. In fact, since $\sum_{i=1}^n x_i^* = M$ we can get λ by solving the equation $g(\lambda) = M$ where $g(\lambda) = \sum_{i=1}^n c_i^{-1}(\lambda)$.

The same applies to $\Gamma_M^{\boldsymbol{a}}$. Call $\lambda^{\boldsymbol{a}}$ the cost at the equilibrium on each edge in $\Gamma_M^{\boldsymbol{a}}$ and $\boldsymbol{x}^{\boldsymbol{a}}$ the equilibrium of $\Gamma_M^{\boldsymbol{a}}$. Then we have $a_i + c_i(x_i^{\boldsymbol{a}}) = \lambda^{\boldsymbol{a}}$ so that

$$M = \sum_{e_i \in E} c_i^{-1}(\lambda^{\boldsymbol{a}} - a_i).$$

Denoting $\underline{a} := \min_{e_i \in E} a_i$ and $\bar{a} : \max_{e_i \in E} a_i$, the monotonicity of $c_i(\cdot)$ gives

$$g(\lambda^{\boldsymbol{a}} - \bar{a}) \leq M \leq g(\lambda^{\boldsymbol{a}} - \underline{a})$$

and since $M = g(\lambda)$ we get the inequality $\lambda^{\boldsymbol{a}} - \bar{a} \leq \lambda \leq \lambda^{\boldsymbol{a}} - \underline{a}$ which implies

$$\lim_{M\to\infty} \frac{\lambda^{\boldsymbol{a}}}{\lambda} = 1. \tag{1}$$

Now, for the optimum we have

$$\mathsf{Opt}(\Gamma_M^{\boldsymbol{a}}) = \min_{x \in \mathscr{F}_M} \sum_{e_i \in E} x_i (a_i + c_i(x_i)) \geq \underline{a} M + \mathsf{Opt}(\Gamma_M)$$

and we derive the estimate

$$\mathsf{PoA}(\Gamma_M^{\boldsymbol{a}}) = \frac{M\lambda^{\boldsymbol{a}}}{\mathsf{Opt}(\Gamma_M^{\boldsymbol{a}})} \leq \frac{M\lambda^{\boldsymbol{a}}}{\underline{a}M + \mathsf{Opt}(\Gamma_M)} = \frac{\lambda^{\boldsymbol{a}}/\lambda}{\underline{a}/\lambda + \frac{\mathsf{Opt}(\Gamma_M)}{M\lambda}} \to 1,$$

which follows from the assumption $\mathsf{Opt}(\Gamma_M)/(M\lambda) = \mathsf{PoA}(\Gamma_M)^{-1} \to 1$, combined with (1) and the fact that $\lambda \to \infty$. □

3.2.2 Regularly Varying Functions

Our next result shows that asymptotically the price of anarchy goes to 1 as soon as the cost functions $c_i(\cdot)$ are symptotically equivalent to some regular benchmark function $c(\cdot)$. The notion of regularity that we use is as follows.

Definition 1 Let $\beta \geq 0$. A function c: $(0, +\infty) \to (0, +\infty)$ is called *β-regularly varying* if for all $a > 0$

$$\lim_{x\to\infty} \frac{c(ax)}{c(x)} = a^\beta \in (0, +\infty).$$

The class of regularly varying functions, introduced by Karamata, contains all polynomials as well as polylogarithmic functions and many others. It basically requires that for all $a > 0$ the quotient $c(ax)/c(x)$ converges for $x \to \infty$ (in which case the limit is necessarily of the form a^β for some β). Roughly speaking, this condition states that $c(\cdot)$ grows at the same rate when looked at different scales. For an in depth survey of this concept and its many applications in probability and analysis, we refer to the monograph [3].

In the following result, we use regularly varying functions as a benchmark for the network's cost functions. The goal is twofold: (i) to rule out pathological cost functions that behave irregularly when the flow goes to infinity, and (ii) to ensure that there is at least one edge with a cost function that does not grow *too fast*. Both conditions can be achieved if the network's cost functions can be asymptotically compared with a regularly varying function. Note however that the cost functions themselves are not required to be regularly varying.

Theorem 3 *Let $c(\cdot)$ be a C^1 increasing function with $x \mapsto c(x) + xc'(x)$ strictly increasing, and assume that $c(\cdot)$ is β-regularly varying for some $\beta > 0$. Consider the game Γ_M and suppose that for each $e_i \in E$ the cost $c_i(\cdot)$ is strictly increasing and continuous and that the following limit exists*

$$\lim_{x\to\infty} \frac{c^{-1} \circ c_i(x)}{x} = \alpha_i \in (0, +\infty] \tag{2}$$

with at least one α_i finite. Then

$$\lim_{M\to\infty} \mathsf{PoA}(\Gamma_M) = 1.$$

Proof If some cost $c_i(\cdot)$ is bounded the result follows directly from Theorem 1, so we may restrict to the case where for all links we have $c_i(x) \to \infty$ when $x \to \infty$. In this case the equilibrium flows x_i^* must diverge to ∞ as $M \to \infty$ and the equilibrium is characterized by $c_i(x_i^*) = \lambda$ for some $\lambda \to \infty$. Thus in this proof λ describes the cost experienced on each edge with positive flow. This allows to derive an upper bound for the cost of the equilibrium. Namely, let $I_0 = \{i : \alpha_i < \infty\}$ and take $a_i > \alpha_i$ for all $i \in I_0$. Then (2) implies that for M large enough and all $i \in I_0$ we have

$$\frac{c^{-1}(\lambda)}{x_i^*} = \frac{c^{-1} \circ c_i(x_i^*)}{x_i^*} < a_i .$$

Hence

$$\sum_{i \in I_0} \frac{c^{-1}(\lambda)}{a_i} \leq \sum_{i \in I_0} x_i^* \leq M$$

so that denoting $a = \left(\sum_{i \in I_0} 1/a_i\right)^{-1}$ we get $\lambda \leq c(Ma)$ and therefore

$$\mathsf{WEq}(\Gamma_M) = M\lambda \leq Mc(Ma).$$

Next we derive a lower bound for the optimal cost

$$\mathsf{Opt}(\Gamma_M) = \min_{x \in \mathscr{F}_M} \sum_{i=1}^{n} x_i c_i(x_i).$$

We note that when $M \to \infty$ the optimal solutions are such that $x_i(M) \to \infty$ for all i. Indeed, suppose by contradiction that there is a link i and a sequence $M_k \to \infty$ such that $x_i^k = x_i(M_k)$ remains bounded. Now, some link j must get an amount of flow $x_j^k \geq M_k/n$. Passing to a subsequence we may assume that j is the same for all k so that $x_j^k \to \infty$. Now, if we transfer a fixed amount ε of flow from j to i, the optimality of x^k implies

$$(x_i^k + \varepsilon)c_i(x_i^k + \varepsilon) + (x_j^k - \varepsilon)c_j(x_j^k - \varepsilon) \geq x_i^k c_i(x_i^k) + x_j^k c_j(x_j^k).$$

Since by monotonicity we have $x_j^k c_j(x_j^k - \varepsilon) \leq x_j^k c_j(x_j^k)$, it follows that

$$(x_i^k + \varepsilon)c_i(x_i^k + \varepsilon) - \varepsilon\, c_j(x_j^k - \varepsilon) \geq x_i^k c_i(x_i^k) \geq 0,$$

which yields a contradiction since the left hand side tends to $-\infty$.

Now, let us choose $b_i < \alpha_i$ for all $i = 1, \ldots, n$. Since $x_i(M) \to \infty$, using (2) we get for all M large enough

$$\min_{x \in \mathscr{F}_M} \sum_{i=1}^{n} x_i c_i(x_i) \geq \min_{x \in \mathscr{F}_M} \sum_{i=1}^{n} x_i c(b_i x_i).$$

The optimality condition for the latter problem yields

$$c(b_i x_i) + b_i x_i\, c'(b_i x_i) = \mu \qquad \text{for all } i \in E.$$

For the sake of brevity we denote $\tilde{c}(x) = c(x) + xc'(x)$ and $y_i = b_i x_i$ so that the optimality condition becomes $\tilde{c}(y_i) = \mu$. This yields $y_i = \tilde{c}^{-1}(\mu)$ and therefore

$$M = \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} \frac{\tilde{c}^{-1}(\mu)}{b_i}.$$

Denoting $b = \left(\sum_{i=1}^{n} \frac{1}{b_i}\right)^{-1}$ we then get $\mu = \tilde{c}(Mb)$ and we obtain the following lower bound for the optimal cost

$$\mathsf{Opt}(\Gamma_M) \geq \min_{x \in \mathscr{F}_M} \sum_{i=1}^{n} x_i c(b_i x_i) = Mc(\tilde{c}^{-1}(\mu)) = Mc(Mb).$$

Combining the previous bounds we can estimate the price of anarchy as

$$\mathsf{PoA}(\Gamma_M) \leq \frac{Mc(Ma)}{Mc(Mb)}.$$

Letting $M \to \infty$ and using the fact that c is β-regularly varying we deduce

$$\limsup_{M \to \infty} \mathsf{PoA}(\Gamma_M) \leq \left(\frac{a}{b}\right)^{\beta}.$$

Finally, we note that by letting $a_i \searrow \alpha_i$ for all $i \in I_0$ and $b_i \nearrow \alpha_i$ for all $i = 1, \ldots, n$ both a and b converge towards the common value $\left(\sum_{i \in I_0} 1/\alpha_i\right)^{-1}$, from where we conclude

$$\limsup_{M \to \infty} \mathsf{PoA}(\Gamma_M) = 1.$$

□

The following results follow easily from Theorem 3.

Corollary 1 *In the game Γ_M if for all $i \in E$ we have $\lim_{x \to \infty} c_i(x)/x = m_i \in (0, +\infty]$ and at least one $m_i < \infty$, then*

$$\lim_{M \to \infty} \mathsf{PoA}(\Gamma_M) = 1.$$

Proof Apply Theorem 3 with c equal to the identity. □

Corollary 2 *In the game Γ_M if for all $i \in E$ we have $\lim_{x \to \infty} c_i'(x) = m_i$ with $m_i \in (0, +\infty]$ and at least one m_i is finite, then*

$$\lim_{M \to \infty} \mathsf{PoA}(\Gamma_M) = 1.$$

Proof Just notice that $\lim_{x \to \infty} c_i(x)/x = \lim_{x \to \infty} c_i'(x) = m_i$. □

The next results examines the case where each cost function is bounded above and below by two affine functions with the same slope, as in Fig. 1.

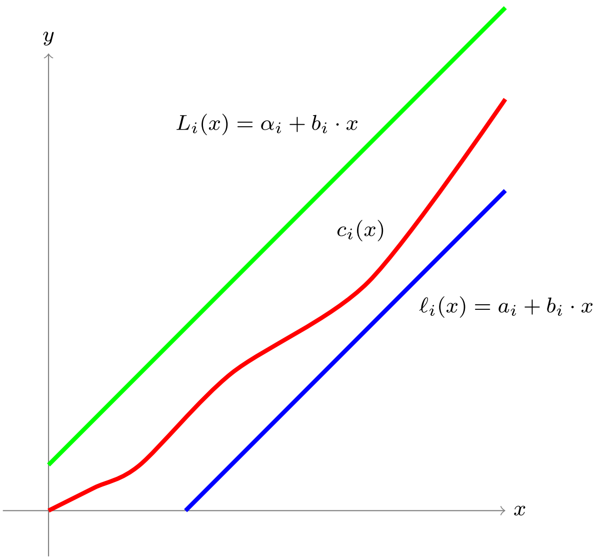

Fig. 1 Affinely bounded costs

Corollary 3 *Consider the game Γ_M and assume that for every $e_i \in E$ and for all x large enough we have*

$$\ell_i(x) := a_i + b_i x \leq c_i(x) \leq \alpha_i + b_i x =: L_i(x).$$

Then

$$\lim_{M\to\infty} \mathsf{PoA}(\Gamma_M) = 1.$$

Proof This follows from Corollary 1. □

Corollary 4 *In the game Γ_M if there is a strictly increasing β-regularly varying function $c(\cdot)$ with $\beta > 0$ such that for all $i \in E$*

$$\lim_{x\to\infty} \frac{c_i(x)}{c(x)} = m_i \in (0, +\infty], \tag{3}$$

and at least one m_i is finite, then

$$\lim_{M\to\infty} \mathsf{PoA}(\Gamma_M) = 1.$$

Proof It suffices to show that (3) implies (2) with $\alpha_i = m_i^{1/\beta}$. To prove this we note that when $m_i < \infty$, then using (3) we see that for all $\varepsilon > 0$ the following inequalities hold for x large enough

$$(m_i - \varepsilon)c(x) \leq c_i(x) \leq (m_i + \varepsilon)c(x),$$

and therefore

$$\frac{c^{-1}((m_i-\varepsilon)c(x))}{x} \le \frac{c^{-1}(c_i(x))}{x} \le \frac{c^{-1}((m_i+\varepsilon)c(x))}{x}.$$

Moreover, by Lemma 1(c) in the Appendix we know that:

$$\lim_{x\to\infty} \frac{c^{-1}((m_i-\varepsilon)c(x))}{x} = (m_i-\varepsilon)^{1/\beta}$$
$$\lim_{x\to\infty} \frac{c^{-1}((m_i+\varepsilon)c(x))}{x} = (m_i+\varepsilon)^{1/\beta}$$

and since

$$\lim_{\varepsilon\downarrow 0}(m_i-\varepsilon)^{1/\beta} = \lim_{\varepsilon\downarrow 0}(m_i+\varepsilon)^{1/\beta} = m_i^{1/\beta} = \alpha_i,$$

we get

$$\frac{c^{-1}(c_i(x))}{x} \to \alpha_i.$$

Similarly, when $m_i = \infty$ we may take m_i' finite and then for x large we have

$$m_i' \le \frac{c_i(x)}{c(x)} \Longrightarrow m_i' c(x) \le c_i(x) \Longrightarrow \frac{c^{-1}(m_i' c(x))}{x} \le \frac{c^{-1}(c_i(x))}{x}.$$

Now, from

$$\lim_{x\to\infty} \frac{c^{-1}(m_i' c(x))}{x} = (m_i')^{1/\beta},$$

letting $m_i' \to \infty$ we obtain once again

$$\lim_{x\to\infty} \frac{c^{-1}(c_i(x))}{x} = \infty = \alpha_i.$$

□

Corollary 5 *If the cost function $c_i(x)$ in the game Γ_M are polynomials, then*

$$\lim_{M\to\infty} \mathsf{PoA}(\Gamma_M) = 1.$$

Proof This follows from Corollary 4. □

4 Asymptotically Ill-Behaved Games

In this section we will present some examples where the price of anarchy is not asymptotic to one as the inflow goes to infinity.

Consider a standard Pigou graph and assume that the costs are as follows:

$$\begin{aligned} c_1(x) &= x, \\ c_2(x) &= a^{k+1} \quad \text{for } x \in (a^k, a^{k+1}], \quad k \in \mathbb{Z}, \end{aligned} \tag{4}$$

with $a \ge 2$, as in Fig. 2. In this game the cost of one edge is the identity, whereas for the other edge it is a step function that touches the identity at intervals that grow

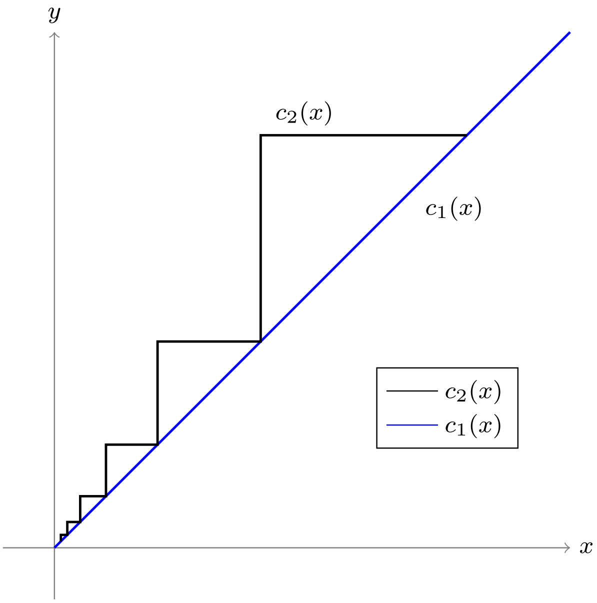

Fig. 2 Step function

exponentially. The cost function c_2 is not continuous, but a very similar game can be constructed by approximating it with a continuous function.

Theorem 4 *Consider the game Γ_M with costs as in (4). Then the price of anarchy presents a periodic behavior on a logarithmic scale. Moreover, we have*

$$\inf \mathsf{PoA}(\Gamma_M) = 1, \quad \sup \mathsf{PoA}(\Gamma_M) = \frac{4+4a}{4+3a}.$$

Remark 1 An immediate consequence of Theorem 4 is that

$$\liminf_{M\to\infty} \mathsf{PoA}(\Gamma_M) = \liminf_{M\to 0} \mathsf{PoA}(\Gamma_M) = 1,$$
$$\limsup_{M\to\infty} \mathsf{PoA}(\Gamma_M) = \limsup_{M\to 0} \mathsf{PoA}(\Gamma_M) = \frac{4+4a}{4+3a}.$$

We can immediately see that

$$\limsup_{M\to\infty} \mathsf{PoA}(\Gamma_M) = \frac{6}{5} \quad \text{for } a = 2$$

and

$$\limsup_{M\to\infty} \mathsf{PoA}(\Gamma_M) \to \frac{4}{3} \quad \text{as } a \to \infty.$$

As a referee pointed out, since when $a \to \infty$ we have $\limsup_{M\to\infty} \mathsf{PoA}(\Gamma_M) \to 4/3$, the reader may wonder if there is a relation with the celebrated $4/3$ bound by Roughgarden and Tardos [28]. The two results would be similar, if the parameter a were allowed to scale with the total inflow M, but in our game the parameter is fixed

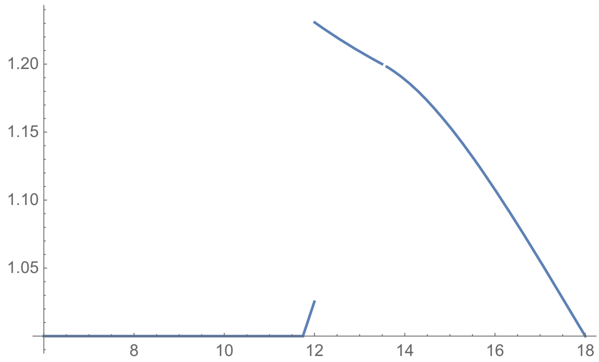

Fig. 3 Price of anarchy for $M \in [2a^k, 2a^{k+1}]$, with $a = 3$, $k = 1$

and the demand grows. The bound 4/3 for the limsup of the price of anarchy can be approximated in our class of games by taking a large enough.

Figure 3 plots the price of anarchy for $M \in [2a^k, 2a^{k+1}]$, when $a = 3$.

The next theorem shows that the price of anarchy may fail to be asymptotic to one, even when the cost functions are all convex.

Theorem 5 *There exist congestion games Γ_M where the cost functions are all increasing and convex and both*

$$\limsup_{M\to\infty} \mathsf{PoA}(\Gamma_M) > 1 \quad \textit{and} \quad \limsup_{M\to 0} \mathsf{PoA}(\Gamma_M) > 1.$$

Our final result shows that the lim sup of the price of anarchy may even be infinite.

Theorem 6 *There exist congestion games Γ_M with $\limsup_{M\to\infty} \mathsf{PoA}(\Gamma_M) = \infty$.*

5 Conclusions

We have examined the asymptotic behavior of the price of anarchy for highy congested nonatomic routing games in parallel networks. Coherently with some empirical results, we have shown that under fairly general conditions on the cost functions the price of anarchy goes to one as the inflow grows to infinity. In the case of general networks, the condition requires the cost of at least one path to be bounded. In the case of parallel networks, it requires all the cost functions to behave well with respect to some regularly varying function. Moreover, we have provided counterexamples to this behavior, even under a very simple network structure, showing that what matters is the shape of the cost functions.

It is worth pointing out that the worst case analysis in Roughgarden and Tardos [28] shows that for any fixed total inflow M one can build a two-link network with affine costs—which depend on M—such that the price of anarchy of the corresponding game is $4/3$. In contrast, here we have proved that once the affine costs are fixed, the price of anarchy converges to 1 as the total flow M increases.

This paper is just a first step to investigate the behavior of the price of anarchy as a function of the demand. The natural next step is the study the asymptotic behavior of the price of anarchy in non-parallel networks. A first result in this direction has been provided by Theorem 1. In order to move beyond this setting the main difficulty lies in the fact that edges and paths are now decoupled and a benchmark for edges does not seem to be sufficient to guarantee the convergence of the price of anarchy.

A further natural next step is represented by the analysis of networks with multiple O/D pairs. In this case the study of the price of anarchy seems even harder because different O/D pairs may contribute in different ways to the total flow that is traversing the network.

Finally, the study of the light traffic case when the flow approaches zero is also an interesting complementary direction. The light traffic case is well motivated by Theorem 4, which shows that, even when the traffic is low, the price of anarchy is not guaranteed to converge to one, unlike many empirical observations would suggest.

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Appendix A: Regularly Varying Functions

The reader is referred to [3] for an extended treatment of regularly varying functions. Here we gather a few basic properties that are useful for our results.

Lemma 1 *Let $\beta > 0$ and let Θ be a continuous and strictly increasing function, then the following properties are equivalent:*

(a) *the function Θ is β-regularly varying,*
(b) *the function Θ^{-1} is $\frac{1}{\beta}$-regularly varying,*
(c) *for all $\gamma > 0$*

$$\lim_{x\to\infty} \frac{1}{x}\Theta^{-1}(\gamma\Theta(x)) = \gamma^{1/\beta}.$$

Proof The equivalence of (a) and (b) is proved in [8, page 22]. The equivalence of (b) and (c) is immediate, since, by setting $u = \Theta(x)$, we have

$$\frac{1}{x}\Theta^{-1}(\gamma \cdot \Theta(x)) = \frac{\Theta^{-1}(\gamma \cdot u)}{\Theta^{-1}(u)} \to \gamma^{1/\beta}.$$

□

Lemma 2 *If Θ is a continuous and strictly increasing β-regularly varying function, then $x \cdot \Theta(x)$ and $\int_0^x \Theta(s)\,\mathrm{d}s$ are $(1+\beta)$-regularly varying functions.*

Proof For the function $x \cdot \Theta(x)$ it suffices to note that

$$\lim_{x\to\infty} \frac{ax\Theta(ax)}{x\Theta(x)} = a \cdot a^\beta = a^{1+\beta}.$$

For $\int_0^x \Theta(s)\,\mathrm{d}s$ a direct application of l'Hôpital rule gives

$$\lim_{x\to\infty} \frac{\int_0^{ax} \Theta(s)\,\mathrm{d}s}{\int_0^x \Theta(s)\,\mathrm{d}s} = \lim_{x\to\infty} \frac{\Theta(ax)a}{\Theta(x)} = a^\beta a = a^{1+\beta}.$$

□

The following two lemmata appear in [3, Proposition 1.5.7].

Lemma 3 *For $i = 1, 2$, let Θ_i be a continuous and strictly increasing β_i-regularly varying function. Then $\Theta_1 \circ \Theta_2$ is $\beta_1 \cdot \beta_2$-regularly varying.*

Lemma 4 *Let Θ_1 and Θ_2 be two continuous and strictly increasing β-regularly varying functions, then $\Theta_1 + \Theta_2$ is β-regularly varying.*

Appendix B: Omitted Proofs

Proofs of Section 4

In the whole subsection, for the sake of simplicity, we call x the flow on e_1 and y the flow on e_2.

Proof (of Theorem 4) Let us study the price of anarchy for $M \in (2a^k, 2a^{k+1}]$, keeping in mind that $a \geq 2$.

Equilibrium cost. In the subinterval $M \in (2a^k, a^k + a^{k+1}]$ we have

$$\begin{aligned} x^* &= M - a^k, & c_1(x^*) &= M - a^k \leq a^{k+1}, \\ y^* &= a^k, & c_2(y^*) &= a^k. \end{aligned}$$

For $M \in (a^k + a^{k+1}, 2a^{k+1}]$ we have

$$\begin{aligned} x^* &= a^{k+1}, & c_1(x^*) &= a^{k+1}, \\ y^* &= M - a^{k+1}, & c_2(y^*) &= a^{k+1}. \end{aligned}$$

Therefore

$$\mathsf{WEq}(\Gamma_M) = \begin{cases} (M - a^k)^2 + a^{2k} & \text{for } M \in (2a^k, a^k + a^{k+1}], \\ Ma^{k+1} & \text{for } M \in (a^k + a^{k+1}, 2a^{k+1}]. \end{cases}$$

Optimal cost. In order to compute the optimal cost

$$\mathsf{Opt}(\Gamma_M) = \min_{0\le y\le M} yc_2(y) + (M-y)^2$$

we decompose the problem over the intervals $I_j = (a^j, a^{j+1}]$ on which $c_2(\cdot)$ is constant, namely, we consider the subproblems

$$C_j = \min_{y\in I_j, y\le M} a^{j+1}y + (M-y)^2.$$

We observe that for $j \ge k+2$ we have $a^j \ge a^{k+2} \ge 2a^{k+1} \ge M$ so that C_j is infeasible and therefore $\mathsf{Opt}(\Gamma_M) = \min\{C_0, C_1, \ldots, C_{k+1}\}$. In fact, we will show that $\mathsf{Opt}(\Gamma_M) = \min\{C_{k-1}, C_k\}$.

Let us compute C_j. Since $(M-y)^2$ is symmetric around M, the constraint $y \le M$ can be dropped and then the minimum C_j is obtained by projecting onto $[a^j, a^{j+1}]$ the unconstrained minimizer $y_j = M - a^{j+1}/2$. We get

$$C_j = \begin{cases} a^{j+1}a^j + (M-a^j)^2 & \text{if } M < a^j + \frac{a^{j+1}}{2}, \\ a^{j+1}\left(M - \frac{a^{j+1}}{2}\right) + \left(\frac{a^{j+1}}{2}\right)^2 & \text{if } a^j + \frac{a^{j+1}}{2} \le M \le a^{j+1} + \frac{a^{j+1}}{2}, \\ a^{j+1}a^{j+1} + (M-a^{j+1})^2 & \text{if } M > a^{j+1} + \frac{a^{j+1}}{2}. \end{cases} \quad (5)$$

□

Claim 7 *For $j \le k-1$ we have $C_j = a^{j+1}a^{j+1} + (M-a^{j+1})^2$ and $C_{j-1} \ge C_j$.*

Proof The expression for C_j follows from (5) if we note that $M > 2a^k \ge \frac{3}{2}a^{j+1}$. In order to prove that $C_{j-1} \ge C_j$ we observe that

$$\begin{aligned} C_{j-1} \ge C_j &\iff (a^j)^2 + (M-a^j)^2 \ge (a^j)^2a^2 + (M-a^ja)^2 \\ &\iff 2(a^j)^2 + M^2 - 2Ma^j \ge 2(a^j)^2a^2 + M^2 - 2Ma^ja \\ &\iff Ma^j(a-1) \ge (a^j)^2(a^2-1) \\ &\iff M \ge a^j(a+1) = a^j + a^{j+1}. \end{aligned}$$

Since $M > 2a^k = a^k + a^k \ge a^j + a^{j+1}$, this holds true. □

Claim 8 $C_{k+1} = a^{k+2}a^{k+1} + (M-a^{k+1})^2 \ge C_{k-1}$.

Proof Since $M \le 2a^{k+1} \le a^{k+1} + \frac{a^{k+2}}{2}$ we get the expression for C_{k+1} from (5). Then

$$\begin{aligned} C_{k-1} \le C_{k+1} &\iff (a^k)^2 + (M-a^k)^2 \le a^{k+2}a^{k+1} + (M-a^{k+1})^2 \\ &\iff 2(a^k)^2 + M^2 - 2Ma^k \le (a^k)^2a^3 + (a^k)^2a^2 + M^2 - 2Ma^{k+1} \\ &\iff 2Ma^k(a-1) \le (a^k)^2(a-1)(a^2+2a+2) \\ &\iff 2M \le a^k(a^2+2a+2). \end{aligned}$$

Since $M \le 2a^{k+1}$ it suffices to have $4a^{k+1} \le a^k(a^2+2a+2)$ which is easily seen to hold. □

Combining the previous claims we get that $\mathsf{Opt}(\Gamma_M) = \min\{C_{k-1}, C_k\}$. It remains to figure out which one between C_{k-1} and C_k attains the minimum. This depends on where M is located within the interval $(2a^k, 2a^{k+1}]$ as explained in our next claim. In the sequel we denote

$$\begin{aligned} \alpha &= 1 + \frac{a}{2} \\ \beta &= 1 + \frac{a}{2} + \sqrt{a-1} \\ \gamma &= \frac{3}{2}a \end{aligned}$$

and we observe that

$$2 \le \alpha \le \beta \le \gamma \le 2a.$$

Claim 9 *For $M \in (2a^k, 2a^{k+1}]$ we have*

$$\mathsf{Opt}(\Gamma_M) = \begin{cases} C_{k-1} = (a^k)^2 + (M - a^k)^2 & \textit{if } M \in (2a^k, \alpha a^k) \\ C_{k-1} = (a^k)^2 + (M - a^k)^2 & \textit{if } M \in [\alpha a^k, \beta a^k) \\ C_k = a^{k+1}(M - \frac{1}{4}a^{k+1}) & \textit{if } M \in [\beta a^k, \gamma a^k] \\ C_k = (a^{k+1})^2 + (M - a^{k+1})^2 & \textit{if } M \in (\gamma a^k, 2a^{k+1}]. \end{cases}$$

Proof Recall that $a \ge 2$. From (5) we have $C_{k-1} = (a^k)^2 + (M - a^k)^2$ whereas the expression for C_k changes depending where M is located.

(a) **Initial interval** $M \in (2a^k, \alpha a^k)$.
Here $M < a^k + \frac{1}{2}a^{k+1}$ so that (5) gives $C_k = a^{k+1}a^k + (M - a^k)^2$. Hence, clearly $C_{k-1} \le C_k$ and $\mathsf{Opt}(\Gamma_M) = C_{k-1}$.

(b) **Final interval** $M \in (\gamma a^k, 2a^{k+1}]$.
Here $M > \gamma a^k = \frac{3}{2}a^{k+1}$ so that (5) gives $C_k = (a^{k+1})^2 + (M - a^{k+1})^2$. Proceeding as in the proof of Claim 7, we have $C_{k-1} \ge C_k$ if and only if $M \ge a^k + a^{k+1}$. The latter holds since $M \ge \frac{3}{2}a^{k+1} \ge a^{k+1} + a^k$. Hence $\mathsf{Opt}(\Gamma_M) = C_k$.

(c) **Intermediate interval** $M \in [\alpha a^k, \gamma a^k]$.

Here $a^k + \frac{1}{2}a^{k+1} \le M \le \frac{3}{2}a^{k+1}$ so that (5) gives

$$C_k = a^{k+1}\left(M - \frac{1}{2}a^{k+1}\right) + \left(\frac{1}{2}a^{k+1}\right)^2 = a^{k+1}\left(M - \frac{1}{4}a^{k+1}\right).$$

Then, denoting $z = M/a^k$ we have

$$\begin{aligned} C_{k-1} \le C_k \iff & 2(a^k)^2 + M^2 - 2Ma^k \le a^{k+1}M - \left(\frac{1}{2}a^{k+1}\right)^2. \\ \iff & z^2 - z(2+a) + \left(2 + \frac{1}{4}a^2\right) \le 0 \\ \iff & 1 + \frac{1}{2}a - \sqrt{a-1} \le z \le 1 + \frac{1}{2}a + \sqrt{a-1}. \end{aligned}$$

The upper limit for z is precisely β while the lower limit is smaller than α. Hence $\mathsf{Opt}(\Gamma_M) = C_{k-1}$ for $M \in [\alpha a^k, \beta a^k]$ and $\mathsf{Opt}(\Gamma_M) = C_k$ for $M \in [\beta a^k, \gamma a^k]$.

Figure 4 illustrates the different intervals in which the equilibrium (above) and the optimum (below) change. Notice that $\mathsf{Opt}(\Gamma_M)$ varies continuously even at breakpoints, whereas $\mathsf{WEq}(\Gamma_M)$ has a jump at $a^k + a^{k+1}$. We now proceed to examine the price of anarchy which will be expressed as a function of $z = M/a^k$.

From the expressions of $\mathsf{WEq}(\Gamma_M)$ and $\mathsf{Opt}(\Gamma_M)$ (see Fig. 4) it follows that $\mathsf{PoA}(\Gamma_M) = 1$ throughout the initial interval $M \in (2a^k, \beta a^k)$. Over the next interval $M \in [\beta a^k, a^k + a^{k+1}]$ we have

$$\mathsf{PoA}(\Gamma_M) = \frac{(a^k)^2 + (M - a^k)^2}{a^{k+1}(M - a^{k+1}/4)} = \frac{1 + (z-1)^2}{a(z - a/4)},$$

which increases from 1 at $z = \beta$ up to $(4 + 4a^2)/(a(4 + 3a))$ at $z = 1 + a$.

At $M = a^k + a^{k+1}$ the equilibrium has a discontinuity and $\mathsf{PoA}(\Gamma_M)$ jumps to $(4 + 4a)/(4 + 3a)$ and then it decreases over the interval $M \in (a^k + a^{k+1}, \frac{3}{2}a^{k+1})$ as

$$\mathsf{PoA}(\Gamma_M) = \frac{a^{k+1}M}{a^{k+1}(M - a^{k+1}/4)} = \frac{z}{z - a/4}.$$

Finally, for $M \in (\frac{3}{2}a^{k+1}, 2a^{k+1}]$ the price of anarchy continues to decrease as

$$\mathsf{PoA}(\Gamma_M) = \frac{a^{k+1}M}{(a^{k+1})^2 + (M - a^{k+1})^2} = \frac{az}{a^2 + (z-a)^2}.$$

going back to 1 at $z = 2a$ which corresponds to $M = 2a^{k+1}$.

Thus the price of anarchy oscillates over each interval $(2a^k, 2a^{k+1}]$ between a minimum value of 1 and a maximum of $(4 + 4a)/(4 + 3a)$. This completes the proof of Theorem 4. □

Proof (of Theorem 5) Consider a parallel network with two edges with a quadratic cost $c_1(x) = x^2$ on the upper edge and a lower edge cost defined by linearly interpolating c_1, that is, for $a \geq 2$ we let (see Fig. 5)

$$c_2(y) = (a^{k-1} + a^k)y - a^{k-1}a^k, \quad \text{for } y \in [a^{k-1}, a^k], \quad k \in \mathbb{Z}.$$

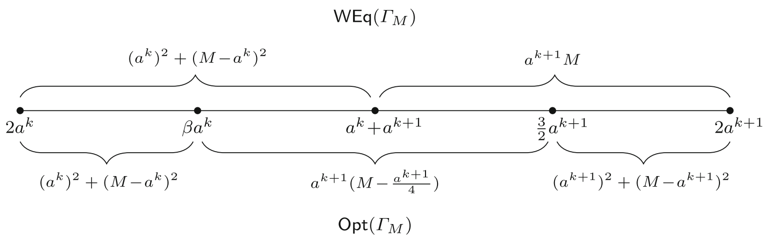

Fig. 4 Breakpoints for optimum and equilibrium

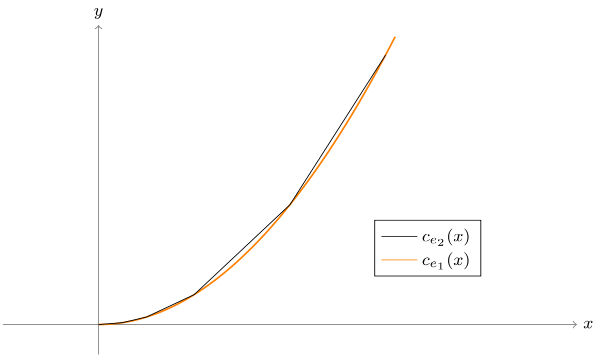

Fig. 5 x^2 and its linear interpolation

Note that c_1 and c_2 are convex. Consider the optimal cost problem

$$\mathsf{Opt}(\Gamma_M) = \min_{\substack{x+y=M \\ x,y\geq 0}} x^3 + yc_2(y).$$

Since the function $h(y) = yc_2(y)$ is non-differentiable, the optimality condition reads $3x^2 \in \partial h(y)$. In particular, the subdifferential at $y = a^k$ is

$$\partial h(a^k) = [a^{2(k-1)}(2a^2+a), a^{2k}(2+a)]$$

and there is a range of values of M for which the optimal solution is $y = a^k$. The smallest such M is obtained when $3x^2 = a^{2(k-1)}(2a^2+a)$. This gives as optimal solution $y = a^k$ and $x = a^{k-1}b$, with $b = \sqrt{(2a^2+a)/3}$, corresponding to $M_k = a^{k-1}[a+b]$ with optimal value

$$\mathsf{Opt}(\Gamma_{M_k}) = a^{3(k-1)}[b^3 + a^3].$$

In order to find the equilibrium for M_k we solve the equation $x^2 = c_2(y)$ with $x + y = M_k$. A routine calculation gives $x = a^{k-1}c$ and $y = a^{k-1}d$ with

$$\begin{aligned} c &= \frac{1}{2}\left[\sqrt{(a+1)^2 + 4a^2 + 4(a+1)b} - (a+1)\right], \\ d &= a + b - c. \end{aligned}$$

Note that $1 < d < a$ so that $y \in (a^{k-1}, a^k)$, and therefore the equilibrium cost is

$$\mathsf{WEq}(\Gamma_{M_k}) = a^{3(k-1)}[c^3 + (a+1)d^2 - ad].$$

Putting together the previous formulae we get

$$\mathsf{PoA}(\Gamma_{M_k}) = \frac{c^3 + (a+1)d^2 - ad}{b^3 + a^3}.$$

For $a = 2$ this expression evaluates to $\mathsf{PoA}(\Gamma_{M_k}) \sim 1.0059$ from which the result follows. □

Proof (of Theorem 6) Take a fixed sequence $\alpha_k > 0$ such that $\alpha_{k+1}/\alpha_k \to \infty$ and consider a game $\Gamma_M = (\mathscr{G}, M, \boldsymbol{c})$, where $\mathscr{G} = (V, E)$ is a simple Pigou network with two parallel links with costs given by

$$c_1(x) = c(x) := \begin{cases} \mathrm{e} & \text{for } x < 1, \\ \mathrm{e}^x/x & \text{for } x \geq 1, \end{cases}$$
$$c_2(y) = \bar{c}(y) := c(\alpha_{k+1}) \ \text{ for } y \in (\alpha_k, \alpha_{k+1}].$$

Since we are interested in asymptotic results, we are concerned only with the case $c(x) = \mathrm{e}^x/x$.

Depending on the location of M, the equilibrium is given by

$$2\alpha_k < M \leq \alpha_k + \alpha_{k+1} \implies y^* = \alpha_k, x^* = M - \alpha_k,$$
$$\alpha_k + \alpha_{k+1} < M \leq 2\alpha_{k+1} \implies y^* = M - \alpha_{k+1}, x^* = \alpha_{k+1}.$$

We note that at the point $M_k = \alpha_k + \alpha_{k+1}$ the equilibrium has a discontinuity. The cost inmediately to the right of this point is

$$\begin{aligned} \mathsf{WEq}(\Gamma_{M_k^+}) &= \lim_{M \downarrow M_k} \mathsf{WEq}(\Gamma_M) \\ &= \alpha_{k+1} c(\alpha_{k+1}) + (M_k - \alpha_{k+1}) c(\alpha_{k+1}) \\ &= M_k c(\alpha_{k+1}). \end{aligned} \tag{6}$$

Let us now turn to computing the optimum

$$\mathsf{Opt}(\Gamma_M) = \min_{0 \leq x \leq M} x c(x) + (M - x)\bar{c}(M - x) = \min_j C_j$$

which we decomposed into the restricted minima C_j given by

$$C_j = \min_{\alpha_j < M - x \leq \alpha_{j+1}} x c(x) + (M - x) c(\alpha_{j+1}).$$

The unconstrained minimum for each j is obtained by solving the equation $\mathrm{e}^x = \frac{\mathrm{e}^{\alpha_{j+1}}}{\alpha_{j+1}}$ so that denoting

$$x_j = \alpha_{j+1} - \ln \alpha_{j+1},$$
$$y_j = M - \alpha_{j+1} + \ln \alpha_{j+1},$$

we have the following expression for the constrained minimizers $\tilde{y}_j$ and the values C_j

$$y_j < \alpha_j \implies \widetilde{y}_j = \alpha_j \quad \implies C_j = \mathrm{e}^{M - \alpha_j} + \frac{\alpha_j}{\alpha_{j+1}} \mathrm{e}^{\alpha_{j+1}},$$
$$y_j > \alpha_{j+1} \implies \widetilde{y}_j = \alpha_{j+1} \implies C_j = \mathrm{e}^{M - \alpha_{j+1}} + \mathrm{e}^{\alpha_{j+1}},$$
$$\alpha_j \leq y_j \leq \alpha_{j+1} \implies \widetilde{y}_j = y_j \quad \implies C_j = \frac{\mathrm{e}^{\alpha_{j+1}}}{\alpha_{j+1}}(1 + M - \alpha_{j+1} + \ln \alpha_{j+1}).$$

We remark that y_j varies continuously with M, and therefore the same holds for $\tilde{y}_j$ and C_j. It follows that $\mathsf{Opt}(\Gamma_M)$ is also continuous in M.

Claim 10 *Let $M = M_k$. For k large enough we have*

$$\mathsf{Opt}(\Gamma_{M_k}) = \min\{C_{k-1}, C_k, C_{k+1}\}.$$

Proof

a) *C_j is decreasing for $j \leq k-1$.* Indeed, for $j \leq k-1$ we have $M > 2\alpha_{j+1}$ so that

$$y_j = M - \alpha_{j+1} + \ln \alpha_{j+1} > \alpha_{j+1} + \ln \alpha_{j+1} > \alpha_{j+1}$$

and therefore

$$C_j = \mathrm{e}^{M-\alpha_{j+1}} + \mathrm{e}^{\alpha_{j+1}}.$$

Denoting $h(x) := \mathrm{e}^{M-x} + \mathrm{e}^x$, the inequality $C_j \leq C_{j-1}$ is equivalent to $h(\alpha_{j+1}) \leq h(\alpha_j)$, which holds because h is decreasing on the interval $[0, M/2]$ and since $\alpha_j \leq \alpha_{j+1} \leq \alpha_k \leq \frac{M}{2}$.

b) *C_j is increasing for $j \geq k+1$.* We first show that if k is large enough then $y_j < \alpha_j$. Considering the expression of y_j, this inequality is equivalent to $M < \alpha_j + \alpha_{j+1} - \ln \alpha_{j+1}$. Now, since $M \leq 2\alpha_{k+1} \leq 2\alpha_j$ it suffices to show that

$$2\alpha_j < \alpha_j + \alpha_{j+1} - \ln \alpha_{j+1}$$

which can also be written as

$$\frac{\ln \alpha_{j+1}}{\alpha_{j+1}} + \frac{\alpha_j}{\alpha_{j+1}} < 1.$$

Now, our choice of the sequence α_j implies that the right hand side tends to zero as $j \to \infty$, proving that $y_j < \alpha_j$ for $j \geq k+1$ provided that k is chosen large enough. In this situation we have for all $j \geq k+1$

$$C_j = \mathrm{e}^{M-\alpha_j} + \frac{\alpha_j}{\alpha_{j+1}} \mathrm{e}^{\alpha_{j+1}}.$$

In order to show that $C_j \leq C_{j+1}$ we note that

$$C_j \leq C_{j+1} \iff \mathrm{e}^{M-\alpha_j} + \frac{\alpha_j}{\alpha_{j+1}} \mathrm{e}^{\alpha_{j+1}} \leq \mathrm{e}^{M-\alpha_{j+1}} + \frac{\alpha_{j+1}}{\alpha_{j+2}} \mathrm{e}^{\alpha_{j+2}}.$$

For $x \geq 1$ the function e^x/x is increasing so that

$$\mathrm{e}^{M-\alpha_j} + \alpha_j \frac{\mathrm{e}^{\alpha_{j+1}}}{\alpha_{j+1}} \leq \mathrm{e}^{M-\alpha_j} + \alpha_j \frac{\mathrm{e}^{\alpha_{j+2}}}{\alpha_{j+2}}.$$

It remains to replace α_j by α_{j+1} on the right for which it suffices to prove that the function

$$g(x) := \mathrm{e}^{M-x} + x \frac{\mathrm{e}^{\alpha_{j+2}}}{\alpha_{j+2}}$$

is increasing for $x \geq \alpha_{k+1}$. Indeed, since $g'(x) = \frac{\mathrm{e}^{\alpha_{j+2}}}{\alpha_{j+2}} - \mathrm{e}^{M-x}$ we have

$$\begin{aligned} g'(x) \geq 0 &\iff M - x \leq \alpha_{j+2} - \ln \alpha_{j+2} \\ &\iff M \leq x + \alpha_{j+2} - \ln \alpha_{j+2}, \end{aligned}$$

which is true for $x \geq \alpha_{k+1}$ iff $M \leq \alpha_{k+1} + \alpha_{j+2} - \ln \alpha_{j+2}$. Since $M \leq 2\alpha_{k+1}$, it is enough to have $\alpha_{k+1} \leq \alpha_{j+2} - \ln \alpha_{j+2}$, and since $x \mapsto x - \ln x$ is increasing for $x \geq 1$, it suffices to prove that $\alpha_{k+1} \leq \alpha_{k+2} - \ln \alpha_{k+2}$, that is,

$$\frac{\ln \alpha_{k+2}}{\alpha_{k+2}} + \frac{\alpha_{k+1}}{\alpha_{k+2}} \leq 1. \tag{7}$$

By our choice of α_k this holds for k large enough, which completes the proof of Claim 10.

□

Claim 11 *Let $M = M_k$. For all k large enough we have*

$$\mathsf{Opt}(\Gamma_{M_k}) = \frac{\mathrm{e}^{\alpha_{k+1}}}{\alpha_{k+1}}(1 + \alpha_k + \ln \alpha_{k+1}). \tag{8}$$

Proof From the proof of Claim 10 we have

$$\begin{aligned} C_{k-1} &= e^{M-\alpha_k} + e^{\alpha_k} = e^{\alpha_{k+1}} + e^{\alpha_k} \\ C_{k+1} &= e^{M-\alpha_{k+1}} + \frac{\alpha_{k+1}}{\alpha_{k+2}} e^{\alpha_{k+2}} = e^{\alpha_k} + \frac{\alpha_{k+1}}{\alpha_{k+2}} e^{\alpha_{k+2}} \end{aligned}$$

and it is easy to see that $C_{k-1} \leq C_{k+1}$ so that in fact $\mathsf{Opt}(\Gamma_{M_k}) = \min\{C_{k-1}, C_k\}$.

Now, the expression of C_k depends on the location of $y_k = M - \alpha_{k+1} + \ln \alpha_{k+1}$ with respect to the interval $[\alpha_k, \alpha_{k+1}]$. Substituting the value of $M = \alpha_k + \alpha_{k+1}$ we get $y_k = \alpha_k + \ln \alpha_{k+1}$ so that clearly $y_k > \alpha_k$. Also, for k large we have $y_k < \alpha_{k+1}$ since

$$\frac{y_k}{\alpha_{k+1}} = \frac{\alpha_k}{\alpha_{k+1}} + \frac{\ln \alpha_{k+1}}{\alpha_{k+1}} \to 0.$$

It follows that

$$C_k = \frac{\mathrm{e}^{\alpha_{k+1}}}{\alpha_{k+1}}(1 + M - \alpha_{k+1} + \ln \alpha_{k+1}) = \frac{\mathrm{e}^{\alpha_{k+1}}}{\alpha_{k+1}}(1 + \alpha_k + \ln \alpha_{k+1})$$

and therefore it remains to show that $C_k \leq C_{k-1}$. The latter is equivalent to

$$\frac{\mathrm{e}^{\alpha_{k+1}}}{\alpha_{k+1}}(1 + \alpha_k + \ln \alpha_{k+1}) \leq \mathrm{e}^{\alpha_{k+1}} + \mathrm{e}^{\alpha_k}$$

which can be rewritten as

$$\frac{1}{\alpha_{k+1}}(1 + \alpha_k + \ln \alpha_{k+1}) \leq 1 + \mathrm{e}^{\alpha_k - \alpha_{k+1}}.$$

Since the right hand side tends to 0, this inequality holds for k large enough. □

Conclusion Let us compute the price of anarchy just to the right of M_k, namely

$$\mathsf{PoA}(\Gamma_{M_k^+}) = \frac{\mathsf{WEq}(\Gamma_{M_k^+})}{\mathsf{Opt}(\Gamma_{M_k^+})}.$$

Since $\mathsf{Opt}(\Gamma_M)$ is continuous in M, using (6) and (8) we get

$$\mathsf{PoA}(\Gamma_{M_k^+}) = \frac{(\alpha_k + \alpha_{k+1})\frac{e^{\alpha_{k+1}}}{\alpha_{k+1}}}{\frac{e^{\alpha_{k+1}}}{\alpha_{k+1}}(1 + \alpha_k + \ln \alpha_{k+1})} = \frac{\alpha_k + \alpha_{k+1}}{1 + \alpha_k + \ln \alpha_{k+1}} = \frac{\frac{\alpha_k}{\alpha_{k+1}} + 1}{\frac{1}{\alpha_{k+1}} + \frac{\alpha_k}{\alpha_{k+1}} + \frac{\ln \alpha_{k+1}}{\alpha_{k+1}}} \to \infty$$

from which we get the conclusion

$$\limsup_{M\to\infty} \mathsf{PoA}(\Gamma_M) = +\infty.$$

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