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Advances in Applied Clifford Algebras

Geometric Algebras of Light Cone Projective Graph Geometries

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Abstract. A null vector is an algebraic quantity with the property that its square is zero. I denote the universal algebra generated by taking all sums and products of null vectors over the real or complex numbers by $\mathcal{N}$. The rules of addition and multiplication in $\mathcal{N}$ are taken to be the same as those for real and complex square matrices. A distinct pair of null vectors is *positively* or *negatively* correlated if their inner product is *positive* or *negative*, respectively. A *basis* of $n+1$ null vectors, with pairwise inner products equal to plus or minus one half, defines the Clifford geometric algebras $\mathbb{G}_{1,n}$, or $\mathbb{G}_{n,1}$, respectively, and provides a foundation for a new Cayley–Grassman linear algebra, a theory of complete graphs, and other applications in pure and applied areas of science.

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1. Introduction

The origin of the ideas in this paper date back most directly to mathematics that was set down in the nineteenth century by H. Grassmann [4], A. Cayley *(Memoir on the Theory of Matrices 1858)*, and W. Clifford [2]. It is regrettable today, after more than 150 years, that Clifford's *geometric algebra* has not found it proper place in the *Halls of Mathematics and Science* [26]. My journey in this saga began in 1965, when I starting working in geometric algebra as a graduate Ph.D. student of Professor David Hestenes at Arizona

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State University [8], and continued with years spent with gracious colleagues in Poland and Mexico. It is my belief that this paper will bring us closer to the day when geometric algebra has finally found its proper place in the Millennial Human Quest for the development of the *geometric concept of number* [21].

In Sect. 2, it is shown that the geometric algebras $\mathbb{G}_{1,n}$ and $\mathbb{G}_{n,1}$ have special bases of all positively, or all negatively correlated null vectors, respectively. In the case of $\mathbb{G}_{1,n}$, the inner products can all be chosen to be $+\frac{1}{2}$, and in the case of $\mathbb{G}_{n,1}$, $-\frac{1}{2}$. For simplicity, the classification of endomorphisms on $\mathbb{R}^{n+1}$ is considered only in the case of a $(+\frac{1}{2})$-positively correlated basis of a geometric algebra $\mathbb{G}_{1,n}$, but the same analysis is valid for studying endomorphisms on $\mathbb{R}^{n+1}$ of a $(-\frac{1}{2})$-negatively correlated basis of a geometric algebra $\mathbb{G}_{n,1}$.

In Sect. 3, basic ideas of linear algebra in $\mathbb{R}^{n+1}$ are developed in the symmetric algebra $\mathcal{A}_{n+1}^{+}$ of a correlated basis of null vectors in $\mathbb{G}_{1,n}^{1}$. The concept of a *star conjugation* of a geometric number is defined and studied. The *vector derivative* ∇ is defined, paying particular attention to its important properties.

In Sect. 4, basic properties of lower dimensional geometric algebras are explored in the correlated basis algebra $\mathcal{A}_4^{+}$ of $\mathbb{G}_{1,3}$. The concept of the star conjugation suggests that a new classification all geometric algebras is possible in the correlated null vector algebra $\mathcal{A}_{n+1}^{+}$ of $\mathbb{G}_{1,n}$, [24].

In Sect. 5, by introducing barycentric coordinates, complete graphs are studied in which every pair of vertices is connected by an edge. *Light Cone Projective Graph Geometry* (LPGG) is built upon the property that for any dimension $n \geq 1$, there exits *positively*, or *negatively correlated light cones*, defined by sets of $(n+1)$ null *basis vectors* $\{a_1, \ldots, a_{n+1}\}$ of $\mathbb{G}_{1,n}$, or $\mathbb{G}_{n,1}$, such that $a_1 \wedge \cdots \wedge a_{n+1} \neq 0$ and $a_i \cdot a_j = \pm\frac{(1-\delta_{ij})}{2}$, respectively.

2. The Geometric Algebras $\mathbb{G}_{1,n}$ and $\mathbb{G}_{n,1}$ of $\mathbb{R}^{n+1}$

A distinct pair of null vectors is *positively* or *negatively* correlated if their inner product is *positive* or *negative*, respectively. The geometric algebras $\mathbb{G}_{1,n}$ and $\mathbb{G}_{n,1}$ arise from null vector bases of $\mathbb{R}^{n+1}$ by constructing positively, or negatively correlated, null vectors in terms of the *standard bases* $\{e_1, f_1, \cdots, f_n\}$ of $\mathbb{G}_{1,n}$, or $\{f_1, e_1, \cdots, e_n\}$ of $\mathbb{G}_{1,n}$, respectively. Renewed interest in these Clifford algebras is due in part to the pivotal Lecture Notes published by Marcel Riesz in 1958, [16]. The geometric algebras $\mathbb{G}_{1,n}$ and $\mathbb{G}_{n,1}$ make up the two *fundamental sequences* of successively larger algebras,

$$\mathbb{R} \subset \mathbb{G}_{1,1} \subset \mathbb{G}_{1,2} \subset \mathbb{G}_{1,3} \subset \cdots \subset \mathbb{G}_{1,n} \subset \cdots \subset \mathcal{N}, \tag{1}$$

and

$$\mathbb{R} \subset \mathbb{G}_{1,1} \subset \mathbb{G}_{2,1} \subset \mathbb{G}_{3,1} \subset \cdots \subset \mathbb{G}_{n,1} \subset \cdots \subset \mathcal{N}, \tag{2}$$

where $\mathcal{N}$ is the universal algebra generated by taking sums and products of null vectors. See [21,22,26], and other references.

Table 1. Multiplication Table

	a_i	a_j	a_ia_j	a_ja_i
a_i	0	a_ia_j	0	a_i
a_j	a_ja_i	0	a_j	0
a_ia_j	a_i	0	a_ia_j	0
a_ja_i	0	a_j	0	a_ja_i

Let $\{a_1, \dots a_{n+1}\} \subset \mathbb{R}^{n+1}$ be a set of positively, or negatively, correlated null vectors satisfying $a_1 \wedge \cdots \wedge a_{n+1} \neq 0$, and the $\binom{n+2}{2}$ properties

$$a_i \cdot a_j \equiv \frac{1}{2}(a_ia_j + a_ja_i) := \pm\frac{1-\delta_{ij}}{2} \quad \text{for} \quad 1 \le i, j \le n+1, \tag{3}$$

respectively, where δ_{ij} is the usual delta function. In terms of these basis null vectors,

$$\mathbb{R}^{n+1} := \{x |\ x = x_1a_1 + \cdots + x_{n+1}a_{n+1},\ x_i \in \mathbb{R}\}. \tag{4}$$

The multiplication tables for sets of positively (PC), or negatively (NC), correlated null vectors a_i, a_j, for $1 \le i, j \le n+1$, follow directly from the properties (3), and generate the positively, or negatively, correlated null vector algebras $\mathcal{A}^+_{1,n} = \mathbb{G}_{1,n}$, and $\mathcal{A}^-_{n,1} = \mathbb{G}_{n,1}$, respectively.

For a set of positively or negatively correlated null vectors $\{a_1, \dots, a_n\}$, define

$$A_k := \sum_{i=1}^{k} a_i. \tag{5}$$

The geometric algebra

$$\mathbb{G}_{1,n} := \mathbb{R}(e_1, f_1, \dots, f_n),$$

where $\{e_1, f_1, \dots, f_n\}$ is the *standard basis* of anticommuting orthonormal vectors, with $e_1^2 = 1$ and $f_1^2 = \cdots = f_n^2 = -1$. The 2^{n+1}-*canonical forms* of the standard multivector basis elements are

$$\Big\{1; e_1, f_1, \cdots, f_n; e_1f_1, \cdots, e_1f_n, \big[_{1\le i<k\le n} f_if_k,\big]; \dots; e_1f_1\cdots f_n\Big\}. \tag{6}$$

Alternatively, the geometric algebra $\mathbb{G}_{1,n}$ can be defined by

$$\mathbb{G}_{1,n} := \mathbb{R}(a_1, \dots, a_{n+1}) =: \mathcal{A}^+_{1,n},$$

where $\{a_1, \dots, a_{n+1}\}$ is a set of positively correlated null vectors satisfying the Multiplication Table 1. In this case, the standard basis vectors of $\mathbb{G}_{1,n}$ can be defined by $e_1 = a_1 + a_2 = A_2$, $f_1 = a_1 - a_2 = A_1 - a_2$, and for $2 \le k \le n$

$$f_k = \alpha_k\Big(A_k - (k-1)a_{k+1}\Big), \tag{7}$$

TABLE 2. Multiplication Table

	a_i	a_j	$a_i a_j$	$a_j a_i$
a_i	0	$a_i a_j$	0	$-a_i$
a_j	$a_j a_i$	0	$-a_j$	0
$a_i a_j$	$-a_i$	0	$-a_i a_j$	0
$a_j a_i$	0	$-a_j$	0	$-a_j a_i$

where $\alpha_k := \frac{-\sqrt{2}}{\sqrt{k(k-1)}}$. The 2^{n+1}-*canonical forms* of the standard multivector basis elements of $\mathcal{A}_{n+1}^+$ are

$$\Big\{1; a_1, \ldots, a_{n+1}; \big[_{1\le i<j\le n+1} a_i a_j, \big]; \ldots; a_1 \cdots a_{n+1}\Big\}. \tag{8}$$

The geometric algebra

$$\mathbb{G}_{n,1} := \mathbb{R}(f_1, e_1, \ldots, e_n),$$

where $\{f_1, e_1, \ldots e_n\}$ is the *standard basis* of anticommuting orthonormal vectors, with $f_1^2 = -1$ and $e_1^2 = \cdots = e_n^2 = 1$. Alternatively, the geometric algebra $\mathbb{G}_{n,1}$ can be defined by

$$\mathbb{G}_{n,1} := \mathbb{R}(a_1, \ldots, a_{n+1}) =: \mathcal{A}_{1,n}^-,$$

where $\{a_1, \ldots, a_{n+1}\}$ is a set of negatively correlated null vectors satisfying the Multiplication Table 2. In this case, the standard basis vectors of $\mathbb{G}_{n,1}$ can be defined by $f_1 = a_1 + a_2 = A_2$, $e_1 = a_1 - a_2 = A_1 - a_2$, and for $2 \le k \le n$

$$e_k = \alpha_k \Big(A_k - (k-1)a_{k+1}\Big), \tag{9}$$

where $\alpha_k := \frac{-\sqrt{2}}{\sqrt{k(k-1)}}$. The 2^{n+1}-*canonical forms* of the standard multivector basis elements of $\mathcal{A}_{n+1}^-$ is the same as (8).

For the remainder of this paper, only properties of the positively correlated null vector algebras $\mathcal{A}_{n+1}^+ := \mathcal{A}_{1,n}$ of the geometric algebras $\mathbb{G}_{1,n}$ are considered. It should be recognized, however, that any of these properties can be easily translated to the corresponding properties of the negatively correlated null vector basis algebras $\mathcal{A}_{n+1}^- := \mathcal{A}_{n,1}$ of $\mathbb{G}_{n,1}$. Indeed, much more general algebras of correlated null vectors in $\mathcal{N}$ can be defined and studied, but with correspondingly more complicated rules of multiplication. In addition to providing a new framework for the study of Linear Algebra on $\mathbb{R}^{n+1}$, the last section of the paper shows how the ideas can be applied to graph theory.

3. Linear Algebra of $\mathbb{R}^{n+1}$ in $\mathcal{A}^+_{n+1}$

The *position vector* $x \in \mathbb{R}^{n+1}$ in the standard basis of $\mathbb{G}_{1,n}$ is

$$x := s_1 e_1 + \sum_{i=1}^{n} s_{i+1} f_i \in \mathbb{G}^1_{1,n}. \tag{10}$$

Alternatively, in the correlated null vector basis algebra $\mathcal{A}^+_{n+1} \equiv \mathbb{G}_{1,n}$,

$$x = \sum_{i=1}^{n+1} x_i a_i \in \mathcal{A}^+_{n+1}. \tag{11}$$

Since geometric algebras are fully compatible with matrix algebras, matrix algebras over geometric algebras are well defined [21]. To relate the bases (10) and (11), in matrix notation

$$x = \begin{pmatrix} s_1 & \dots & s_{n+1} \end{pmatrix} \begin{pmatrix} e_1 \\ f_1 \\ \cdot \\ \cdot \\ \cdot \\ f_n \end{pmatrix} = \begin{pmatrix} x_1 & \dots & x_{n+1} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \cdot \\ \cdot \\ \cdot \\ a_{n+1} \end{pmatrix}, \tag{12}$$

or in *abbreviated form*, $x = s_{(n+1)} F_{(n+1)} = x_{(n+1)} A_{(n+1)}$. The *quadratic form* B of $\mathbb{G}_{1,n}$ is specified by $F^{-1}_{(n+1)} := F^t_{(n+1)} B$, where

$$B := F_{(n+1)} \cdot F^t_{(n+1)} = \begin{pmatrix} e_1 \\ f_1 \\ \cdot \\ \cdot \\ \cdot \\ f_n \end{pmatrix} \cdot \begin{pmatrix} e_1 & f_1 & \dots & f_n \end{pmatrix} \tag{13}$$

$$= \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & 0 & -1 \end{pmatrix},$$

where $F^t_{(n+1)}$ denotes the *row transpose* of the column $F_{(n+1)}$.

Let $v, w \in \mathbb{G}^1_{1,n}$ be vectors. Expressed in the *standard basis* of $\mathbb{G}_{1,n}$, the geometric product

$$\begin{aligned} vw &= v_{(n+1)} F_{(n+1)} F^t_{(n+1)} w^t_{(n+1)} \\ &= v_{(n+1)} F_{(n+1)} \cdot F^t_{(n+1)} w^t_{(n+1)} + v_{(n+1)} F_{(n+1)} \wedge F^t_{(n+1)} w^t_{(n+1)} \end{aligned}$$

$$= v_{(n+1)} B w^t_{(n+1)} + v_{(n+1)} \begin{pmatrix} 0 & e_1 f_1 & e_1 f_2 & \cdots & e_1 f_n \\ f_1 e_1 & 0 & f_1 f_2 & \cdots & f_1 f_n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ f_n e_1 & f_n f_1 & f_n f_2 & \cdots & 0 \end{pmatrix} w^t_{(n+1)}. \qquad (14)$$

Dotting each side of the equation (12) on the right by the row matrix

$$F^{-1}_{(n+1)} := \begin{pmatrix} e_1 & -f_1 & \dots & -f_n \end{pmatrix},$$

and noting that

$$\begin{pmatrix} e_1 \\ f_1 \\ \cdot \\ \cdot \\ \cdot \\ f_n \end{pmatrix} \cdot \begin{pmatrix} e_1 & -f_1 & \dots & -f_n \end{pmatrix}$$

is an expression for the $(n+1) \times (n+1)$ identity matrix, immediately gives $s_{(n+1)} = x_{(n+1)} T$, where the matrix of transition T is defined by the *Gramian matrix*

$$T := A_{(n+1)} \cdot F^{-1}_{(n+1)} = \begin{pmatrix} a_1 \\ a_2 \\ \cdot \\ \cdot \\ \cdot \\ a_{n+1} \end{pmatrix} \cdot \begin{pmatrix} e_1 & -f_1 & \dots & -f_n \end{pmatrix} \qquad (15)$$

in terms of the inner products $a_i \cdot f_j$. These inner products are directly calculated using (7). The transition matrix T_8, and its inverse T_8^{-1}, for the geometric algebra $\mathbb{G}_{1,7}$ is given in Appendix A.

Note, that whereas T is the transition matrix

$$T F_{(n+1)} = T \begin{pmatrix} e_1 \\ f_1 \\ \cdot \\ \cdot \\ f_n \end{pmatrix} = \begin{pmatrix} a_1 \\ \cdot \\ \cdot \\ a_{n+1} \end{pmatrix} = A_{(n+1)}, \qquad (16)$$

of the column basis vectors $F_{(n+1)}$ to the column basis vectors $A_{(n+1)}$, T^{-1} is the coordinate transition matrix

$$s_{(n+1)} T^{-1} = \begin{pmatrix} s_1 & \cdots & s_{n+1} \end{pmatrix} T^{-1} = \begin{pmatrix} x_1 & \cdots & x_{n+1} \end{pmatrix} = x_{(n+1)}, \qquad (17)$$

from the *row vector* coordinates of x to the row vector coordinates of x in the basis $A_{(n+1)}$. Great care must be taken to avoid confusion.

Converting the calculation in (14) to a calculation for $v, w \in \mathcal{A}^+_{n+1}$,

$$\begin{aligned} vw &= v_{(n+1)} F_{(n+1)} F^t_{(n+1)} w^t_{(n+1)} \\ &= v_{(n+1)} T^{-1} T F_{(n+1)} F^t_{(n+1)} T^t (T^t)^{-1} w^t_{(n+1)} \\ &= v^a_{(n+1)} A_{(n+1)} A^t_{(n+1)} (w^a_{(n+1)})^t, \end{aligned}$$

giving

$$vw = v^a_{(n+1)}A_{(n+1)} \cdot A^t_{(n+1)}(w^a_{(n+1)})^t + v^a_{(n+1)}A_{(n+1)} \wedge A^t_{(n+1)}(w^a_{(n+1)})^t$$

$$= v^a_{(n+1)} \begin{pmatrix} 0 & a_1a_2 & a_1a_3 & \cdots & a_1a_{n+1} \\ a_2a_1 & 0 & a_2a_3 & \cdots & a_2a_{n+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n+1}a_1 & a_{n+1}a_2 & a_{n+1}a_3 & \cdots & 0 \end{pmatrix} (w^a_{(n+1)})^t. \tag{18}$$

3.1. Bivector Endomorphisms in $\mathcal{A}^+_{n+1}$

The standard treatment of the relationship between Clifford's geometric algebras and Cayley's matrix algebras is well-known, [22, p.74], [13, p.217]. Something that has always disturbed me is that this relationship is an isomorphism only for square matrix algebras of order $2^n \times 2^n$. This sorry state of affairs is at least partially rectified in the algebras $\mathcal{A}^+_{n+1} = \mathbb{G}_{1,n}$.

Recalling the definition of (5), it is easy to show, by induction, that for $k \geq 2$,

$$(A_k)^2 = \binom{k}{2}. \tag{19}$$

For $k = 2$, $A_2^2 = (a_1 + a_2)^2 = a_1^2 + 2a_1 \cdot a_2 + a_2^2 = 1 = \binom{2}{2}$. Assuming true for $k = n$, for $k = n + 1$,

$$A^2_{n+1} = A^2_n + 2a_{n+1} \cdot A_n + a^2_{n+1} = \binom{n}{2} + 2\frac{n}{2} = \binom{n+1}{2},$$

completing the proof.

Letting $\hat{A}_k := \frac{\sqrt{2}}{\sqrt{k(k-1)}}A_k$, it follows that $\hat{A}_k^2 = 1$. For $g \in \mathcal{A}^+_{n+1}$, and $k \geq 2$, define the *LPGG star k-conjugation* of $g \in \mathcal{A}^+_{n+1}$ by

$$g^\star := \hat{A}_k g \hat{A}_k. \tag{20}$$

For $g, h \in \mathcal{A}^+_{n+1}$, $(g^\star)^\star = g$, $(h^\star)^\star = h$, and

$$(gh)^\star = \hat{A}_k g \hat{A}_k \hat{A}_k h \hat{A}_k = g^\star h^\star. \tag{21}$$

Defining the *A-matrix* of $g \in \mathcal{A}^+_{n+1}$ by

$$[g]_a := \hat{A}_{(n+1)} g \hat{A}^t_{(n+1)} = [a_i g a_j]_a, \tag{22}$$

where $\hat{A}_{(n+1)} := \frac{\sqrt{2}}{\sqrt{(n+1)n}}A_{(n+1)}$, it follows that

$$g^\star = \hat{A}_{n+1} g \hat{A}_{n+1} = \mathcal{I}^t_{(n+1)}[g]_a \mathcal{I}_{(n+1)},$$

where $\mathcal{I}_{(n+1)}$ and $\mathcal{I}^t_{(n+1)}$ are the $n+1$ column and row matrices

$$\mathcal{I}_{(n+1)} := \begin{pmatrix} 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{pmatrix}, \quad \mathcal{I}^t_{(n+1)} := \begin{pmatrix} 1 \; 1 \cdots 1 \end{pmatrix},$$

respectively.

The GA product of $g, h \in \mathcal{A}^+_{n+1}$, in terms of their $(n+1)\times(n+1)$ A-matrices $[g]_a, [h]_a$, then takes the unusual form

$$gh = \hat{A}_{n+1}\mathcal{I}^t_{(n+1)}\Big([g]_a\mathcal{I}_{(n+1)}\mathcal{I}^t_{(n+1)}[h]_a\Big)\mathcal{I}_{(n+1)}\hat{A}_{n+1}, \tag{23}$$

mediated by the square singular $(n+1)$-matrix $\mathcal{I}_{(n+1)}\mathcal{I}^t_{(n+1)}$. Equation (23) is a generalization of the closely related formula (18) for the multiplication of the vectors $v, w \in \mathcal{A}^+_{n+1}$.

It follows from (20) and (23) that a real or complex $(n+1)\times(n+1)$-matrix $[g_{ij}]$ is the matrix of a scalar plus a bivector $g \in \mathcal{A}^+_{n+1}$, that is

$$[g]_a = \hat{A}_{n+1}[g^\star]_a\hat{A}_{n+1}$$

$$= \begin{pmatrix} 0 & g_{12}a_1a_2 & g_{13}a_1a_3 & \cdots & g_{1,n+1}a_1a_{n+1} \\ g_{21}a_2a_1 & 0 & g_{23}a_2a_3 & \cdots & g_{2,n+1}a_2a_{n+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ g_{1,n+1}a_{n+1}a_1 & g_{2,n+1}a_{n+1}a_2 & g_{n,n+1}a_{n+1}a_3 & \cdots & 0 \end{pmatrix}. \tag{24}$$

Comparing the matrix $[g]_a$ in (24) to the matrix in (14), seems to contradict that the trace of a matrix is invariant under a change of basis. However this is not the case since the terms $g_{ij}a_ia_j$ of (24) consists of scalars and bivectors. Plücker relations are important in understanding the structure of bivectors [23], particularly bivectors in $\mathbb{G}_{1,n}$, and in study of conformal mappings [20].

3.2. The Gradient ∇

A crucial tool for carrying out calculations in the geometric algebra $\mathbb{G}_{1,n}$ is the *gradient* ∇. In the references [5,8,21], the gradient ∇, alongside the geometric algebra $\mathbb{G}_n$, has been developed as a basic tool for formulating and proving basic theorems of linear algebra in $\mathbb{R}^n$. Since the properties of the gradient are independent of the quadratic form of the geometric algebra used, instead of using the Euclidean geometric algebra $\mathbb{G}_{n+1}$ of $\mathbb{R}^{n+1}$, we can equally well define it in terms of the geometric algebra $\mathbb{G}_{1,n}$. It follows that all theorems of linear algebra developed in [5,8,21] are equally valid in $\mathbb{G}_{1,n}$ without modification. In the standard basis of $\mathbb{G}_{1,n}$,

$$\nabla := e_1\frac{\partial}{\partial s_1} - f_1\frac{\partial}{\partial s_2} - \cdots - f_n\frac{\partial}{\partial s_{n+1}}. \tag{25}$$

With the transition matrix (15) in hand, the expression for the gradient in the null vector basis of $\mathcal{A}_{1,n}$,

$$\nabla = \sum_{i=1}^{n+1} (\nabla x_i) \frac{\partial}{\partial x_i}, \tag{26}$$

is nothing more than a simple expression of the *chain rule* in calculus. In terms of the abbreviated notation for (12), it is not difficult to derive the transformation rules relating the *columns of basis vectors* $A_{(n+1)}$ and $F_{(n+1)}$.

Using (15), and solving

$$x = s_{(n+1)} F_{(n+1)} = x_{(n+1)} A_{(n+1)}, \tag{27}$$

gives the important relations

- $s_{(n+1)} = x_{(n+1)} T \iff x_{(n+1)} = s_{(n+1)} T^{-1}$
- $x \cdot F^{-1}_{(n+1)} = s_{(n+1)} = x_{(n+1)} T \iff F^{-1}_{(n+1)} = \nabla s_{(n+1)} = \nabla x_{(n+1)} T$
- $x_{(n+1)} = s_{(n+1)} T^{-1} \iff A^{-1}_{(n+1)} := \nabla x_{(n+1)} = F^{-1}_{(n+1)} T^{-1}$.
- $A^{-1}_{(n+1)} = F^{-1}_{(n+1)} T^{-1} \iff A^{-1}_{(n+1)} T = F^{-1}_{(n+1)}$.
- $\nabla x = n + 1 = F^{-1}_{(n+1)} F_{(n+1)} = A^{-1}_{(n+1)} A_{(n+1)}$.
- $F_{(n+1)} = F_{(n+1)} \cdot \nabla x = F_{(n+1)} \cdot \nabla x_{(n+1)} A_{(n+1)}$
  $= F_{(n+1)} \cdot \nabla s_{(n+1)} T^{-1} A_{(n+1)} = T^{-1} A_{(n+1)}$.

Note, whereas $F_{(n+1)}$ and $A_{(n+1)}$ have been defined as *column* matrices of vectors, $F^{-1}_{(n+1)}$ and $A^{-1}_{(n+1)}$ are *row* matrices of vectors. Taking the outer product of basis vectors in the relation $F_{(n+1)} = T^{-1} A_{(n+1)}$, gives

$$\wedge F_{(n+1)} = \det T^{-1} \wedge A_{(n+1)},$$

or equivalently, after calculating and simplifying,

$$e_1 f_1 \cdots f_n = -\frac{(\sqrt{2})^{n+1}}{\sqrt{n}} a_1 \wedge \cdots \wedge a_{n+1}, \tag{28}$$

relating the pseudoscalar elements of the geometric algebra $\mathbb{G}_{1,n}$ expressed in the standard basis and in the null vector basis of $\mathcal{A}^+_{n+1}$.

3.3. Decomposition Formulas for ∇

Usually the concept of *duality* is defined between two distinct vector spaces, or in terms of the operation of multiplication in an algebraic structure such as the geometric algebra $\mathbb{G}_n$ of Euclidean space. The geometric algebras $\mathbb{G}_{1,n}$ and $\mathbb{G}_{n,1}$, defined in terms of the null vector basis algebras $\mathcal{A}^+_{1,n}$ and $\mathcal{A}^-_{n,1}$, whose rules of multiplication have been given in the Multiplication Tables 1 and 2, suggests a new concept of duality. The *dual n-sum* $\overset{\vee}{a}_i$ of $a_i \in \mathcal{A}^+_{n+1}$ is the n-sum

$$\overset{\vee}{a}_i := a_1 + \cdots + \overset{\vee}{i} + \cdots a_{n+1} = A_{n+1} - a_i, \tag{29}$$

formed leaving out the i^{th} term of the basis null vectors $\{a_1, \ldots, a_{n+1}\} \subset \mathcal{A}^+_{n+1}$. Another related notation which we will use is the *wedge-dual*,

$$\wedge \overset{\vee}{a}_{(i)} := a_1 \wedge \cdots \wedge \overset{\vee}{i} \wedge \cdots \wedge a_{n+1}, \tag{30}$$

formed leaving out wedge the i^{th}-term.

Calculations with the gradient ∇ in $\mathcal{A}_{n+1}^+$ can often be simplified using the following *decompostion formulas*. Defining the *dual sum* and *null gradients*

$$\check{\nabla} := \sum_{i=1}^{n+1} {}^{\vee}a_i \partial_i \quad \text{and} \quad \hat{\nabla} := \sum_{i=1}^{n+1} a_i \partial_i, \tag{31}$$

respectively, the gradient ∇ defined by (26), $\check{\nabla}$ and $\hat{\nabla}$ defined in (31), satisfy the following decomposition formulas:

- $\nabla = \frac{2}{n}\Big(A_{n+1}\partial_{(n+1)} - n\hat{\nabla}\Big) = \frac{2}{n}\Big(\check{\nabla} - (n-1)\hat{\nabla}\Big)$, where $\partial_{(n+1)} := \sum_{i=1}^{n+1} \partial_i$.
- $A_{n+1} \cdot \nabla = (n+1)\partial_{(n+1)} - 2A_{n+1} \cdot \hat{\nabla}$
- $\check{\nabla} + \hat{\nabla} = A_{n+1}\partial_{(n+1)} \iff A_{n+1} \cdot \check{\nabla} + A_{n+1} \cdot \hat{\nabla} = \frac{(n+1)n}{2}\partial_{(n+1)}$
- $\hat{\nabla}^2 = \sum_{i<j}^{n+1} \partial_i\partial_j, \ \check{\nabla}^2 = \frac{(n-1)n}{2}\sum_{i=1}^{n+1} \partial_i^2 + (n^2-n+1)\sum_{i<j}^{n+1} \partial_i\partial_j$
- $\nabla^2 = \frac{4}{n^2}\Big(\check{\nabla} - (n-1)\hat{\nabla}\Big)^2 = \check{\nabla}^2 - 2(n-1)\check{\nabla} \cdot \hat{\nabla} + \hat{\nabla}^2$,
  where $\check{\nabla} \cdot \hat{\nabla} = \frac{n}{2}\sum_{i=1}^{n+1} \partial_i^2 + (n-1)\sum_{i<j} \partial_i\partial_j$.

Verifications of the above formulas are omitted. They depend heavily on the combinatorial-like identities

$$A_{n+1}^2 = \frac{(n+1)n}{2}, \quad a_i \cdot A_{n+1} = A_{n+1} \cdot a_i = \frac{n}{2}, \tag{32}$$

and the *additive duality* formula for distinct i, j, $1 \le i, j \le n+1$,

$${}^{\vee}a_i \cdot {}^{\vee}a_j = \frac{n^2 - n + 1}{2}, \tag{33}$$

as easily follow from (19) and (29).[1]

4. Lower Dimensional Geometric Algebras

This section characterizes geometric sub-algebras of $\mathcal{A}_3^+ \equiv \mathbb{G}_{1,2}$ in $\mathbb{R}^3$.

The pseudoscalar

$$i := e_1 f_1 f_2 = -2a_1 \wedge a_2 \wedge a_3, \tag{34}$$

is in the center of the algebra, commuting with all elements. The algebra

$$\mathbb{G}_3 := \mathbb{R}(e_1, e_2, e_3),$$

is obtained from the algebra $\mathbb{G}_{1,2}$, simply by defining $e_2 = if_1 = e_1 f_2 \in \mathbb{G}_{1,2}^2$ and $e_3 = -if_2 = e_1 f_1 \in \mathbb{G}_{1,2}^2$, and reinterpreting these anticommuting elements to be vectors in $\mathbb{G}_3^1$.

The *matrix coordinates* $[e_1], [e_2], [e_3]$ of e_1, e_2, e_3, known as the famous *Pauli matrices*, opened the door to the study of quantum mechanics [22, p.108]. It has found many recent applications in computer science and robotics, [9]. The geometric algebra $\mathbb{G}_3$ is isomorphic to the even subalgebra of the

[1] A more comprehensive treatment of these formulas is found in my preprint "Calculus of Compatible Nilpotents", https://hal.science/hal-04108375 (May 2023).

spacetime algebra $\mathbb{G}_{1,3} = \mathcal{A}_4^+$ of $\mathbb{R}^4$. Its matrix version is known as the *Dirac algebra*.

The null vector basis algebra $\mathcal{A}_3^+ = \mathbb{G}_{1,2}$ is defined by 3 null vectors $\{a_1, a_2, a_3\}$ with the property that $\wedge A_{(3)} \neq 0$, $a_i \cdot a_j = \frac{(1-\delta ij)}{2}$, and the Multiplication Table 1. The relations between the standard basis of $\mathbb{G}_{1,2}$, and the basis of $\mathcal{A}_3^+$, are summarized by the 3×3 transition matrix T_3, and its inverse,

$$T_3 := \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad T_3^{-1} := \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ -1 & -1 & 1 \end{pmatrix}. \tag{35}$$

Using the relations given after (27),

$$\begin{pmatrix} e_1 \\ f_1 \\ f_2 \end{pmatrix} = T_3^{-1} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} a_1 + a_2 \\ a_1 - a_2 \\ -a_1 - a_2 + a_3 \end{pmatrix} \tag{36}$$

and

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = T_3 \begin{pmatrix} e_1 \\ f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(e_1 + f_1) \\ \frac{1}{2}(e_1 - f_1) \\ e_1 + f_2 \end{pmatrix}. \tag{37}$$

The canonical forms relating the vectors, bivectors and trivectors are:

- $e_1 = a_1 + a_2, \ f_1 = a_1 - a_2, f_2 = -a_1 - a_2 + a_3$
- $e_1 f_1 = (a_1 + a_2)(a_1 - a_2) = a_2 a_1 - a_1 a_2 = 1 - 2a_1 a_2$
- $e_1 f_2 = (a_1 + a_2)(-a_1 - a_2 + a_3) = -1 + a_1 a_3 + a_2 a_3$
- $f_1 f_2 = (1 - 2a_1 a_2 + a_1 a_3 - a_2 a_3)$
- $e_1 f_1 f_2 = (a_1 + a_2)(1 - 2a_1 a_2 + a_1 a_3 - a_2 a_3) = a_1 - a_2 + a_3 - 2a_1 a_2 a_3$

One of the simplest endomorphisms, $f : \mathbb{R}^2 \to \mathbb{R}^2$, defined by $v_1, v_2 \in \mathbb{R}^2$, is

$$f(x) := 2(v_1 \wedge v_2)x = 2\Big((x \cdot v_2)v_1 - (x \cdot v_1)v_2\Big), \tag{38}$$

where $v_i = v_{i1}a_1 + v_{i2}a_2$ for $i \in \{1, 2\}$. The endomorphism $f(x)$ has the *eigenvectors* a_1 and a_2, with the eigenvalues $\pm \det \begin{pmatrix} v_{11} & v_{12} \\ v_{12} & v_{22} \end{pmatrix}$,

$$f(a_1) = 2(v_1 \wedge v_2)a_1 = \det \begin{pmatrix} v_{11} & v_{12} \\ v_{12} & v_{22} \end{pmatrix} a_1, \tag{39}$$

$$f(a_2) = 2(v_1 \wedge v_2)a_2 = -\det \begin{pmatrix} v_{11} & v_{12} \\ v_{12} & v_{22} \end{pmatrix} a_2, \tag{40}$$

respectively, as is easily verified.

Now calculate,

$$(v_1 \wedge v_2)(v_1 \wedge v_2 \wedge x) = (v_1 \wedge v_2) \cdot (v_1 \wedge v_2 \wedge x)$$
$$= (v_1 \wedge v_2)^2 x + (v_1 \wedge v_2) \cdot (v_2 \wedge x)v_1 + (v_1 \wedge v_2) \cdot (x \wedge v_1)v_2 = 0.$$

Dividing both sides of this last equation by $(v_1 \wedge v_2)^2$, gives

$$\frac{(v_1 \wedge v_2 \wedge x)}{(v_1 \wedge v_2)} = x - \frac{(x \wedge v_2)}{(v_1 \wedge v_2)} v_1 + \frac{(x \wedge v_1)}{(v_1 \wedge v_2)} v_2 = 0, \tag{41}$$

expressing the position vector $x \in \mathbb{R}^2$ uniquely in terms of its LPGG projective coordinates. Of course, the trivector $v_1 \wedge v_2 \wedge x = 0$, because we are in the geometric algebra $\mathbb{G}_{1,1}$ of $\mathbb{R}^2$. Multiplying equation (41) by $4(v_1 \wedge v_2)^2$ immediately gives what I call the *Cayley–Grassmann identity*,

$$4(v_1 \wedge v_2)(v_1 \wedge v_2 \wedge x) = f^2(x) - 4(v_1 \wedge v_2)(x \wedge v_2)v_1 + 4(v_1 \wedge v_2)(x \wedge v_1)v_2$$

$$= f^2(x) - 2f(x) \cdot v_2 v_1 + 2f(x) \cdot v_1 v_2 = 0. \tag{42}$$

The *matrix* of $[f(x)]$ of $f(x)$ is given by

$$[f(x)] = [2(v_1 \wedge v_2)x] = 2[(v_1 \wedge v_2)][x], \tag{43}$$

which is the product of the matrix

$$[v_1 \wedge v_2] = \frac{1}{2}\Big([v_1 v_2 - v_2 v_1]\Big) = \frac{1}{2}\Big([v_1][v_2] - [v_2][v_1]\Big), \tag{44}$$

where $[v_1 \wedge v_2], [v_1], [v_2]$ are the matrices of $v_1 \wedge v_2, v_1, v_2$, respectively, and $[x]$ is the matrix of x. These matrices are given below. With (36) and (37) in hand, the matrix $[x]$ of the position vector $x \in \mathbb{R}^3$

$$[x] = x_1[a_1] + x_2[a_2] + x_3[a_3] = \begin{pmatrix} x_3 i & x_2 - x_3 \\ x_1 - x_3 & -x_3 i \end{pmatrix}$$

with respect to the basis $\mathcal{A}_3^+$, and

$$[x] = s_1[e_1] + s_2[f_1] + s_3[f_2] = \begin{pmatrix} s_3 i & s_1 - s_2 \\ s_1 + s_2 & -s_3 i \end{pmatrix}$$

with respect to the standard basis of $\mathbb{G}_{1,2}$.

The 2×2 matrices are defined with respect to the *spectral basis*

$$\begin{pmatrix} a_2 a_1 & a_2 \\ a_1 & a_1 a_2 \end{pmatrix}$$

of $\mathbb{G}_{1,1}$, as detailed in [22] and [21, p.78]. The matrices of $[a_1]$ and $[a_2]$ of a_1 and a_2, are

$$[a_1] = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad [a_2] = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

respectively, and

$$[x] = \begin{pmatrix} 0 & x_2 \\ x_1 & 0 \end{pmatrix}, \ [v_1] = \begin{pmatrix} 0 & v_{12} \\ v_{11} & 0 \end{pmatrix}, \ [v_2] = \begin{pmatrix} 0 & v_{22} \\ v_{21} & 0 \end{pmatrix},$$

which are used with (44) to calculate

$$[v_1 \wedge v_2] = \frac{1}{2}\Big([v_1][v_2] - [v_2][v_1]\Big) = \frac{1}{2}\begin{pmatrix} v_{12}v_{21} - v_{11}v_{22} & 0 \\ 0 & v_{11}v_{22} - v_{12}v_{21} \end{pmatrix},$$

and

$$[f(x)] = 2[(v_1 \wedge v_2)][x] = \begin{pmatrix} 0 & (v_{12}v_{21} - v_{11}v_{22})x_2 \\ -(v_{12}v_{21} - v_{11}v_{22})x_1 & 0 \end{pmatrix}.$$

Usually an endomorphism $f(x)$ on $\mathbb{R}^2$ is represented by a 2×2 matrix $[f]$ acting on a column matrix $[x] = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. Alternatively, the matrix $[x]$ of x can be represented by the matrix endomorphism $[x] := \begin{pmatrix} x_1 & 0 \\ x_2 & 0 \end{pmatrix}$, [13, p.52]. The matrix of the endomorphism $f(x)$ in the LPGG of $\mathcal{V}_2^+(v_1, v_2)$ comes as the single real matrix $[f(x)]$, which can be broken into the product of two 2×2 matrices. By (43),

$$[f(x)] = 2[(v_1 \wedge v_2)][x] = \begin{pmatrix} v_{12}v_{21} - v_{11}v_{22} & 0 \\ 0 & -(v_{12}v_{21} - v_{11}v_{22}) \end{pmatrix} \begin{pmatrix} 0 & x_2 \\ x_1 & 0 \end{pmatrix}.$$

For $k \in \{1, 2, 3\}$, let

$$v_k := v_{k1}a_1 + v_{k2}a_2 + v_{k3}a_3 \in \mathbb{R}^3.$$

Consider the endomorphism

$$f : \mathbb{R}^3 \to \mathcal{A}_3^+, \tag{45}$$

defined by

$$f(x) := 2(v_1 \wedge v_2 \wedge v_3)x = 2\det[v_{ij}](a_1 \wedge a_2 \wedge a_3)x$$
$$= \det[v_{ij}]\Big((x_1 + x_2)a_1 \wedge a_2 + (x_2 + x_3)a_2 \wedge a_3 + (x_1 + x_3)a_3 \wedge a_1\Big).$$

It is interesting to note that each of the bivectors in the above expression are anticommutative and square to $\frac{1}{4}$. In view of (34), this is not surprising. Indeed, the mapping (45) can simply be expressed as the duality relation $f(x) = -ix$. It follows that over the complex numbers, every vector $x \in \mathcal{A}_3^+$ is an eigenvector.

Let $g : \mathbb{R}^3 \to \mathcal{A}_3^+$, be defined by

$$g(x) := \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & g_{12}a_1a_2 & g_{13}a_1a_3 \\ g_{21}a_2a_1 & 0 & g_{23}a_2a_3 \\ g_{31}a_3a_1 & g_{32}a_3a_2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} x = Gx, \tag{46}$$

where

$$G = \frac{1}{2}tr(G) + \mathrm{g}_1 a_2 \wedge a_3 + \mathrm{g}_2 a_3 \wedge a_1 + \mathrm{g}_3 a_1 \wedge a_2, \tag{47}$$

for $\mathrm{g}_1 := (g_{23} - g_{32})$, $\mathrm{g}_2 := (g_{31} - g_{13})$, $\mathrm{g}_3 := (g_{12} - g_{21})$, and

$$tr(G) := g_{12} + g_{13} + g_{21} + g_{23} + g_{31} + g_{32}.$$

The same mapping (46) can equally well be considered over $\mathbb{C}$,

$$g : \mathbb{C}^3 \to \mathcal{A}_3^+(\mathbb{C}). \tag{48}$$

The *minimal polynomial* of G is easily calculated. Starting with (47),

$$\big(G - \frac{1}{2}tr(G)\big)^2 = \big(g_1 a_2 \wedge a_3 + g_2 a_3 \wedge a_1 + g_3 a_1 \wedge a_2\big)^2.$$

Since every bivector in the geometric algebras $\mathbb{G}_{1,2} \widetilde{=} \mathbb{G}_3$ is a simple bivector, or *blade*, it follows that the right-hand side of this equation is a complex scalar. Thus,

$$\big(g_1 a_2 \wedge a_3 + g_2 a_3 \wedge a_1 + g_3 a_1 \wedge a_2\big)^2 = \frac{1}{4}(g_1^2 + g_2^2 + g_3^2) - \frac{1}{2}(g_1 g_2 + g_1 g_3 + g_2 g_3)$$

Putting these two equations together, and simplifying, gives the minimal polynomial of the *Cayley-Hamilton Theorem,*

$$\varphi(G) = \big(G - \frac{1}{2} tr(G)\big)^2 - \frac{1}{4}\Big(\mathrm{g}_1 + \mathrm{g}_2 + \mathrm{g}_3\Big)^2 + g_1 g_2 + g_1 g_3 + g_2 g_3 \equiv 0.$$

Setting $\varphi(r) = 0$ and solving for r, gives the two roots r_- and r_+,

$$r_{\mp} := \frac{1}{2}\Big(tr(G) \mp \sqrt{tr(G)^2 - 4H}\Big),$$

for

$$H = \frac{1}{4}\Big(tr(G)^2 - (g_1 + g_2 + g_3)^2\Big) + g_1 g_2 + g_1 g_3 + g_2 g_3.$$

In the *spectral basis*, see [19,21,22], G takes the form

$$G = r_- p_1(G) + r_+ p_2(G), \tag{49}$$

where

$$p_1(t) := \frac{-2t + tr(G) + \sqrt{tr(G)^2 - 4H}}{2\sqrt{tr(G)^2 - 4H}}$$

and

$$p_2(t) := \frac{2t - tr(G) + \sqrt{tr(G)^2 - 4H}}{2\sqrt{tr(G)^2 - 4H}}.$$

There are whole classes of endomorphisms $f : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ defined in terms of *double coverings* with respect to group elements of the *Lipchitz Group* $\Gamma_{1,n}$ contained in the geometric algebra $\mathbb{G}_{1,n}$, [13, p.220]. For an invertible element $h \in \Gamma_{1,n}$, the endomorphism f is defined by $f(x) := hxh^{-1} \in \mathbb{R}^{n+1}$. In [27], I begin the study of endomorphisms generated by elements in the Lorentz group $O_{1,n} \subset \Gamma_{1,n}$ acting on representations of the symmetric group of permutations, useful in quantum computation [11, p.60].

5. Simplices in $\mathcal{A}_{n+1}^+$

It has been shown in previous sections how the development of linear algebra can be carried out in $\mathbb{R}^{n+1}$, using the tools of $\mathbb{G}_{1,n} \equiv \mathcal{A}_{n+1}^+$. Restricting to barycentric coordinates, gives new tools for application in graph theory. In *Simplicial Calculus with Geometric Algebra*, many ideas of simplicial geometry were set down in the context of geometric algebra [18], in spite of pesky problems with the *Schwarz paradox* [14,17]. The present work is in many ways a continuation of this earlier work.

Let $\mathcal{A}_{n+1}^+$ be the null vector algebra of the geometric algebra $\mathbb{G}_{1,n}$, defined by the Multiplication Table 1, and where the null vectors a_i satisfy for $1 \le i, j \le n+1$,

$$a_i \cdot a_j = \frac{(1-\delta_{ij})}{2}. \tag{50}$$

For $x \in \mathbb{R}^{n+1}$ the position vector (11), the *convex null n-simplex* in $\mathbb{R}^{n+1}$ is defined by

$$\mathcal{S}_n^+ := \mathcal{S}_n^+(a_1, \ldots, a_{n+1}) = \{x \in \mathbb{R}^{n+1} |\ x_1 + \cdots + x_{n+1} = 1,\ x_i \ge 0\}, \tag{51}$$

by the requirement that the coordinates $x_{(s)}$ of $x \in \mathbb{R}^{n+1}$, are *homogeneous barycentric coordinates*, [28].

By the *content* of $\mathcal{S}_n^+$, we mean

$$a_{\triangle_n} := \frac{1}{n!} \wedge_{i=2}^{(n+1)} (a_i - a_1) = \frac{1}{n!}(a_2 - a_1) \wedge (a_3 - a_1) \wedge \cdots \wedge (a_{n+1} - a_1)$$

$$= \frac{1}{n!}\Big(\wedge {}^{\vee}a_{(1)} - \wedge {}^{\vee}a_{(2)} + \cdots + (-1)^n \wedge {}^{\vee}a_{(n+1)}\Big), \tag{52}$$

in terms of the *wedge dual* notation introduced in (30). Wedging (54) on the left by $x \in \mathcal{S}_n^+$, gives

$$x \wedge a_{\triangle_n} = \frac{1}{n!}\Big(\sum_{i=1}^{n+1} x_i\Big) \wedge A_{(n+1)} = \frac{1}{n!} \wedge A_{(n+1)}. \tag{53}$$

Similarly, dotting (54) on the left by x gives

$$x \cdot a_{\triangle_n} = \frac{1}{n!} x \cdot \Big(\wedge {}^{\vee}a_{(1)} - \wedge {}^{\vee}a_{(2)} + \cdots + (-1)^n \wedge {}^{\vee}a_{(n+1)}\Big). \tag{54}$$

Let $v_1, \ldots, v_{k+1} \in \mathbb{R}^{k+1}$ be a set of $k+1$ vertices of a *k-simplex* $\mathcal{V}_k^+$ in $\mathcal{A}_{k+1}^+$. That is

$$v_i := \sum_{j=1}^{k+1} v_{ij} a_j = [v_{ij}] A_{(k+1)}, \tag{55}$$

where $[V]_{k+1} := [v_{ij}]$ is the *matrix* of $\mathcal{V}_k^+$. The *rows* of the *simplicial matrix* $[V]_{k+1}$ are the barycentric coordinates of the vertices $v_i \in \mathcal{V}_{k+1}^+$. It follows that $[V]_{k+1}$ is a non-negative matrix with the property that the sum of the coordinates in each row is equal to 1. Alternatively, since $v_1 \wedge \cdots \wedge v_{k+1} \ne 0$, and not requiring the coordinates to be barycentric, the matrix $[V]_{k+1}$ becomes the transition matrix from the basis of null vectors $\mathcal{A}_{k+1}$ to the basis vectors $v_i \in \mathcal{V}_{k+1}^+$, for which all the relations found after (27) remain valid.

The *content* of $\mathcal{V}_{k+1}^+$ is

$$v_{\triangle_k} = \wedge_{i=2}^{k+1}(v_i - v_1) = (v_2 - v_1) \wedge \cdots \wedge (v_{k+1} - v_1) \ne 0, \tag{56}$$

in the geometric algebra $\mathcal{A}_{k+1}^+$ of $\mathbb{R}^{k+1}$. Similar to (53) and (52), we have

$$x v_{\triangle_k} = x \cdot v_{\triangle_k} + x \wedge v_{\triangle_k},$$

but there is no obvious simplification as found for null simplices in (53).

5.1. LPGG Calculus of $S_n^+ \subset \mathbb{R}^{n+1}$

I will now give a brief introduction to the general theory of *LPGG Calculus*. Standard *geometric calculus* has been in continual development over the last half Century [8,9,13,21]. Every *signed graph* $\mathcal{V}_n^{\pm}$ of n-vertices can be studied in terms of any of the geometric algebras determined by the sequences of signs (60), (62), found in Appendix B. I will limit my discussion here to *signed positive* $\frac{1}{2}$*-graphs* $\mathcal{V}_m^+$ in $\mathbb{R}^{n+1}$, using the barycentric coordinates of the convex null symplex

$$S_n^+ := \mathcal{S}_n^+(a_1, \ldots, a_{n+1}) \subset \mathcal{N},$$

and the geometric algebra $\mathcal{A}_{n+1}^+ \equiv \mathbb{G}_{1,n}$ for $n \geq 1$. For the *special case* $n = 0$, we choose a single non-trivial null vector $a \in \mathcal{N} \notin \mathbb{G}_{1,0}$, and define

$$S_0^+ := \{a\} \subset \mathcal{N}$$

to represent a graph with single vertex.

For $m \leq n$, let $v_1, \ldots v_{m+1} \in \mathbb{R}^{n+1}$ denote the vertices of a signed simplex

$$\mathcal{V}_m^+ := \mathcal{V}_m^+(v_1, \ldots, v_{m+1}) \subset \mathcal{S}_n^+,$$

where $v_1 \wedge \cdots \wedge v_{m+1} \neq 0$. Define $a_{(n+1)} \equiv \{a\}_{(n+1)}$ by

$$\{a\}_{(n+1)} := \{a_1, \ldots, a_{n+1}\},$$

and by $\{\overset{\vee}{a}_i\}_{(n)}$, the set of n correlated null vectors obtained by leaving out a_i,

$$\{\overset{\vee}{a}_i\}_{(n)} := \{a_1, \ldots \overset{\vee}{a}_i \ldots, a_{n+1}\}.$$

When no confusion can arise, we shorten $\{a\}_{(n)}$ to $a_{(n)}$. For $n = 3$,

$$\{\overset{\vee}{a}_1\}_{(3)} = \{a_2, a_3\}, \; \{\overset{\vee}{a}_2\}_{(3)} = \{a_1, a_3\}, \text{ and } \{\overset{\vee}{a}_3\}_{(3)} = \{a_1, a_2\}.$$

Since the set of vectors $\{v\}_{(m+1)}$ are *linearly independent*,

$$\wedge v_{(m+1)} := v_1 \wedge \cdots \wedge v_{m+1} \neq 0,$$

$\mathcal{V}_m^+$ defines an m-simplex with $m + 1 = \binom{m+1}{m}$-*faces*. Each $(m+1)$-face is geometrically represented by the oriented m-vector

$$\wedge \overset{\vee}{v}_{(i)} := v_1 \wedge \cdots \overset{\vee}{i} \cdots \wedge v_{m+1}.$$

Also, define the $(m+1)$-sum and the m-sum, by

$$\sum v_{(m+1)} := v_1 + \cdots + v_{m+1}, \text{ and } \sum v_{(\overset{\vee}{i})} := v_1 + \cdots \overset{\vee}{i} \cdots + v_{m+1}.$$

The *signed complete graph* $\mathcal{V}_m^+$ is said to be *closed* if $\sum_i v_{(\overset{\vee}{i})} = 0$, and of *order* k, if k is the largest number of linearly independent vertices of $\mathcal{V}_m^+$. Naturally, we use the *barycentric coordinates* associated with $\mathcal{S}_n^+$, and, without loss of generality, assume that the *position vector* $x \in \mathcal{V}_{m+1}^+$, is given by

$$x = (x_1, \cdots, x_{m+1}) := \sum_{i=1}^{m+1} x_i a_i \in \mathbb{R}^{m+1} \subset \mathbb{R}^{n+1},$$

although other coordinate systems can be used.

I now restrict attention to studying the graph $\mathcal{V}_{m+1}^+$ of a particular m-dimensional *polytope*. For $x \in \mathcal{V}_{m+1}^+$, calculate

$$x^2 = \left(\sum_{i=1}^{m+1} x_i a_i\right)^2 = \sum_{0 \le i < j \le m+1} x_i x_j,$$

for all i, j, $0 < i \neq j \le m+1$, and where the vertices of the m-polytope satisfy

$$v_1 \wedge \cdots \wedge v_{m+1} \neq 0.$$

Since $\mathcal{V}_m^+ \subset \mathcal{S}_n^+$, the barycentric coordinates x_i of x will all be positive, so that

$$|x|^2 = x^2 = \sum_{0 \le i < j \le m+1} x_i x_j \ge 0. \tag{57}$$

For $x \in \mathcal{V}_m^+$, define

$$|x| = \sqrt{\sum_{0 \le i < j \le m+1} x_i x_j} \ge 0.$$

The points $x \in \mathcal{V}_m^+ \subset \mathcal{S}_n^+$, for which $|x| = 0$, are exactly those points x of the graph on the light cone. For all interior points of $\mathcal{V}_m^+$, where $|x| > 0$, define the unit vector

$$\hat{x} := \frac{x}{|x|}. \tag{58}$$

Since $x \in \mathcal{S}_n^+$ is barycentric, its coordinates satisfy $\sum x_i = 1$. Taking the partial derivative ∂_i of this equation, gives $\partial_i \sum_{j=1}^{n+1} x_j = 0$. By employing *higher order barycentric coordinates*, based upon *Hermite interpolation*, this constraint can be satisfied. Without going into details, for $x \in \mathcal{S}_n^+$, I want to preserve the property that $\partial_i x = a_i$ for each $0 < i \le n+1$, [12,19]. The same effect can be achieved by assuming, when differentiating x, we have relaxed the condition that the coordinates of $x \in \mathbb{R}^{n+1}$ are barycentric.

Recalling (26) and (31)

$$\nabla = \frac{2}{n}\Big(A_{n+1}\partial_{(n+1)} - n\hat{\nabla}\Big) = \frac{2}{n}\Big(\overset{\vee}{\nabla} - (n-1)\hat{\nabla}\Big). \tag{59}$$

For $x \in \mathbb{R}^{n+1}$, $\nabla x^2 = 2x$, $\nabla |x| = \hat{x}$, $\nabla |x| = \hat{x}$, and $\nabla \hat{x} = \frac{n}{|x|}$. These formulas remain valid at all points $x \in \mathcal{S}_n^+$, [21, p.66].

5.2. Platonic Solids

Applying the decomposition formula (59),

$$\overset{\vee}{\nabla} x = \sum_i \overset{\vee}{a}_i \frac{\partial x}{\partial i} = \sum_i \overset{\vee}{a}_i a_i = \binom{n}{2} = \frac{n(n-1)}{2},$$

which is the number of linear independent edges of S_n^+.

The *Laplacian* $\check{\nabla}^2$ for the light cone projective geometry of $\mathcal{S}_n^+$ is

$$\check{\nabla}^2 = \sum_{i=1}^{n} \partial_i^2 + \binom{n}{2} \sum_{1 \le i \le j \le n} \partial_i \partial_j .$$

Just as in Euclidean and pseudo-Euclidean geometry, the Laplacian $\check{\nabla}^2$ in $\mathcal{S}_n^+$, is scalar valued.

For the signed simplex $\mathcal{S}_3^+$,

$$\check{\nabla}^2 = \big(\partial_1^2 + \partial_2^2 + \partial_3^2 + \partial_2\partial_3 + \partial_1\partial_3 + \partial_1\partial_2\big).$$

For $x \in \mathcal{S}_n^+$, we calculate

$$\check{\nabla}^2 x^2 = \sum_{i=1}^{n} \partial_i^2 x^2 + \binom{n}{2} \sum_{1 \le i \le j \le n} \partial_i \partial_j x^2 = \binom{n}{2}^2 .$$

I conclude with a *Conjecture* for *n-Platonic Solids* in $(n+1)$-dimensional space $\mathbb{R}^{n+1}$.

Conjecture: The number of n-Platonic Solids in any dimension n is equal to the number of distinct n-Platonic Solids found in the n-simplex $\mathcal{S}_n^+ \subset \mathbb{R}^{n+1}$ with its vertices located at the null vectors $a_1, \ldots, a_{n+1} \in \mathcal{S}_n^+$.

The number is known to be given by the sequence

$$\{1, 1, \infty, 5, 6, 3, 3, 3, \ldots\},$$

[1,6,7,10,25].

I want to welcome the reader to this beautiful new, but not really so new, theory. Be careful—the calculations can be treacherous.

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Appendix A: Geometric Algebra Identities in $\mathcal{A}^+_{1,n}$

Some basis identities of the geometric algebra

$$\mathbb{G}_{1,n} \equiv \mathcal{A}^+_{1,n} = \mathbb{R}^{n+1} := \mathbb{R}(a_1, \ldots, a_{n+1}),$$

where $a_i \cdot a_j = \frac{1-\delta_{ij}}{2}$.

1. $x^2 = x_1 x_2,\ x \cdot v_1 = \frac{1}{2}(x_1 v_{12} + x_2 v_{11}),\ x \cdot v_2 = \frac{1}{2}(x_2 v_{22} + x_2 v_{21})$
2. $v_1 \cdot v_2 = \frac{1}{2}(v_{11} v_{22} + v_{12} v_{21}),\ v_1 \wedge v_2 = \det \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix} a_1 \wedge a_2$
3. $(a_1 \wedge a_2) = \frac{1}{2}(a_1 - a_1) \wedge (a_1 + a_2) = \frac{1}{2} f_1 e_1,\ (a_1 \wedge a_2)^2 = \frac{1}{4},$
4. For $y = y_1 a_1 + y_2 a_2$, $x \wedge y = \det \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} a_1 \wedge a_2$
5. $(x \wedge y)^2 = \det \begin{pmatrix} y \cdot x & y^2 \\ x^2 & x \cdot y \end{pmatrix}$

Change of Basis Formulas for $\mathbf{n+1=8}$

$$T_8 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 \\ 1 & 0 & \frac{1}{2} & \frac{1}{2\sqrt{3}} & \sqrt{\frac{2}{3}} & 0 & 0 & 0 \\ 1 & 0 & \frac{1}{2} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{6}} & \frac{\sqrt{5}}{2\sqrt{2}} & 0 & 0 \\ 1 & 0 & \frac{1}{2} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{6}} & \frac{1}{2\sqrt{10}} & \sqrt{\frac{3}{5}} & 0 \\ 1 & 0 & \frac{1}{2} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{6}} & \frac{1}{2\sqrt{10}} & \frac{1}{2\sqrt{15}} & \frac{\sqrt{7}}{2\sqrt{3}} \end{pmatrix}$$

$$T_8^{-1} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{3}} & 0 & 0 & 0 & 0 \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \sqrt{\frac{3}{2}} & 0 & 0 & 0 \\ -\frac{1}{\sqrt{10}} & -\frac{1}{\sqrt{10}} & -\frac{1}{\sqrt{10}} & -\frac{1}{\sqrt{10}} & -\frac{1}{\sqrt{10}} & 2\sqrt{\frac{2}{5}} & 0 & 0 \\ -\frac{1}{\sqrt{15}} & -\frac{1}{\sqrt{15}} & -\frac{1}{\sqrt{15}} & -\frac{1}{\sqrt{15}} & -\frac{1}{\sqrt{15}} & -\frac{1}{\sqrt{15}} & \sqrt{\frac{5}{3}} & 0 \\ -\frac{1}{\sqrt{21}} & -\frac{1}{\sqrt{21}} & -\frac{1}{\sqrt{21}} & -\frac{1}{\sqrt{21}} & -\frac{1}{\sqrt{21}} & -\frac{1}{\sqrt{21}} & -\frac{1}{\sqrt{21}} & 2\sqrt{\frac{3}{7}} \end{pmatrix}$$

Appendix B: Classification of Geometric Algebras

There is an extremely interesting relationship between plus and minus signs of the squares of the standard basis elements of $\mathbb{G}_{p,q}$, and the 8-fold periodicity structure of Clifford geometric algebras. Consider the following:

1. $\{+\}, \{-\},\ \ e_1 \in \mathbb{G}_{1,0},\ f_1 \in \mathbb{G}_{0,1}$ $\qquad \prod signs\ -$
2. $\{++\},\ \{+-\},\ \{--\},\ \ \mathbb{G}_{p,q},\ \ p+q=2$ $\qquad \prod signs\ -$
3. $\{+++\},\ \{++-\},\ \{+--\},\ \{---\}\ \ p+q=3,\ etc.$ $\qquad \prod signs\ +$
4. $\{++++\},\ \{+++-\},\ \{++--\},\ \{+---\},\ \{----\}$ $\qquad \prod signs\ +$

5. $\binom{n+1}{2} = \binom{6}{2} = 15$ $\qquad \prod$ (15) $-$
6. $\binom{n+1}{2} = \binom{7}{2} = 21$ $\qquad \prod$ (21) $-$

This obviously gives the infinite sequence

$$--, ++, --, ++, --, ++ \ldots. \tag{60}$$

Real geometric algebras $\mathbb{G}_{p,q}$ are constructed by extending the real number system $\mathbb{R}$ by $n = p+q$ anti-commuting vectors e_i, f_j which have squares ± 1, respectively

$$\mathbb{G}_{p,q} := \mathbb{R}[e_1, \cdots, e_p, f_1, \cdots f_q], \tag{61}$$

[8, 21]. A more concise treatment of this construction, and its relationship to real and complex square matrices is [22].

Geometric algebras enjoy a very special 8-fold periodicity relationship [24]. A basic understanding of this important periodicity relationship can be obtained by studying the signs of the squares of the pseudoscalar elements for the geometric algebra $\mathbb{G}_{p,q}$ of successively higher dimensions. The $\pm$ signs over *pseudoscalar elements* indicate the sign of the *square* of that element.

0. $\{a \in \mathcal{N} |\ a^2 = 0\}$. The *null vector* $a \neq 0$ has the property that $a^2 = 0$.
1. $\{\overset{+}{e}_1\},\ \{\overset{-}{f}_1\}:\ \ \mathbb{G}_{1,0},\ \mathbb{G}_{0,1};$
2. $\{e_1\overset{-}{e}_2\}\ \{e_1\overset{+}{f}_1\},\ \{f_1\overset{-}{f}_2\}:\ \mathbb{G}_{2,0},\ \mathbb{G}_{1,1},\ \mathbb{G}_{0,2};$
3. $\{e_1\overset{-}{e}_2e_3\},\ \{e_1\overset{+}{e}_2f_1\},\ \{e_1\overset{-}{f}_1f_2\},\ \{f_1\overset{+}{f}_2f_3\}:\ \mathbb{G}_{3,0},\ \mathbb{G}_{2,1},\ \mathbb{G}_{1,2},\ \mathbb{G}_{0,3};$
4. $\{e_1\overset{+}{e}_2e_3e_4\},\ \{e_1e_2\overset{-}{e}_3f_1\},\ \{e_1e_2\overset{+}{f}_1f_2\},\ \{e_1\overset{-}{f}_1f_2f_3\},\ \{f_1\overset{+}{f}_2f_3f_4\}$
5. $\{e_1e_2\overset{+}{e}_3e_4e_5\},\ \{e_1e_2\overset{-}{e}_3e_4f_1\},\ \{e_1e_2\overset{+}{e}_3f_1f_2\},\ \{e_1e_2\overset{-}{f}_1f_2f_3\},$
   $\{e_1f_1\overset{+}{f}_2f_3f_4\}, \{f_1f_2\overset{-}{f}_3f_4f_5\}.$
6. $\{e_1e_2\overset{-}{e}_3e_4e_5e_6\},\ \{e_1e_2e_3\overset{+}{e}_4e_5f_1\},\ \{e_1e_2e_3\overset{-}{e}_4f_1f_2\},\ \{e_1e_2e_3\overset{+}{f}_1f_2f_3\},$
   $\{e_1e_2\overset{-}{f}_1f_2f_3f_4\}, \{e_1f_1\overset{+}{f}_2f_3f_4f_5, \{f_1f_1\overset{-}{f}_2f_3f_4f_5\}.$

This obviously gives the sequence,

$$+, -, -+-, -+-+, +-+-+, -+-+-+, \cdots \tag{62}$$

The sequences (60) and (62) follow directly from the well known periodicity laws of all real and complex geometric algebras [13]. The two sequences beautifully reflect how any geometric algebra $\mathbb{G}_{p,q}$, for $n = p+q$ can be represented either as a real or complex matrix algebra of dimension 2^n. In the case of the complex matrix algebra, the imaginary number i can be interpreted as the pseudoscalar element $e_1f_1 \cdots e_nf_nf_{n+1}$ in the center of the real geometric algebra $\mathbb{G}_{n,n+1}$.

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